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HAUSDORFF AND PACKING DIMENSIONS, INTERSECTION MEASURES, AND SIMILARITIES

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Abstract. Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports. We study the Hausdorff, $\dim_H$, and packing dimension, $\dim_p$, properties of the intersection measures $\mu \cap f_\sharp \nu$ when f runs through the similarities of $\mathbf{R}^n$ and $f_\sharp \nu$ is the image of ν under f. These measures can be regarded as natural measures on $\operatorname{spt} \mu \cap f(\operatorname{spt} \nu)$, where spt is the support of a measure. Using the relations between Hausdorff dimensions of sets and measures, we show that if $\dim_H(\mu \times \nu) = \dim_H \mu + \dim_H \nu > n$ and if the t-energy of ν is finite for all $0 < t < \dim_H \nu < n$, then for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ we have

$$\operatorname{ess\,inf}\{\dim_H \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu : z \in \mathbf{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbf{R}^n) > 0\} = \dim_H \mu + \dim_H \nu - n.$$

Here θ_n is the unique orthogonally invariant Radon probability measure on the orthogonal group of $\mathbf{R}^n$, denoted by $\mathscr{O}_n$, $\mathscr{L}^1$ is the Lebesgue measure on the open interval $(0,\infty)$, and $\tau_z \circ g \circ \delta_r \colon \mathbf{R}^n \to \mathbf{R}^n$ is the similarity $\tau_z \circ g \circ \delta_r(a) = rga + z$. By relating packing dimensions of intersection measures to certain integral kernels, we prove that if the s-energy of μ is finite and the t-energy of ν is finite for some $0 < s < n$ and $0 < t < n$ with $s + t > n$, then for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ we have

$$\operatorname{ess\,inf}\{\dim_p \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu : z \in \mathbf{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbf{R}^n) > 0\} = d_{\mu,\nu},$$

where $d_{\mu,\nu}$ is a constant depending only on the measures μ and ν. We also deduce corresponding equalities for the upper Hausdorff and upper packing dimensions.

1. Introduction

Let A and B be Borel sets in $\mathbf{R}^n$. The relations between the Hausdorff dimensions, $\dim_H$, of A, B, and $f(B)$, when f runs through the similarities of $\mathbf{R}^n$, were studied by Mattila in [9]. He showed that if A is $\mathscr{H}^s$ measurable and B is $\mathscr{H}^t$ measurable such that $0 < \mathscr{H}^s(A) < \infty$ and $0 < \mathscr{H}^t(B) < \infty$ for some $0 < s < n$ and $0 < t < n$ with $s + t \geq n$, then

$$\dim_H A \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})B \geq s + t - n \tag{1.1}$$

for $\mathscr{H}^s \times \mathscr{H}^t \times \theta_n \times \mathscr{L}^1$ almost all $(x,y,g,r) \in A \times B \times \mathscr{O}_n \times (0,\infty)$. Here $\mathscr{H}^s$ is the s-dimensional Hausdorff measure, $\tau_x \circ g \circ \delta_r \circ \tau_{-y} \colon \mathbf{R}^n \to \mathbf{R}^n$ is the similarity

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$\tau_x \circ g \circ \delta_r \circ \tau_{-y}(a) = rg(a-y)+x$, θ_n is the unique orthogonally invariant Radon probability measure on the orthogonal group of $\mathbf{R}^n$ denoted by $\mathscr{O}_n$, and $\mathscr{L}^1$ is the Lebesgue measure on the open interval $(0,\infty)$. In general the opposite inequality in (1.1) is false, but it holds under the additional assumption that the set B has positive t-dimensional lower density at all of its points (see [9, Theorem 6.13]). For more information on results related to these questions see also [7].

Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports. The purpose of this paper is to study Hausdorff and packing dimension analogues of (1.1) for intersection measures $\mu \cap f_\sharp \nu$ when f runs through the similarities of $\mathbf{R}^n$ and $f_\sharp \nu$ is the image of ν under f. These intersection measures introduced by Mattila in [9] can be regarded as natural measures on $\operatorname{spt}\mu \cap f(\operatorname{spt}\nu)$, where spt is the support of a measure. Using the relations between the Hausdorff dimensions of sets and measures, we show that if the t-energy of ν is finite for all $0 < t < \dim_H \nu < n$, then for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ we have

$$\dim_H \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq \dim_H \nu + \dim_H \nu - n$$

for $\mathscr{L}^n$ almost all $z \in \mathbf{R}^n$ with $\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbf{R}^n) > 0$. Here $(\tau_z \circ g \circ \delta_r)_\sharp \nu$ is the image of ν under the similarity $\tau_z \circ g \circ \delta_r \colon \mathbf{R}^n \to \mathbf{R}^n$, $\tau_z \circ g \circ \delta_r(a) = rga + z$ and $\mathscr{L}^n$ is the Lebesgue measure on $\mathbf{R}^n$. If we suppose in addition to the above assumptions that $\dim_H(\mu \times \nu) = \dim_H \mu + \dim_H \nu > n$, then for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ we have

$$\begin{aligned} &\operatorname{ess\,inf}\{ \dim_H \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu : z \in \mathbf{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbf{R}^n) > 0\} \\ &\qquad = \dim_H \mu + \dim_H \nu - n. \end{aligned} \tag{1.2}$$

We also prove corresponding results for the upper Hausdorff dimension.

We continue the work by Järvenpää ([4] and [5]) on packing dimension, $\dim_p$, properties of intersection measures. In [4] it is shown that if the $(n-t)$-energy of μ is finite and the t-energy of ν is finite for some $0 < t < n$, and if $\dim_H \mu + \dim_H \nu > n$, then

$$\dim_p \mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu \geq \max\left\{ \frac{\dim_H \nu \dim_p \mu (\dim_H \mu + \dim_H \nu - n)}{n \dim_H \mu - (n - \dim_H \nu) \dim_p \mu}, \frac{\dim_H \mu \dim_p \nu (\dim_H \mu + \dim_H \nu - n)}{n \dim_H \nu - (n - \dim_H \mu) \dim_p \nu} \right\}$$

for $\mu \times \nu \times \theta_n \times \mathscr{L}^1$ almost all $(x,y,g,r) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathscr{O}_n \times (0,\infty)$. In [5] a corresponding result is proved when we take isometries as the transformation group in place of similarities. The methods we are using when considering packing dimensions of intersection measures are influenced by the theory for projections of measures introduced by Falconer and Howroyd in [1] and by the methods in [6] where sections of measures were considered instead of general intersections. We show that the following analogue of (1.1) holds: if the s-energy of μ is finite and the t-energy of ν is finite for some $0<s<n$ and $0<t<n$ with $s+t>n$, then for $\theta_n \times \mathscr{L}^1$ almost all $(g,r)\in \mathscr{O}_n\times(0,\infty)$ we have

(1.3)
$$\operatorname{ess\,inf}\bigl\{\dim_p \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu : z\in \mathbf{R}^n \text{ with } \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(\mathbf{R}^n)>0\bigr\} = d_{\mu,\nu}.$$

Here $d_{\mu,\nu}$ is a constant obtained by convolving the product measure $\mu\times\nu$ with a certain kernel. Corresponding results for the upper packing dimension are also deduced. If we consider isometries instead of similarities, then (1.2) holds if we assume that $\dim_H(\mu\times\nu)=\dim_H\mu+\dim_H\nu>n$ and that the t-energy of ν is finite for all $\frac{1}{2}(n+1)<t<n$. This can be proved using the methods of Section 3. For isometries the methods of Section 5 cannot be used. Then an integration with respect to r is not involved, which makes things more difficult.

Equalities (1.2) and (1.3) cannot be strengthened to a result saying that $\dim_H \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu$ or $\dim_p \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu$ would be almost surely constant. To see this, let μ_1 and μ_2 be suitably chosen measures on $\mathbf{R}^n$ supported by two disjoint balls such that there is $A\subset\mathscr{O}_n\times(0,\infty)$ with positive $\theta_n\times\mathscr{L}^1$ measure such that for any $(g,r)\in A$ either $W_{(z,-z)/2}\cap\bigl(\operatorname{spt}\mu_1\times(g\circ\delta_r)(\operatorname{spt}\nu)\bigr)=\emptyset$ or $W_{(z,-z)/2}\cap\bigl(\operatorname{spt}\mu_2\times(g\circ\delta_r)(\operatorname{spt}\nu)\bigr)=\emptyset$ for all $z\in\mathbf{R}^n$ (for the notation see Chapter 2). Then by choosing the dimensions (both Hausdorff and packing) of the measures μ_1 and μ_2 in a suitable way, we can find for any $(g,r)\in A$ sets $B^1_{g,r}\subset\mathbf{R}^n$ and $B^2_{g,r}\subset\mathbf{R}^n$ with positive $\mathscr{L}^n$ measures such that the dimension of the measure $\pi_\sharp\bigl[\bigl(\mu_1\times(g\circ\delta_r)_\sharp\nu+\mu_2\times(g\circ\delta_r)_\sharp\nu\bigr)_{W,(z,-z)/2}\bigr]$ is big for $z\in B^1_{g,r}$ and small for $z\in B^2_{g,r}$.

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2. Notation and preliminaries

We denote by $d(x,A)=\inf\{|x-a| : a\in A\}$ the distance between $x\in\mathbf{R}^n$ and a non-empty set $A\subset\mathbf{R}^n$. The open interval in $\mathbf{R}\cup\{-\infty,\infty\}$ with end points $a,b\in\mathbf{R}\cup\{-\infty,\infty\}$ is denoted by (a,b). For the corresponding closed interval we use the notation $[a,b]$. Further, $B(x,r)$ is the closed ball of centre $x\in\mathbf{R}^n$ and radius $0<r<\infty$, and $\alpha(n)=\mathscr{L}^n\bigl(B(0,1)\bigr)$ where $\mathscr{L}^n$ is the Lebesgue measure on $\mathbf{R}^n$. For $0\le s<\infty$, the s-dimensional Hausdorff measure is denoted by $\mathscr{H}^s$.

If μ is a measure on a set X, we denote by $f_\sharp\mu$ the image of μ under a function $f\colon X\to Y$, that is,

$$f_\sharp\mu(A)=\mu\bigl(f^{-1}(A)\bigr)$$

for all $A \subset Y$. The restriction of μ to a set $B \subset X$ is denoted by $\mu \mid B$, that is,

$$\mu \mid B(A) = \mu(B \cap A)$$

for all $A \subset X$. For $0 < t < n$, the t-energy of a Radon measure μ on $\mathbf{R}^n$ is defined by

$$I_t(\mu) = \iint |x-y|^{-t}\, d\mu x\, d\mu y.$$

Note that if μ is a finite Radon measure on $\mathbf{R}^n$ and $I_t(\mu) < \infty$, then $I_s(\mu) < \infty$ for all $0 < s < t$. Let μ and ν be measures on a set X. The measure μ is said to be absolutely continuous with respect to ν, if $\mu(A) = 0$ for any $A \subset X$ with $\nu(A) = 0$. In this case we write $\mu \ll \nu$.

Let m and n be integers with $0 < m < n$. The Grassmann manifold, which consists of all $(n-m)$-dimensional linear subspaces of $\mathbf{R}^n$, is denoted by $G_{n,n-m}$. For all $V \in G_{n,n-m}$, let $V^\perp \in G_{n,m}$ be the orthogonal complement of V, and $P_{V^\perp}\colon \mathbf{R}^n \to V^\perp$ the orthogonal projection onto $V^\perp$. We use the notation $\mathscr{O}_n$ for the orthogonal group of $\mathbf{R}^n$ consisting of all linear maps $g\colon \mathbf{R}^n \to \mathbf{R}^n$ preserving distance, that is, $|g(x) - g(y)| = |x-y|$ for all $x, y \in \mathbf{R}^n$. The unique invariant Radon probability measure on $\mathscr{O}_n$ is denoted by θ_n.

A map $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is a similarity, if there is $0 < r < \infty$ such that $|f(x) - f(y)| = r|x-y|$ for all $x, y \in \mathbf{R}^n$. In this case for some $z \in \mathbf{R}^n$, $g \in \mathscr{O}_n$, and $0 < r < \infty$ we have

$$f = \tau_z \circ g \circ \delta_r,$$

where $\tau_z\colon \mathbf{R}^n \to \mathbf{R}^n$ is the translation $\tau_z(x) = x + z$, and $\delta_r\colon \mathbf{R}^n \to \mathbf{R}^n$ is the homothety $\delta_r(x) = rx$.

For the definition of intersection measures we need the following definition of sliced measures. Let m and n be integers with $0 < m < n$. Let μ be a finite Radon measure on $\mathbf{R}^n$, $V \in G_{n,n-m}$, and $V_a = \{v + a : v \in V\}$ for all $a \in V^\perp$. For $\mathscr{H}^m$ almost all $a \in V^\perp$ there is a Radon measure $\mu_{V,a}$ on V_a such that

$$\int \varphi\, d\mu_{V,a} = \lim_{\delta \to 0} (2\delta)^{-m} \int_{V_a(\delta)} \varphi\, d\mu \tag{2.1}$$

for all non-negative continuous functions φ on $\mathbf{R}^n$ with compact support (see [10, Chapter 10]). Here

$$V_a(\delta) = \{y \in \mathbf{R}^n : d(y, V_a) \le \delta\}.$$

The measure $\mu_{V,a}$ is the slice of μ by the plane V_a. Obviously,

$$\operatorname{spt} \mu_{V,a} \subset \operatorname{spt} \mu \cap V_a,$$

where spt is the support of a measure. Further, if $P_{V^\perp\sharp}\mu \ll \mathscr{H}^m \mid V^\perp$ and f is a non-negative Borel function on $\mathbf{R}^n$ with $\int f\,d\mu < \infty$, then

$$\int_{V^\perp}\int f\,d\mu_{V,a}\,d\mathscr{H}^m a = \int f\,d\mu \tag{2.2}$$

(see [8, Lemma 3.4 (4)]).

Now we are ready to define intersection measures. We use the method from [9], which is the same as used in [3, 4.3.20] in connection with the construction of intersection currents. Let W be the diagonal of $\mathbf{R}^n \times \mathbf{R}^n$, that is,

$$W = \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : x = y\}.$$

Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports. The intersection measure $\mu \cap f_\sharp \nu$, where $f = \tau_z \circ g \circ \delta_r$ for some $z \in \mathbf{R}^n$, $g \in \mathscr{O}_n$, and $0 < r < \infty$, is constructed by slicing the product measure $\mu \times (g \circ \delta_r)_\sharp \nu$ by the n-plane

$$W_{(z,-z)/2} = \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : x - y = z\},$$

and by projecting this sliced measure to $\mathbf{R}^n$ by the projection $\pi\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$, $\pi(x,y) = x$. Hence

$$\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu = 2^{n/2}\alpha(n)^{-1}\pi_\sharp\big[\big(\mu \times (g \circ \delta_r)_\sharp \nu\big)_{W,(z,-z)/2}\big]$$

provided that the sliced measure $\big(\mu \times (g \circ \delta_r)_\sharp \nu\big)_{W,(z,-z)/2}$ exists. This is the case for $\mathscr{L}^n$ almost all $z \in \mathbf{R}^n$. Clearly,

$$\operatorname{spt} \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \subset \operatorname{spt} \mu \cap (\tau_z \circ g \circ \delta_r) \operatorname{spt} \nu.$$

If φ is a non-negative lower semicontinuous function on $\mathbf{R}^n$, then (2.1) gives

$$\begin{aligned} &\int \varphi\,d\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \\ &\qquad \le \varliminf_{\delta \to 0} \alpha(n)^{-1}\delta^{-n} \int_{\{(x,y): |S_{g,r}(x,y) - z| \le \delta\}} \varphi(x)\,d(\mu \times \nu)(x,y). \end{aligned} \tag{2.3}$$

Here $S_{g,r}\colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is defined by $S_{g,r}(x,y) = x - rgy$.

Note that if $S_{g,r\sharp}(\mu \times \nu) \ll \mathscr{L}^n$, then $P_{W^\perp\sharp}\big(\mu \times (g \circ \delta_r)_\sharp \nu\big) \ll \mathscr{H}^n | W^\perp$, and the disintegration formula (2.2) implies that

$$\int_{W^\perp}\int f\,d\big(\mu \times (g \circ \delta_r)_\sharp \nu\big)_{W,a}\,d\mathscr{H}^n a = \int f\,d\big(\mu \times (g \circ \delta_r)_\sharp \nu\big) \tag{2.4}$$

provided that f is a non-negative Borel function with $\int f\,d\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)<\infty$. Further, by (2.4),

$$\mathscr{H}^n\Big(\big\{a\in W^\perp:\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,a}(\mathbf{R}^n\times\mathbf{R}^n)>0\big\}\Big)>0,$$

which gives

$$\mathscr{L}^n\Big(\big\{z\in\mathbf{R}^n:\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(\mathbf{R}^n)>0\big\}\Big)>0.$$

The following definitions of dimensions will be used throughout this paper. The Hausdorff and packing dimensions of a finite Radon measure μ on $\mathbf{R}^n$ are defined by

$$\dim_H\mu=\sup\big\{u\geq 0:\limsup_{h\to 0}h^{-u}\mu\big(B(x,h)\big)=0\text{ for }\mu\text{ almost all }x\in\mathbf{R}^n\big\}$$

and

$$\dim_p\mu=\sup\big\{u\geq 0:\liminf_{h\to 0}h^{-u}\mu\big(B(x,h)\big)=0\text{ for }\mu\text{ almost all }x\in\mathbf{R}^n\big\}.$$

We will also consider the upper Hausdorff and upper packing dimensions defined as follows

$$\dim_H^*\mu=\sup\Big\{u\geq 0:\mu\Big(\big\{x\in\mathbf{R}^n:\limsup_{h\to 0}h^{-u}\mu\big(B(x,h)\big)=0\big\}\Big)>0\Big\}$$

and

$$\dim_p^*\mu=\sup\Big\{u\geq 0:\mu\Big(\big\{x\in\mathbf{R}^n:\liminf_{h\to 0}h^{-u}\mu\big(B(x,h)\big)=0\big\}\Big)>0\Big\}.$$

Equivalently, these dimensions can be determined by using Hausdorff and packing dimensions of sets. In fact,

$$\begin{aligned}\dim_H\mu&=\inf\{\dim_H A:A\text{ is a Borel set and }\mu(A)>0\},\\ \dim_p\mu&=\inf\{\dim_p A:A\text{ is a Borel set and }\mu(A)>0\},\\ \dim_H^*\mu&=\inf\{\dim_H A:A\text{ is a Borel set and }\mu(\mathbf{R}^n\setminus A)=0\},\end{aligned}$$

and

$$\dim_p^*\mu=\inf\{\dim_p A:A\text{ is a Borel set and }\mu(\mathbf{R}^n\setminus A)=0\}.$$

The following lemma gives a relation between finiteness of energies and Hausdorff dimensions of measures. It is an immediate consequence of the relation between Riesz capacities and Hausdorff dimensions of Borel sets.

Lemma 2.5. *If μ is a Radon measure on $\mathbf{R}^n$ with $0<\mu(\mathbf{R}^n)<\infty$ and with $I_t(\mu)<\infty$, then $\dim_H\mu\geq t$.*

3. Hausdorff dimension and intersection measures

Lemma 3.1. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports. Assume that $\dim_H(\mu\times\nu) > n$. If $(g,r) \in \mathscr{O}_n \times (0,\infty)$ is such that $S_{g,r_\sharp}(\mu\times\nu) \ll \mathscr{L}^n$, then*

$$\operatorname{ess\,inf}\{ \dim_H \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu : z\in\mathbf{R}^n \text{ with } \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(\mathbf{R}^n) > 0\} \\ \leq \dim_H(\mu\times\nu) - n.$$

Proof. Consider $u \geq 0$ such that $\dim_H \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu \geq u$ for $\mathscr{L}^n$ almost all $z\in\mathbf{R}^n$ with $\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(\mathbf{R}^n) > 0$. Since $\dim_H \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu \leq \dim_H\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}$, we have

$$u \leq \dim_H\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}$$

for $\mathscr{H}^n$ almost all $(z,-z)/2\in W^\perp$ with $\big(\mu\times(g\circ\delta_r)_\sharp\big)_{W,(z,-z)/2}(\mathbf{R}^n\times\mathbf{R}^n) > 0$. The fact that $\dim_H\big(\mu\times(g\circ\delta_r)_\sharp\nu\big) = \dim_H(\mu\times\nu) > n$, gives with [6, Lemma 3.1]

$$u \leq \dim_H \mu\times(g\circ\delta_r)_\sharp\nu - n = \dim_H(\mu\times\nu) - n$$

and the claim follows. Note that [6, Lemma 3.1] holds for W since $P_{W^\perp\sharp}\big(\mu\times(g\circ\delta_r)_\sharp\nu\big) \ll \mathscr{H}^n \mid W^\perp$. □

For the purpose of proving that the opposite inequality holds in Lemma 3.1 under some additional assumptions we need the following result.

Lemma 3.2. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports and let $B\subset\mathbf{R}^n$ be a Borel set. If $(g,r)\in\mathscr{O}_n\times(0,\infty)$ is such that $S_{g,r_\sharp}(\mu\times\nu)\ll\mathscr{L}^n$, then*

$$(\mu\mid B)\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu = \big(\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu\big)\mid B$$

for $\mathscr{L}^n$ almost all $z\in\mathbf{R}^n$.

Proof. Let $A\subset\mathbf{R}^n$. Since $P_{W^\perp\sharp}\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)\ll\mathscr{H}^n\mid W^\perp$, we have by [6, Lemma 3.2] for $\mathscr{L}^n$ almost all $z\in\mathbf{R}^n$

$$\begin{aligned}
\big((\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu)\mid B\big)(A) &= 2^{n/2}\alpha(n)^{-1}\pi_\sharp\big[\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}\big](B\cap A)\\
&= 2^{n/2}\alpha(n)^{-1}\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}\big((B\times\mathbf{R}^n)\cap(A\times\mathbf{R}^n)\big)\\
&= 2^{n/2}\alpha(n)^{-1}\big(\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}\mid(B\times\mathbf{R}^n)\big)(A\times\mathbf{R}^n)\\
&= 2^{n/2}\alpha(n)^{-1}\big(\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)\mid(B\times\mathbf{R}^n)\big)_{W,(z,-z)/2}(A\times\mathbf{R}^n)\\
&= 2^{n/2}\alpha(n)^{-1}\big((\mu|B)\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}(A\times\mathbf{R}^n)\\
&= 2^{n/2}\alpha(n)^{-1}\pi_\sharp\big[\big((\mu|B)\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}\big](A)\\
&= (\mu\mid B)\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(A). \;\square
\end{aligned}$$

Lemma 3.3. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports. Assume that $I_t(\nu) < \infty$ for all $0 < t < \dim_H \nu < n$. Then for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ we have*

$$\dim_H \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq \dim_H \mu + \dim_H \nu - n$$

for $\mathscr{L}^n$ almost all $z \in \mathbf{R}^n$ with $\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbf{R}^n) > 0$.

Proof. Let $0 < t < \dim_H \nu < n$. It is enough to prove that for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ we have

$$\dim_H \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq \dim_H \mu + t - n$$

for $\mathscr{L}^n$ almost all $z \in \mathbf{R}^n$ with $\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbf{R}^n) > 0$. The claim follows by taking a sequence (t_i) tending to $\dim_H \nu$ from below.

We may assume that $\dim_H \mu + t > n$. For all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ define

$$C_{g,r} = \{z \in \mathbf{R}^n : \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbf{R}^n) > 0\}.$$

Since

$$\begin{aligned}\{z \in C_{g,r} :& \dim_H \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu < \dim_H \mu + t - n\} \\ &= \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \Big\{ z \in C_{g,r} : \text{ there is a Borel set } A \subset \mathbf{R}^n \text{ such that} \\ &\qquad \dim_H A < \dim_H \mu + t - n - \frac{1}{i} \text{ and } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(A) > \frac{1}{j} \Big\},\end{aligned}$$

it suffices to show that for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ the set

$$\begin{aligned}E_{g,r} = \{z \in C_{g,r} :& \text{ there is a Borel set } A \subset \mathbf{R}^n \text{ such that} \\ & \dim_H A < u + t - n \text{ and } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(A) > \varepsilon\}\end{aligned}$$

has $\mathscr{L}^n$ measure zero for fixed $u < \dim_H \mu$ and $\varepsilon > 0$.

Let $u < v < \dim_H \mu$. Then $\limsup_{h \to 0} h^{-v} \mu\big(B(x,h)\big) = 0$ for μ almost all $x \in \mathbf{R}^n$, and so,

$$\lim_{i \to \infty} \mu(\mathbf{R}^n \setminus B^i) = 0, \tag{3.4}$$

where

$$B^i = \Big\{ x \in \mathbf{R}^n : \mu\big(B(x,h)\big) \leq h^v \text{ for all } 0 < h \leq \frac{1}{i} \Big\}$$

is a Borel set. Further, $I_u(\mu \mid B^i) < \infty$ for all i. By [9, Theorem 6.6] we have for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ that $S_{g,r_\sharp}\big((\mu \mid B_i) \times \nu\big) \ll \mathscr{L}^n$ for all i. This together with (3.4) gives $S_{g,r_\sharp}(\mu\times\nu)\ll\mathscr{L}^n$ for $\theta_n\times\mathscr{L}^1$ almost all $(g,r)\in\mathscr{O}_n\times(0,\infty)$. By [9, Theorem 6.7] and Lemma 3.2, for $\theta_n\times\mathscr{L}^1$ almost all $(g,r)\in\mathscr{O}_n\times(0,\infty)$ we have for $\mathscr{L}^n$ almost all $z\in\mathbf{R}^n$

$$I_{u+t-n}\big((\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu)\mid B^i\big)<\infty \tag{3.5}$$

for all i. For all i and $(g,r)\in\mathscr{O}_n\times(0,\infty)$ define

$$D^i_{g,r}=\big\{z\in C_{g,r}:\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}\big((\mathbf{R}^n\setminus B^i)\times\mathbf{R}^n\big)>\alpha(n)2^{-n/2}\varepsilon\big\}.$$

Now the disintegration formula (2.4) gives that for $\theta_n\times\mathscr{L}^1$ almost all $(g,r)\in\mathscr{O}_n\times(0,\infty)$

$$\begin{aligned}\mu\times(g\circ\delta_r)_\sharp\nu\big((\mathbf{R}^n\setminus B^i)\times\mathbf{R}^n\big)&\\ &=\int\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,a}\big((\mathbf{R}^n\setminus B^i)\times\mathbf{R}^n\big)\,d(\mathscr{H}^n\mid W^\perp)a\\ &=2^{n/2}\alpha(n)^{-1}\int\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}\big((\mathbf{R}^n\setminus B^i)\times\mathbf{R}^n\big)\,d\mathscr{L}^nz\\ &\geq\varepsilon\mathscr{L}^n(D^i_{g,r})\end{aligned}$$

for all i, which gives by (3.4)

$$\lim_{i\to\infty}\mathscr{L}^n(D^i_{g,r})=0. \tag{3.6}$$

Let $(g,r)\in\mathscr{O}_n\times(0,\infty)$ such that (3.5) and (3.6) hold. We will prove that for $\mathscr{L}^n$ almost all $z\in E_{g,r}$ we have $z\in D^i_{g,r}$ for all i. Then the claim follows by (3.6). Let $z\in E_{g,r}$ such that (3.5) holds. Then there is a Borel set $A\subset\mathbf{R}^n$ such that $\dim_H A<u+t-n$ and $\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(A)>\varepsilon$. Consider a positive integer i. Now $\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(A\setminus B^i)>\varepsilon$. In fact, if $\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(A\setminus B^i)<\varepsilon$, then $\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(A\cap B^i)>0$, and by (3.5) and Lemma 2.5, $\dim_H(A\cap B^i)\geq u+t-n$, which is a contradiction. Hence $\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(\mathbf{R}^n\setminus B^i)>\varepsilon$, and so $z\in D^i_{g,r}$. □

Lemmas 3.1 and 3.3 together imply the following theorem. Note that as shown above under the assumptions of Theorem 3.7 we have $S_{g,r_\sharp}(\mu\times\nu)\ll\mathscr{L}^n$ for $\theta_n\times\mathscr{L}^1$ almost all $(g,r)\in\mathscr{O}_n\times(0,\infty)$.

Theorem 3.7. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports. Assume that $\dim_H(\mu\times\nu)=\dim_H\mu+\dim_H\nu>n$ and $I_t(\nu)<\infty$ for all $0<t<\dim_H\nu<n$. Then for $\theta_n\times\mathscr{L}^1$ almost all $(g,r)\in\mathscr{O}_n\times(0,\infty)$ we have*

$$\begin{aligned}\operatorname{ess\,inf}\{\dim_H\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu: z\in\mathbf{R}^n \textit{ with } \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(\mathbf{R}^n)>0\}&\\ =\dim_H\mu+\dim_H\nu-n.&\end{aligned}$$

This result is analogous to the packing dimension case (see Theorem 5.9) since

$$\begin{aligned}&\dim_H \mu + \dim_H \nu - n = \dim_H(\mu \times \nu) - n\\ = \sup\Big\{ u \geq 0 : \limsup_{h\to 0} h^{-u} &\int_{B(x,h)\times B(y,h)} |(x,y)-(a,b)|^{-n}\, d(\mu\times\nu)(a,b) = 0\\ &\text{for } \mu\times\nu \text{ almost all } (x,y)\in \mathbf{R}^n \times \mathbf{R}^n \Big\}.\end{aligned}$$

This can be verified in the same way as [6, Remark 3.9].

4. Upper Hausdorff dimension and intersection measures

The following lemma is an analogue of Lemma 3.1 for the upper Hausdorff dimension.

Lemma 4.1. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports. Assume that $\dim_H(\mu\times\nu) > n$. If $(g,r)\in \mathscr{O}_n \times (0,\infty)$ is such that $S_{g,r_\sharp}(\mu\times\nu) \ll \mathscr{L}^n$, then*

$$\sup\big\{u \geq 0 : \mathscr{L}^n\big(\{z\in\mathbf{R}^n : \dim_H^* \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu \geq u\}\big) > 0\big\} \leq \dim_H^*(\mu\times\nu) - n.$$

Proof. Since $\dim_H\big(\mu\times(g\circ\delta_r)_\sharp\nu\big) = \dim_H(\mu\times\nu) > n$ and $\dim_H^*\big(\mu\times(g\circ\delta_r)_\sharp\nu\big) = \dim_H^*(\mu\times\nu)$, we have by [6, Lemma 4.2]

$$\sup\big\{u\geq 0 : \mathscr{H}^n\big(\big\{a\in W^\perp : \dim_H^*\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,a} \geq u\big\}\big) > 0\big\} \leq \dim_H^*(\mu\times\nu) - n.$$

Note that [6, Lemma 4.2] holds for W since $P_{W^\perp\sharp}\big(\mu\times(g\circ\delta_r)_\sharp\nu\big) \ll \mathscr{H}^n \mid W^\perp$. Using the fact that $\dim_H^*\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu \leq \dim_H^*\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}$, this gives the claim. □

Lemma 4.2. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports. Assume that $I_t(\nu) < \infty$ for all $0 < t < \dim_H^*\nu < n$. Then for $\theta_n\times\mathscr{L}^1$ almost all $(g,r)\in\mathscr{O}_n\times(0,\infty)$ we have*

$$\sup\big\{u \geq 0 : \mathscr{L}^n\big(\{z\in\mathbf{R}^n : \dim_H^* \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu \geq u\}\big) > 0\big\} \geq \dim_H^*\mu + \dim_H^*\nu - n.$$

Proof. Let $0 < t < \dim_H^*\nu < n$. It is sufficient to prove that for $\theta_n\times\mathscr{L}^1$ almost all $(g,r)\in\mathscr{O}_n\times(0,\infty)$ we have

$$\sup\big\{u \geq 0 : \mathscr{L}^n\big(\{z\in\mathbf{R}^n : \dim_H^* \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu \geq u\}\big) > 0\big\} \geq \dim_H^*\mu + t - n.$$

The claim follows by taking a sequence (t_i) tending to $\dim_H^*\nu$ from below.

We may assume that $\dim_H^* \mu + t > n$. We will prove that for any $u < \dim_H^* \mu - t - n$ we have for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$

$$\mathscr{L}^n(E_{g,r}) > 0$$

for

$$E_{g,r} = \{z \in \mathbf{R}^n : \dim_H^* \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq u\}.$$

The desired result follows then by taking a sequence (u_i) tending to $\dim_H^* \mu + t - n$.

Let $u < v < \dim_H^* \mu + t - n$. Then for some positive integer i we have $\mu(C_i) > 0$, where

$$C_i = \left\{ x \in \mathbf{R}^n : \mu\big(B(x,h)\big) \leq h^{v-t+n} \text{ for all } 0 < h \leq \frac{1}{i} \right\}$$

is a Borel set. Further, $\mu \times (g \circ \delta_r)_\sharp \nu(C_i \times \mathbf{R}^n) > 0$ for all $(g,r) \in \mathscr{O}_n \times (0,\infty)$. Now $I_{u-t+n}(\mu \mid C_i) < \infty$, whence [9, Theorem 6.6] implies that $S_{g,r_\sharp}\big((\mu \mid C_i) \times \nu\big) \ll \mathscr{L}^n$ for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$. This together with the disintegration formula (2.4) gives for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ that

$$\mathscr{L}^n(D^i_{g,r}) > 0, \tag{4.3}$$

where

$$D^i_{g,r} = \big\{ z \in \mathbf{R}^n : \big((\mu \mid C_i) \times (g \circ \delta_r)_\sharp \nu\big)_{W,(z,-z)/2}(C_i \times \mathbf{R}^n) > 0 \big\}.$$

By [9, Theorem 6.7], we have for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$

$$I_u\big((\mu \mid C_i) \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu\big) < \infty \tag{4.4}$$

for $\mathscr{L}^n$ almost all $z \in \mathbf{R}^n$. It is enough to show that for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ we have $z \in \widetilde{E}_{g,r}$ for $\mathscr{L}^n$ almost all $z \in D^i_{g,r}$, where

$$\widetilde{E}_{g,r} = \{z \in \mathbf{R}^n : \dim_H^* (\mu \mid C_i) \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq u\}.$$

Then (4.3) implies the claim. Consider $(g,r) \in \mathscr{O}_n \times (0,\infty)$ such that (4.3) and (4.4) hold. Let $z \in D^i_{g,r}$ such that (4.4) holds. If $z \notin \widetilde{E}_{g,r}$, then there is a Borel set $A \subset \mathbf{R}^n$ such that $\dim_H A < u$ and $(\mu \mid C_i) \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbf{R}^n \setminus A) = 0$. Now $A \cap C_i$ is a Borel set and $\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(A \cap C_i) > 0$. This gives by Lemma 2.5 and (4.4) that $\dim_H(A \cap C_i) \geq u$, which is a contradiction. Thus $z \in \widetilde{E}_{g,r}$. □

Now we obtain:

Theorem 4.5. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports. Assume that* $\dim_H \mu + \dim_H \nu > n$, $\dim_H^*(\mu \times \nu) = \dim_H^* \mu + \dim_H^* \nu$, *and* $I_t(\nu) < \infty$ *for all* $0 < t < \dim_H^* \nu < n$. *Then for* $\theta_n \times \mathscr{L}^1$ *almost all* $(g, r) \in \mathscr{O}_n \times (0, \infty)$ *we have*

$$\sup\bigl\{u \geq 0 : \mathscr{L}^n\bigl(\{z \in \mathbf{R}^n : \dim_H^* \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq u\}\bigr) > 0\bigr\} = \dim_H^* \mu + \dim_H^* \nu - n.$$

Proof. The claim follows from Lemmas 4.1 and 4.2. Lemma 4.1 is applicable since $\dim_H(\mu \times \nu) \geq \dim_H \mu + \dim_H \nu > n$ and since, using the methods of the proof of Lemma 3.3, we see that $S_{g,r_\sharp}(\mu \times \nu) \ll \mathscr{L}^n$ for $\theta_n \times \mathscr{L}^1$ almost all $(g, r) \in \mathscr{O}_n \times (0, \infty)$.

This result is analogous to the upper packing dimension case (see Theorem 6.3) since we have here

$$\begin{aligned} &\dim_H^* \mu + \dim_H^* \nu - n = \dim_H^*(\mu \times \nu) - n \\ &= \sup\biggl\{u \geq 0 : \mu \times \nu\biggl(\biggl\{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n : \\ &\qquad \limsup_{h \to 0} h^{-u} \int_{B(x,h) \times B(y,h)} |(x, y) - (a, b)|^{-n} \, d(\mu \times \nu)(a, b) = 0\biggr\}\biggr) > 0\biggr\}. \end{aligned}$$

5. Packing dimension and intersection measures

Let μ and ν be finite Radon measures on $\mathbf{R}^n$. As in the projection (see [1]) and section (see [6]) cases, we relate intersection measures to certain integral kernels. We will show that it is the limiting behaviour of the product measure $\mu \times \nu$ against the kernel considered in [6] in connection with sections of measures that determines the packing dimensions of intersection measures almost everywhere.

The quantities

$$\begin{aligned} d_{\mu,\nu} = \sup\biggl\{u \geq 0 : &\liminf_{h \to 0} h^{-u} \int_{B(x,h) \times B(y,h)} |(x, y) - (a, b)|^{-n} \, d(\mu \times \nu)(a, b) = 0 \\ &\text{for } \mu \times \nu \text{ almost all } (x, y) \in \mathbf{R}^n \times \mathbf{R}^n\biggr\} \end{aligned}$$

and the upper analogue

$$\begin{aligned} d_{\mu,\nu}^* = \sup\biggl\{u \geq 0 : \mu \times \nu\biggl(\biggl\{&(x, y) \in \mathbf{R}^n \times \mathbf{R}^n : \\ &\liminf_{h \to 0} h^{-u} \int_{B(x,h) \times B(y,h)} |(x, y) - (a, b)|^{-n} \, d(\mu \times \nu)(a, b) = 0\biggr\}\biggr) > 0\biggr\} \end{aligned}$$

are what we need when considering intersection measures.

The following lemma is a modification of [4, Lemma 3.5].

Lemma 5.1. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports such that $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$ for some $0 < s < n$ and $0 < t < n$ with $s + t \geq n$. Let $0 < r_1 < r_2 < \infty$. Assume that there exists $u > 0$ such that for $\mu \times \nu$ almost all $(x, y) \in \mathbf{R}^n \times \mathbf{R}^n$ there is $0 < c < \infty$ such that*

$$\int_{r_1}^{r_2} \int \mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big)\, d\theta_n g\, d\mathscr{L}^1 r \leq c h^u$$

for arbitrarily small $h > 0$. Then for $\theta_n \times \mathscr{L}^1$ almost all $(g, r) \in \mathscr{O}_n \times [r_1, r_2]$ we have

$$\dim_p \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq u$$

for $\mathscr{L}^n$ almost all $z \in \mathbf{R}^n$ with $\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbf{R}^n) > 0$.

Proof. Let $\varepsilon > 0$. For $h > 0$ and $(x, y) \in \mathbf{R}^n \times \mathbf{R}^n$, define

$$J_h(x,y) = \int_{r_1}^{r_2} \int \mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big)\, d\theta_n g\, d\mathscr{L}^1 r$$

provided that the right hand side is defined (see [4, Lemma 3.4]). If $J_h(x, y) \leq c h^u$, then

$$\begin{aligned}\theta_n \times \mathscr{L}^1\big(\big\{(g,r) \in \mathscr{O}_n \times [r_1, r_2] : \mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big) \geq h^{u-\varepsilon}\big\}\big)& \\ \leq h^{\varepsilon - u} J_h(x,y) \leq c h^{\varepsilon}.&\end{aligned}$$

Hence, for $\mu \times \nu$ almost all $(x, y) \in \mathbf{R}^n \times \mathbf{R}^n$ we have

$$\theta_n \times \mathscr{L}^1\big(\big\{(g,r) \in \mathscr{O}_n \times [r_1, r_2] : \liminf_{h \to 0} h^{\varepsilon - u} \mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big) > 1\big\}\big) = 0,$$

which gives, by Fubini's theorem, that for $\theta_n \times \mathscr{L}^1$ almost all $(g, r) \in \mathscr{O}_n \times [r_1, r_2]$

$$\mu \times \nu(A_{g,r}) = 0, \tag{5.2}$$

where

$$A_{g,r} = \big\{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : \liminf_{h \to 0} h^{\varepsilon - u} \mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big) > 1\big\}$$

is a Borel set. For all $g \in \mathscr{O}_n$ and $r_1 \leq r \leq r_2$, define a Borel set

$$B_{g,r} = \big\{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : \liminf_{h \to 0} h^{\varepsilon - u} \mu \cap (\tau_{x-y} \circ g \circ \delta_r)_\sharp \nu\big(B(x,h)\big) > 1\big\}.$$

Then (5.2) implies that for $\theta_n \times \mathscr{L}^1$ almost all $(g, r) \in \mathscr{O}_n \times [r_1, r_2]$

$$\mu \times (g \circ \delta_r)_\sharp \nu(B_{g,r}) = \mu \times \nu(A_{g,r}) = 0,$$

and so, using [9, Theorem 6.6] and (2.4), we have for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times [r_1, r_2]$

$$\big(\mu \times (g \circ \delta_r)_\sharp \nu\big)_{W,(z,-z)/2}(B_{g,r}) = 0,$$

for $\mathscr{L}^n$ almost all $z \in \mathbf{R}^n$, that is,

$$\liminf_{h \to 0} h^{\varepsilon - u} \mu \cap (\tau_{x-y} \circ g \circ \delta_r)_\sharp \nu\big(B(x,h)\big) \le 1 \tag{5.3}$$

for $\big(\mu \times (g \circ \delta_r)_\sharp \nu\big)_{W,(z,-z)/2}$ almost all $(x,y) \in \mathbf{R}^n \times \mathbf{R}^n$. Since $(x,y) \in W_{(z,-z)/2}$ if and only if $x - y = z$, inequality (5.3) gives for $\mathscr{L}^n$ almost all $z \in \mathbf{R}^n$

$$\liminf_{h \to 0} h^{\varepsilon - u} \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu\big(B(x,h)\big) \le 1$$

for $\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu$ almost all $x \in \mathbf{R}^n$. Thus, for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times [r_1, r_2]$ we have

$$\dim_p \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \ge u - 2\varepsilon$$

for $\mathscr{L}^n$ almost all $z \in \mathbf{R}^n$ with $\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbf{R}^n) > 0$, and the claim follows by taking a sequence (ε_i) tending to zero. □

In the proof of the following lemma we need the estimates given in [4, Lemma 3.8].

Lemma 5.4. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports such that $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$ for some $0 < s < n$ and $0 < t < n$ with $s + t \ge n$. Then for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ we have*

$$\operatorname{ess\,inf}\{\dim_p \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu : z \in \mathbf{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbf{R}^n) > 0\} \ge d_{\mu,\nu}.$$

Proof. Consider $(x,y) \in \mathbf{R}^n \times \mathbf{R}^n$ such that $\int |a - x|^{t-n} \, d\mu a < \infty$ and $\int |b - y|^{-t} \, d\nu b < \infty$. Let $0 < r_1 < r_2 < \infty$ and $h > 0$. By (2.3), Fatou's lemma and Fubini's theorem we have

$$\begin{aligned}\int_{r_1}^{r_2} \int \mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big) \, d\theta_n g \, d\mathscr{L}^1 r \\ \le \liminf_{\delta \to 0} \alpha(n)^{-1} \delta^{-n} \int_{B(x,2h) \times \mathbf{R}^n} I_\delta(a,b) \, d(\mu \times \nu)(a,b),\end{aligned}$$

where

$$I_\delta(a,b) = \int_{\{r \in [r_1, r_2] : |\,|a-x| - r|b-y|\,| \le \delta\}} \theta_n(\{g \in \mathscr{O}_n : |a - x - rg(b-y)| \le \delta\}) \, d\mathscr{L}^1 r.$$

Define

$$A_\delta = \{(a,b) \in \mathbf{R}^n \times \mathbf{R}^n : r_1|b-y| - \delta \le |a-x| \le r_2|b-y| + \delta\}.$$

If $(a,b) \notin A_\delta$, then $\{r \in [r_1, r_2] : \big||a-x| - r|b-y|\big| \le \delta\} = \emptyset$. Hence,

$$\int_{r_1}^{r_2} \int \mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big)\, d\theta_n g\, d\mathscr{L}^1 r$$
$$\le \liminf_{\delta \to 0} \alpha(n)^{-1} \delta^{-n} \int_{(B(x,2h)\times \mathbf{R}^n)\cap A_\delta} I_\delta(a,b)\, d(\mu \times \nu)(a,b).$$

Let

$$A_\delta^1 = \{(a,b) \in A_\delta : |a-x| \le 2\delta\},$$
$$A_\delta^2 = \{(a,b) \in A_\delta : r_1|b-y| \le 2\delta\},$$

and

$$A_\delta^3 = \{(a,b) \in A_\delta : |a-x| > 2\delta \text{ and } r_1|b-y| > 2\delta\}.$$

Then $A_\delta = A_\delta^1 \cup A_\delta^2 \cup A_\delta^3$. Further, if $(a,b) \in A_\delta^1$, then

$$\delta^{-n} \le 2^{n-t} 3^t r_1^{-t} |a-x|^{t-n} |b-y|^{-t},$$

and so,

$$\limsup_{\delta \to 0} \alpha(n)^{-1} \delta^{-n} \int_{A_\delta^1} I_\delta(a,b)\, d(\mu \times \nu)(a,b) = 0.$$

Similarly,

$$\limsup_{\delta \to 0} \alpha(n)^{-1} \delta^{-n} \int_{A_\delta^2} I_\delta(a,b)\, d(\mu \times \nu)(a,b) = 0.$$

Thus,

$$\int_{r_1}^{r_2} \int \mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big)\, d\theta_n g\, d\mathscr{L}^1 r$$
$$\le \limsup_{\delta \to 0} \alpha(n)^{-1} \delta^{-n} \int_{(B(x,2h)\times \mathbf{R}^n)\cap A_\delta^3} I_\delta(a,b)\, d(\mu \times \nu)(a,b).$$

If $(a,b) \in A_\delta^3$, then $\frac{1}{2} r_1 |b-y| \le |a-x| \le 2r_2|b-y|$, and so by [10, Lemma 3.8] there is a constant c_1 depending only on n such that for any $a \ne x$ and $b \ne y$

$$\begin{aligned} I_\delta(a,b) &\le c_1 \delta^{n-1} |a-x|^{1-n} \mathscr{L}^1\big(\{r \in [r_1,r_2] : \big||a-x| - r|b-y|\big| \le \delta\}\big) \\ &\le c_1 \delta^{n-1} |a-x|^{1-n} 2\delta |b-y|^{-1} \\ &\le 4 c_1 r_2 \delta^n |a-x|^{-n}. \end{aligned}$$

Thus, for $\mu \times \nu$ almost all $(x,y) \in \mathbf{R}^n \times \mathbf{R}^n$ we have

$$
\begin{aligned}
&\int_{r_1}^{r_2} \int \mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big)\, d\theta_n g\, d\mathscr{L}^1 r \\
&\qquad \le \limsup_{\delta \to 0} 4c_1 \alpha(n)^{-1} r_2 \int_{(B(x,2h)\times \mathbf{R}^n)\cap A_\delta^3} |a-x|^{-n}\, d(\mu \times \nu)(a,b) \\
&\qquad \le \limsup_{\delta \to 0} c_2 \int_{(B(x,2h)\times \mathbf{R}^n)\cap A_\delta^3} |(x,y)-(a,b)|^{-n}\, d(\mu \times \nu)(a,b) \\
&\qquad \le c_2 \int_{B(x,c_3h)\times B(y,c_3 h)} |(x,y)-(a,b)|^{-n}\, d(\mu \times \nu)(a,b),
\end{aligned} \tag{5.5}
$$

where c_2 is a constant depending on n, r_1, and r_2 and $c_3 = \max\{2, 4/r_1\}$. The last inequality follows from the fact that $\big(B(x,2h)\times \mathbf{R}^n\big) \cap A_\delta^3 \subset B(x,2h) \times B(y,4h/r_1)$.

Consider $u \ge 0$ such that

$$\liminf_{h \to 0} h^{-u} \int_{B(x,h)\times B(y,h)} |(x,y)-(a,b)|^{-n}\, d(\mu \times \nu)(a,b) = 0$$

for $\mu \times \nu$ almost all $(x,y) \in \mathbf{R}^n \times \mathbf{R}^n$. Then, by (5.5) and Lemma 5.1 we have for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times [r_1, r_2]$

$$\operatorname{ess\,inf}\{\dim_p \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu : z \in \mathbf{R}^n \text{ with } \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu(\mathbf{R}^n) > 0\} \ge u.$$

The claim follows by taking a sequence (u_i) tending to $d_{\mu,\nu}$. □

In order to prove that the opposite inequality holds in Lemma 5.4 we need the following version of [6, Lemma 2.4] concerning sections of the product measures $\mu \times (g \circ \delta_r)_\sharp \nu$.

Lemma 5.6. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports. If $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$ for some $0 < s < n$ and $0 < t < n$ such that $s + t > n$, then for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ there exists for any $\varepsilon > 0$ a compact set $C_\varepsilon \subset \mathbf{R}^n \times \mathbf{R}^n$ with $\mu \times (g\circ\delta_r)_\sharp \nu\big((\mathbf{R}^n \times \mathbf{R}^n) \setminus C_\varepsilon\big) < \varepsilon$ and $H_\varepsilon > 0$ such that for $\mathscr{H}^n$ almost all $(z,-z)/2 \in W^\perp$ we have*

$$\big(\big(\mu \times (g \circ \delta_r)_\sharp \nu\big) \mid C_\varepsilon\big)_{W,(z,-z)/2}\big(B(x,h) \times B(y,h)\big) \le c h^{(s+t-n)/2}$$

for all $(x,y) \in W_{(z,-z)/2}$ and $0 < h \le H_\varepsilon$. Here c is a constant depending only on s, t, and n.

Proof. Let $q=s+t-n$. By [9, Theorem 6.7] we have for $\theta_n\times\mathscr{L}^1$ almost all $(g,r)\in\mathscr{O}_n\times(0,\infty)$

$$I_q\big(\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu\big)<\infty \tag{5.7}$$

for $\mathscr{L}^n$ almost all $z\in\mathbf{R}^n$. Consider $(g,r)\in\mathscr{O}_n\times(0,\infty)$ such that (5.7) holds and $S_{g,r_\sharp}(\mu\times\nu)\ll\mathscr{L}^n$ (see [9, Theorem 6.6]). We will prove that for $\mu\times(g\circ\delta_r)_\sharp\nu$ almost all $(x,y)\in\mathbf{R}^n\times\mathbf{R}^n$ we have

$$\int|(x,y)-(a,b)|^{-q}\,d\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(x-y,y-x)/2}(a,b)<\infty.$$

To show that this holds, define

$$E=\Big\{(x,y)\in\mathbf{R}^n\times\mathbf{R}^n:\int|(x,y)-(a,b)|^{-q}\,d\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(x-y,y-x)/2}(a,b)=\infty\Big\}.$$

If $\mu\times(g\circ\delta_r)_\sharp\nu(E)>0$, then by (2.4) we have $\mathscr{L}^n(F)>0$, where

$$F=\big\{z\in\mathbf{R}^n:\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}(E)>0\big\}.$$

For all $z\in F$ we have

$$\begin{aligned}
&I_q\big(\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu\big)\\
&\qquad=2^n\alpha(n)^{-2}\int\int|x-a|^{-q}\,d\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}(a,b)\\
&\qquad\qquad\times\,d\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}(x,y)\\
&\qquad\ge2^n\alpha(n)^{-2}\int_E\int|(x,y)-(a,b)|^{-q}\,d\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}(a,b)\\
&\qquad\qquad\times\,d\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}(x,y)=\infty,
\end{aligned}$$

which is a contradiction by (5.7).

Let $\varepsilon>0$. For all positive integers i, define a Borel set

$$B_i=\Big\{(x,y)\in\mathbf{R}^n\times\mathbf{R}^n:\int|(x,y)-(a,b)|^{-q}\,d\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(x-y,y-x)/2}(a,b)\le i\Big\}.$$

Since

$$\lim_{i\to\infty}\mu\times(g\circ\delta_r)_\sharp\nu\big((\mathbf{R}^n\times\mathbf{R}^n)\setminus B_i\big)=0,$$

we find a compact set $C_\varepsilon\subset\mathbf{R}^n\times\mathbf{R}^n$ such that $\mu\times(g\circ\delta_r)_\sharp\nu\big((\mathbf{R}^n\times\mathbf{R}^n)\setminus C_\varepsilon\big)<\varepsilon$ and $C_\varepsilon\subset B_{i_\varepsilon}$ for some positive integer i_ε (see [10, Theorem 1.10]).

Let $H_\varepsilon = i_\varepsilon^{-2/q}/3$. Consider $(z,-z)/2 \in W^\perp$ such that $\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(z,-z)/2}$ and $\big((\mu\times(g\circ\delta_r)_\sharp\nu) \mid C_\varepsilon\big)_{W,(z,-z)/2}$ are defined. If $(z,-z)/2 \notin P_{W^\perp}(C_\varepsilon)$, then

$$\big((\mu\times(g\circ\delta_r)_\sharp\nu) \mid C_\varepsilon\big)_{W,(z,-z)/2}\big(B(x,h)\times B(y,h)\big) = 0$$

for all $(x,y) \in W_{(z,-z)/2}$ and $h > 0$. This follows from (2.1) and from the fact that $W_{(z,-z)/2}(\delta) \cap C_\varepsilon = \emptyset$ for all small $\delta > 0$, since C_ε is compact. If $(z,-z)/2 \in P_{W^\perp}(C_\varepsilon)$ and $(x,y) \in W_{(z,-z)/2} \cap C_\varepsilon$, then for any $0 < h \le 3H_\varepsilon$, we have

(5.8)
$$\begin{aligned}
&\big((\mu\times(g\circ\delta_r)_\sharp\nu) \mid C_\varepsilon\big)_{W,(z,-z)/2}\big(B(x,h)\times B(y,h)\big)\\
&\quad\le \big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(x-y,y-x)/2}\big(B(x,h)\times B(y,h)\big)\\
&\quad\le 2^{q/2}h^q\int_{B(x,h)\times B(y,h)} |(x,y)-(a,b)|^{-q}\, d\big(\mu\times(g\circ\delta_r)_\sharp\nu\big)_{W,(x-y,y-x)/2}(a,b)\\
&\quad\le 2^{q/2} i_\varepsilon h^q \le 2^{q/2}h^{q/2}.
\end{aligned}$$

Finally, let $(z,-z)/2 \in P_{W^\perp}(C_\varepsilon)$ and $(x,y) \in W_{(z,-z)/2} \setminus C_\varepsilon$. Then there exists $h_{x,y} > 0$ such that $\big(B(x,2h)\times B(y,2h)\big) \cap C_\varepsilon \cap W_{(z,-z)/2} = \emptyset$ for all $0 < h < h_{x,y}$ and $\big(B(x,2h)\times B(y,2h)\big) \cap C_\varepsilon \cap W_{(z,-z)/2} \ne \emptyset$ for all $h \ge h_{x,y}$. If $0 < h < h_{x,y}$, then by (2.1)

$$\big((\mu\times(g\circ\delta_r)_\sharp\nu) \mid C_\varepsilon\big)_{W,(z,-z)/2}\big(B(x,h)\times B(y,h)\big) = 0.$$

If $h_{x,y} \le h \le H_\varepsilon$, then $B(x,h)\times B(y,h) \subset B(a,3h)\times B(b,3h)$ for some $(a,b) \in C_\varepsilon \cap W_{(z,-z)/2}$, and (5.8) gives the claim. □

Theorem 5.9. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports such that $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$ for some $0 < s < n$ and $0 < t < n$ with $s+t > n$. Then for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times (0,\infty)$ we have*

$$\operatorname{ess\,inf}\{\dim_p \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu : z \in \mathbf{R}^n \text{ with } \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(\mathbf{R}^n) > 0\} = d_{\mu,\nu}.$$

Proof. Consider $(g,r) \in \mathscr{O}_n\times(0,\infty)$ such that Lemma 5.6 holds and $S_{g,r_\sharp}(\mu\times\nu) \ll \mathscr{L}^n$. Let $u \ge 0$ such that $\dim_p \mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu \ge u$ for $\mathscr{L}^n$ almost all $z \in \mathbf{R}^n$ with $\mu\cap(\tau_z\circ g\circ\delta_r)_\sharp\nu(\mathbf{R}^n) > 0$. Then for $\mathscr{H}^n$ almost all $(z,-z)/2 \in W^\perp$ with $\big(\mu\times(g\circ\delta_r)_\sharp)\nu\big)_{W,(z,-z)/2}$ $(\mathbf{R}^n\times\mathbf{R}^n) > 0$ we have

$$u \le \big(\dim_p\big(\mu\times(g\circ\delta_r)_\sharp\big)\nu\big)_{W,(z,-z)/2}.$$

Now we use a result given in [6, Theorem 5.16]. What is actually proved there is the following: if k and p are integers with $0 < p < k$ and if m is a Radon measure on $\mathbf{R}^k$ with compact support and with $I_p(m) < \infty$, then

$$\begin{aligned}&\operatorname{ess\,inf}\{\dim_p m_{V,a} : a \in V^\perp \text{ with } m_{V,a}(\mathbf{R}^k) > 0\}\\ &\leq \sup\left\{v \geq 0 : \liminf_{h\to 0} h^{-v} \int_{B(x,h)} |x-y|^{-p}\, dm(y) = 0 \text{ for } m \text{ almost all } x \in \mathbf{R}^k\right\}\end{aligned}$$

provided that $V \in G_{k,k-p}$ such that [6, Lemma 2.4] holds and $P_{V^\perp\sharp} m \ll \mathscr{H}^p \mid V^\perp$. Note that the assumption that $I_{p+d}(m) < \infty$ for some $d > 0$ in [6, Theorem 5.16] is needed only to make sure that [6, Lemma 2.4] holds for almost all $(k-p)$-planes. If we know that this is the case for $V \in G_{k,k-p}$, then this assumption is unnecessary, since in the proof of [6, Theorem 5.16] we need only the fact that $I_p(m) < \infty$ when applying [6, Lemma 5.9]. Here we choose $m = \mu \times (g \circ \delta_r)_\sharp \nu$. Then $I_n(m) < \infty$, since $I_n(\mu \times \nu) < \infty$. Further, $P_{W^\perp} m \ll \mathscr{H}^n \mid W^\perp$ and [6, Lemma 2.4] holds for W. So, by the above result

$$u \leq c_{\mu\times(g\circ\delta_r)_\sharp\nu},$$

where

$$\begin{aligned}c_{\mu\times(g\circ\delta_r)_\sharp\nu} = \sup\Big\{v \geq 0 : \liminf_{h\to 0} h^{-v} &\int_{B(x,h)\times B(y,h)} |(x,y)-(a,b)|^{-n}\\ &\times d\big(\mu \times (g\circ\delta_r)_\sharp\nu\big)(a,b) = 0\\ &\text{for } \mu \times (g\circ\delta_r)_\sharp\nu \text{ almost all } (x,y) \in \mathbf{R}^n \times \mathbf{R}^n\Big\}.\end{aligned}$$

We will show that $c_{\mu\times(g\circ\delta_r)_\sharp\nu} \leq d_{\mu,\nu}$, which gives the claim by Lemma 5.4. If $v \geq 0$ is such that

$$\liminf_{h\to 0} h^{-v} \int_{B(x,h)\times B(y,h)} |(x,y)-(a,b)|^{-n}\, d\big(\mu \times (g\circ\delta_r)_\sharp\nu\big)(a,b) = 0$$

for $\mu \times (g\circ\delta_r)_\sharp\nu$ almost all $(x,y) \in \mathbf{R}^n \times \mathbf{R}^n$, then

$$\liminf_{h\to 0} h^{-v} \int_{B(x,h)\times B(y,h/r)} |(x,ry)-(a,rb)|^{-n}\, d(\mu\times\nu)(a,b) = 0 \tag{5.10}$$

for $\mu\times\nu$ almost all $(x,y) \in \mathbf{R}^n \times \mathbf{R}^n$. Consider $(x,y) \in \mathbf{R}^n \times \mathbf{R}^n$ such that (5.10) holds. If $r \leq 1$, then $|(x,y)-(a,b)|^{-n} \leq |(x,ry)-(a,rb)|^{-n}$, and so,

$$\liminf_{h\to 0} h^{-v} \int_{B(x,h)\times B(y,h)} |(x,y)-(a,b)|^{-n}\, d(\mu\times\nu)(a,b) = 0.$$

If $r \geq 1$, then $|(x,y)-(a,b)|^{-n} \leq r^n |(x,ry)-(a,rb)|^{-n}$, which gives

$$\begin{aligned}&\liminf_{h\to 0} (h/r)^{-v} \int_{B(x,h/r)\times B(y,h/r)} |(x,y)-(a,b)|^{-n}\, d(\mu\times\nu)(a,b)\\ &\leq \liminf_{h\to 0} r^{n-v} h^{-v} \int_{B(x,h)\times B(y,h/r)} |(x,ry)-(a,rb)|^{-n}\, d(\mu\times\nu)(a,b) = 0.\end{aligned}$$

Hence $c_{\mu\times(g\circ\delta_r)_\sharp\nu} \leq d_{\mu,\nu}$. □

6. Upper packing dimension and intersection measures

In this chapter we prove analogues of the results of the previous chapter for the upper packing dimension. For this purpose we need the following lemma.

Lemma 6.1. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports such that $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$ for some $0 < s < n$ and $0 < t < n$ with $s + t \geq n$. Let $0 < r_1 < r_2 < \infty$. Assume that there is $u > 0$ such that*

$$\mu \times \nu\Bigg(\bigg\{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : \text{ there is } 0 < c < \infty \text{ such that for arbitrarily}$$
$$\text{small } h > 0 \int_{r_1}^{r_2} \int \mu \cap (\tau_z \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big)\, d\theta_n g\, d\mathscr{L}^1 r \leq c h^u \bigg\}\Bigg) > 0.$$

Then for any $\varepsilon > 0$ we have for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times [r_1, r_2]$

$$\mathscr{L}^n\big(\{z \in \mathbf{R}^n : \dim_p^* \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq u - \varepsilon\}\big) > 0.$$

Proof. Let $\varepsilon > 0$. For $h > 0$ and $(x,y) \in \mathbf{R}^n \times \mathbf{R}^n$, define

$$J_h(x,y) = \int_{r_1}^{r_2} \int \mu \cap (\tau_z \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big)\, d\theta_n g\, d\mathscr{L}^1 r$$

provided that the right hand side is defined (see [4, Lemma 3.4]). Define

$$A_u = \{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : \text{there is } 0 < c < \infty \text{ such that } J_h(x,y) \leq c h^u \text{ for arbitrarily small } h > 0\}.$$

Then $\mu \times \nu(A_u) > 0$ and for all $(x,y) \in A_u$ we have

$$\liminf_{h \to 0} h^{\varepsilon - u} \mu \cap (\tau_z \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big) \leq 1$$

for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times [r_1, r_2]$. This together with Fubini's theorem gives that for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times [r_1, r_2]$ we have $\mu \times \nu(B_{g,r}) > 0$, where

$$B_{g,r} = \big\{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : \liminf_{h \to 0} h^{\varepsilon - u} \mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\big(B(x,h)\big) \leq 1\big\}.$$

Since $\mu \times \nu(B_{g,r}) = \mu \times (g \circ \delta_r)_\sharp \nu(C_{g,r})$ for

$$C_{g,r} = \big\{(x,y) \in \mathbf{R}^n \times \mathbf{R}^n : \liminf_{h \to 0} h^{\varepsilon - u} \mu \cap (\tau_{x-y} \circ g \circ \delta_r)_\sharp \nu\big(B(x,h)\big) \leq 1\big\},$$

[9, Theorem 6.6] and the disintegration formula (2.4) imply that for $\theta_n \times \mathscr{L}^1$ almost all $(g,r) \in \mathscr{O}_n \times [r_1, r_2]$ we have $\mathscr{L}^n(D_{g,r}) > 0$, where

$$D_{g,r} = \big\{z \in \mathbf{R}^n : \big(\mu \times (g \circ \delta_r)_\sharp \nu\big)_{W,(z,-z)/2}(C_{g,r}) > 0\big\}.$$

For any $z \in D_{g,r}$ we have

$$\mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu\big(\{x \in \mathbf{R}^n : \liminf_{h \to 0} h^{2\varepsilon - u} \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu\big(B(x,h)\big) = 0\}\big) > 0,$$

that is,

$$\dim_p^* \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq u - 2\varepsilon,$$

and the claim follows. □

Lemma 6.2. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports such that $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$ for some $0 < s < n$ and $0 < t < n$ with $s + t \geq n$. Then for $\theta_n \times \mathscr{L}^1$ almost all $(g, r) \in \mathscr{O}_n \times (0, \infty)$ we have*

$$\sup\bigl\{u \geq 0 : \mathscr{L}^n\bigl(\{z \in \mathbf{R}^n : \dim_p^* \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq u\}\bigr) > 0\bigr\} \geq d^*_{\mu,\nu}.$$

Proof. Let $0 < r_1 < r_2 < \infty$. Consider $v \geq 0$ such that $\mu \times \nu(A_v) > 0$, where

$$A_v = \left\{(x, y) \in \mathbf{R}^n \times \mathbf{R}^n : \liminf_{h \to 0} h^{-v} \int\limits_{B(x,h) \times B(y,h)} |(x, y) - (a, b)|^{-n} \, d\mu \times \nu(a, b) = 0\right\}.$$

As in the proof of Lemma 5.4, we see that for $\mu \times \nu$ almost all $(x, y) \in A_v$ we have for some constants c_1 and c_2

$$\begin{aligned}\int_{r_1}^{r_2} \int \mu \cap (\tau_x \circ g \circ \delta_r \circ \tau_{-y})_\sharp \nu\bigl(B(x, h)\bigr) \, d\theta_n g \, d\mathscr{L}^1 r \\ \leq c_1 \int_{B(x, c_2 h) \times B(y, c_2 h)} |(x, y) - (a, b)|^{-n} \, d\mu \times \nu(a, b) \leq c_1 (c_2 h)^v\end{aligned}$$

for arbitrarily small $h > 0$. For any $\varepsilon > 0$ Lemma 6.1 gives that for $\theta_n \times \mathscr{L}^1$ almost all $(g, r) \in \mathscr{O}_n \times [r_1, r_2]$ we have

$$\sup\bigl\{u \geq 0 : \mathscr{L}^n\bigl(\{z \in \mathbf{R}^n : \dim_p^* \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq u\}\bigr) > 0\bigr\} \geq v - \varepsilon.$$

The claim follows by taking sequences (ε_i) tending to zero and (v_i) tending to $d^*_{\mu,\nu}$ □

Theorem 6.3. *Let μ and ν be Radon measures on $\mathbf{R}^n$ with compact supports. Assume that $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$ for some $0 < s < n$ and $0 < t < n$ with $s + t > n$. Then for $\theta_n \times \mathscr{L}^1$ almost all $(g, r) \in \mathscr{O}_n \times (0, \infty)$ we have*

$$\sup\bigl\{u \geq 0 : \mathscr{L}^n\bigl(\{z \in \mathbf{R}^n : \dim_p^* \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq u\}\bigr) > 0\bigr\} = d^*_{\mu,\nu}.$$

Proof. Consider $(g, r) \in \mathscr{O}_n \times (0, \infty)$ such that Lemma 5.6 holds and $S_{g,r_\sharp}(\mu \times \nu) \ll \mathscr{L}^n$. Let $u \geq 0$ such that $\mathscr{L}^n\bigl(\{z \in \mathbf{R}^n : \dim_p^* \mu \cap (\tau_z \circ g \circ \delta_r)_\sharp \nu \geq u\}\bigr) > 0$. Then

$$\mathscr{H}^n\bigl(\{a \in W^\perp : \dim_p^*\bigl(\mu \times (g \circ \delta_r)_\sharp \nu\bigr)_{W,(z,-z)/2} \geq u\}\bigr) > 0.$$

As in the proof of Theorem 5.9 we use a result from [6] to prove the claim. What we need here and what is actually proved in [6, Theorem 6.4] is as follows: if k and

p are integers with $0 < p < k$ and m is a Radon measure on $\mathbf{R}^k$ with compact support and with $I_p(m) < \infty$, then

$$\begin{aligned}\sup\bigl\{v \geq 0 : \mathscr{H}^p(\{a \in V^\perp : \dim_p^* m_{V,a} \geq v\}) > 0\bigr\} \\ \leq \sup\biggl\{v \geq 0 : m\biggl(\biggl\{x \in \mathbf{R}^k : \liminf_{h\to 0} h^{-v} \int_{B(x,h)} |x-y|^{-p}\, dmy = 0\biggr\}\biggr) > 0\biggr\}\end{aligned}$$

provided that $V \in G_{k,k-p}$ such that [6, Lemma 2.4] holds and $P_{V^\perp_\sharp} m \ll \mathscr{H}^p \mid V^\perp$. In our case this result implies that

$$u \leq c^*_{\mu\times(g\circ\delta_r)_\sharp\nu},$$

where

$$\begin{aligned}c^*_{\mu\times(g\circ\delta_r)_\sharp\nu} = \sup\biggl\{v \geq 0 : \mu \times (g\circ\delta_r)_\sharp\nu\biggl(\biggl\{(a,b) \in \mathbf{R}^n \times \mathbf{R}^n : \\ \liminf_{h\to 0} h^{-v} \int_{B(x,h)\times B(y,h)} |(x,y)-(a,b)|^{-n}\, d\mu \times (g\circ\delta_r)_\sharp\nu(a,b) = 0\biggr\}\biggr) > 0\biggr\}.\end{aligned}$$

Hence it suffices to show that $c^*_{\mu\times(g\circ\delta_r)_\sharp\nu} \leq d^*_{\mu,\nu}$. This can be verified in the same way as the fact that $c_{\mu\times(g\circ\delta_r)_\sharp\nu} \leq d_{\mu,\nu}$ was shown in the proof of Theorem 5.9. □

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