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ON THE LENGTH SPECTRUMS OF NON-COMPACT RIEMANN SURFACES

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Abstract. In this paper we prove that the length spectrum metric is topologically equivalent to the Teichmüller metric in Teichmüller space $T(g,m,n)$. This result solved a problem suggested by Sorvali [9] in 1972.

0. Introduction

Let G be a torsion-free non-elementary Fuchsian group acting on the upper half-plane U. Denote

(1) $$L^\infty(U,G) = \{\mu \in L^\infty(U,\mathbf{C}) : (\mu\circ g)\bar{g}'/g' = \mu \text{ a.e. for any } g\in G\},$$

(2) $$L^\infty(U,G)_1 = \{\mu \in L^\infty(U,G),\ \|\mu\|_\infty < 1\}.$$

As we know, for any $\mu\in L^\infty(U,G)_1$, there exists a quasiconformal mapping $\omega_\mu\colon U\to U$ such that the Beltrami coefficient of ω_μ is μ and ω_μ fixes 0, 1, ∞. Let $\Lambda(G)$ be the limit set of the Fuchsian group G. Without loss of generality, we may assume that $\Lambda(G)$ contains 0, 1, ∞. We can define the Teichmüller space of the Fuchsian group G to be $T(G) = L^\infty(U,G)_1/\sim$, where $\mu\sim\eta$ if and only if $\omega_\mu|_{\Lambda(G)} = \omega_\eta|_{\Lambda(G)}$. Letting $X = U/G$, we can give a definition of the Teichmüller space $T(X)$ [5]. Denote by $[X,f_1,X_1]$ the marked Riemann surface based on X, where X is a Riemann surface and $f\colon X\to X_1$ is a quasiconformal mapping. Two marked Riemann surfaces are equivalent if there exists a conformal mapping $\sigma\colon X_1\to X_2$ such that the mapping $g^{-1}\circ\sigma\circ f\colon X\to X$ is homotopic to the identity. The Teichmüller space $T(X)$ is the set of the above equivalent classes. These two definitions of Teichmüller spaces are equivalent. For any marked Riemann surface $[X,f_1,X_1]$, let $\omega_\mu\colon U\to U$ be the lifting of f_1; then $[X,f_1,X_1]$ can be represented by $[G,\chi,\omega_\mu\circ G\circ\omega_\mu^{-1}]$, where $\chi\colon G\to\omega_\mu\circ G\circ\omega_\mu^{-1}$, is a group isomorphism, defined by $\chi(g) = \omega_\mu\circ g\circ\omega_\mu^{-1}$, $g\in G$. Therefore any point in the Teichmüler space $T(X) = T(G)$ can be written as the formula $[G,\chi_1,G_1]$, this formula being unique up to an inner automorphism. The equivalent relations

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between any two of these formulas are induced by the equivalent relations in the above definitions of Teichmüller spaces.

On the other hand, the Riemann surface X can be viewed as some Riemann surface structure on a topological surface S. We obtain various Riemann surfaces when the Riemann surface structures on S vary. Let Σ_X be the homotopy class of closed curves on X, and let Σ'_X be the elements in Σ_X which are not homotopic to a single puncture. Let Σ''_X be the elements in Σ'_X which is the homotopy class of simple closed curves. There exists a Poincaré metric on X because the Riemann surface X is hyperbolic. For any $\alpha \in \Sigma_X$, let $l_X(\alpha)$ be the shortest length of elements of α in the Poincaré metric on X. The sequence $\{l_X(\alpha)\}$ corresponds to the element $\alpha \in \Sigma_X$ called the length spetrum of the Riemann surface X. For a Riemann surface $X = U/G$ and any $\alpha \in \Sigma_X$, let $g \in G$ cover α; then $l_X(\alpha) = \log \lambda(g)$, where $\lambda(g)$ is the multiplier of g [5], [1]. In the Teichmüller space $T(G)$, Sorvali defined a metric d [9], [5]. If $[G, \chi_1, G_1]$ and $[G, \chi_2, G_2]$ are two points in $T(G)$, denote

$$d([G,\chi_1,G_1],[G,\chi_2,G_2]) = \log \rho([G,\chi_1,G_1],[G,\chi_2,G_2]), \tag{3}$$

where ρ is the infinum of a $(a \geq 1)$ with a satisfying

$$|\lambda(g)|^{1/a} \leq |\lambda(\chi_2 \circ \chi_1^{-1} \circ g)| \leq |\lambda(g)|^a, \quad \text{for any } g \in G_1. \tag{4}$$

Because $|\lambda(g)| = |\lambda(\chi_2 \circ \chi_1^{-1} \circ g)| = 1$ for any parabolic element $g \in G$, ρ is the infinum of a $(a \geq 1)$ with a satisfying

$$|\lambda(g)|^{1/a} \leq |\lambda(\chi_2 \circ \chi_1^{-1} \circ g)| \leq |\lambda(g)|^a \tag{5}$$

for any $g \in G_1$ and g is hyperbolic.

Then we have

$$\rho = \sup_{g \in G_1,\ g \text{ is hyperbolic}} \left\{ \frac{\log \lambda(g)}{\log \lambda(\chi_2 \circ \chi_1^{-1} \circ g)}, \frac{\log \lambda(\chi_2 \circ \chi_1^{-1} \circ g)}{\log \lambda(g)} \right\}. \tag{6}$$

From the relation between the Poincaré length of closed curves and the multiplier of the elements in G, we can define the following function [5], [9]:

$$d([X,f_1.X_1].[X,f_2,X_2]) = \log \rho([X,f_1,X_1],[X,f_2,X_2]), \tag{7}$$

where

$$\rho([X,f_1,X_1],[X,f_2,X_2]) = \sup_{\alpha \in \Sigma'_{X_1}} \left\{ \frac{l_{X_2}\big(f_2 \circ f_1^{-1}(\alpha)\big)}{l_{X_1}(\alpha)}, \frac{l_{X_1}(\alpha)}{l_{X_2}\big(f_2 \circ f_1^{-1}(\alpha)\big)} \right\}. \tag{8}$$

According to the results in [1], [5] and [9], we know that (3) and (7) define a metric on $T(G)$ and $T(X)$, respectively. We also know that these two metrics are the same.

In 1972, Sorvali [9] suggested the following problem: if G is a finitely generated Fuchsian group, is the metric d topologically equivalent to the Teichmüller metric d_T in $T(G) = T(X)$? In 1975, Sorvali [10] studied the above problem for torus and proved that the two metrics are topologically equivalent in this case. In the same paper [10], Sorvali suggested the same problem for compact Riemann surfaces. In 1986, Li Zhong [5] proved that for any compact Riemann surface $X = U/G$, the two metrics are topologically equivalent in $T(X)$, and this result solved the problem presented by Sorvali in [10]. But for the Teichmüller space corresponding to a non-compact Riemann surface, the above problem remains open.

In this paper we will prove that for any torsion-free non-elementary finitely generated Fuchsian group G, the metric d and the Teichmüller metric d_T in $T(G)$ are topologically equivalent. This result solves the above problem considered by Sorvali [9] in 1972. First we prove that, for any conformally finite Riemann surface X, the metrics d and d_T are topologically equivalent in $T(X)$. Then, using the Nielsen extension and Schottky double, we prove that the above result remains true for any torsion-free non-elementary finitely generated Fuchsian group (or any non-elementary topologically finite Riemann surface).

1. Conformally finite Riemann surfaces

If there is no special claim, the Riemann surfaces $X = U/G$ in this section are of type (g, p), where g is the number of genus and p is the number of punctures. The Fuchsian group G is finitely generated and of the first kind.

Let $\mathrm{QD}(X)$ be the set of holomorphic quadratic differentials on the Riemann surface X, and let $\mathrm{PQD}(X)$ be the set of its projective classes [4]. As we know, the real dimension of $\mathrm{QD}(X)$ is $6g-6+2p$ and that of $\mathrm{PQD}(X)$ is $6g-7+2p$ [4], [5]. For any $\phi \in \mathrm{QD}(X)$, it determines a pair of transversely measured foliations. These are the horizontal trajectory together with its vertical measure and the vertical trajectory together with its horizontal measure. Let $\mathrm{MF}(\widetilde{X})$ be the set of measured foliations on a topological surface $\widetilde{X}$, and let $\mathrm{PMF}(\widetilde{X})$ be the set of its projective classes. We know that the real dimensions of $\mathrm{MF}(\widetilde{X})$ and $\mathrm{PMF}(\widetilde{X})$ are $6g-6+2p$ and $6g-7+2p$, respectively. $\mathrm{PQD}(X)$ and $\mathrm{PMF}(\widetilde{X})$ may be viewed as the unit spheres in $\mathrm{QD}(X)$ and $\mathrm{PMF}(\widetilde{X})$, respectively. Therefore $\mathrm{PQD}(X)$ and $\mathrm{PMF}(\widetilde{X})$ are compact. The Riemann surface X may be viewed as a topological surface $\widetilde{X}$ together with some complex structure.

We have the mapping [4]

$$H\colon \mathrm{QD}(X) \to \mathrm{MF}(\widetilde{X}), \tag{9}$$

where H maps ϕ onto its horizontal trajectory together with its vertical measure. We know that H is a homeomorphism.

For any $\alpha \in \Sigma''_X$, we can define its extremal length $E_X(\alpha)$ [4]. Kerckhoff [4] generalized the definition of extremal length from simple closed curves to measured foliations. This is a natural generalization. Actually any $\alpha \in \Sigma''_X$ may be viewed as a measured foliation [4]. For any $F \in \mathrm{MF}(\widetilde{X})$, $E_X(F)$ is realized by the metric determined by the holomorphic quadratic differential $H^{-1}(F)$.

We first introduce two lemmas. In these two lemmas, the Riemann surfaces are hyperbolic, not necessarily of a conformally finite type.

Lemma 1 [1]. *For any $\alpha \in \Sigma'_{X_1}$ the dilatation of a quasiconformal mapping $h\colon X_1 \to X_2$ satisfies the inequality*:

$$K[h] \geq \frac{l_{X_2}\big(f(\alpha)\big)}{l_{X_1}(\alpha)}. \tag{10}$$

Lemma 2. *For any two points $\tau_1, \tau_2 \in T(X)$, we have*

$$d_T(\tau_1, \tau_2) \geq d(\tau_1, \tau_2). \tag{11}$$

The following theorem is a natural generalization of a result of Kerckhoff [4].

Theorem 1. *For any two points $[X, f_1, X_1]$ and $[X, f_2, X_2]$ in $T(X)$, we have*

$$d_T([X, f_1, X_1], [X, f_2, X_2]) = \frac{1}{2} \log \sup_{\alpha \in \Sigma''_X} \frac{E_{X_2}\big(f_2 \circ f_1^{-1}(\alpha)\big)}{E_{X_1}(\alpha)}. \tag{12}$$

Remark. Kerckhoff [4] obtained a similar result for compact Riemann surfaces. Because the proof of the above theorem is the same as that of Kerckhoff, we omit the details. On the other hand, we do not know whether the above result remains valid for any non-conformal finite-type Riemann surface.

As a generalization of the Poincaré length of a simple closed curve, we may define for any $F \in \mathrm{MF}(\widetilde{X})$ its Poincaré length $l_X(F)$ [8]. By the definition of extremal length and the Gauss–Bonnet theorem, we have

Lemma 3. *For any $F \in \mathrm{MF}(\widetilde{X})$, we have*

$$\frac{E_X(F)}{l_X^2(F)} \geq \frac{1}{2\pi|\chi(X)|}, \tag{13}$$

where $\chi(X)$ is the Euler number of X.

Morerover, the Poincaré length and the extremal length of a measured foliation have the following relation.

Theorem 2. *There exist constants $M_1(X)$ and $M_2(X)$, depending only on X, such that for any $F \in \mathrm{MF}(\widetilde{X})$,*

$$M_1(X) \le \frac{E_X(F)}{l_X^2(F)} \le M_2(X). \tag{14}$$

Proof. As functions defined on $\mathrm{MF}(\widetilde{X})$, $E_X(F)$ and $l_X^2(F)$ are continuous and take positive values in $MF(\widetilde{X}) - \{0\}$. For any $r > 0$, we have ([4]),

$$E_X(rF) = r^2 E_X(F), \tag{15}$$

$$l_X^2(rF) = r^2 l_X^2(F). \tag{16}$$

Therefore the function $E_X(F)/l_X^2(F)$ is a positive continuous function on the compact set $\mathrm{PMF}(\widetilde{X})$. So it can attain its maximum and minimum. Denote them by $M_2(X)$ and $M_1(X)$, respectively. This completes the proof of the theorem. □

From the above results we have:

Theorem 3. *Let $[X, f_1, X_1]$ and $[X, f_2, X_2]$ be two points in $T(X)$. Then*

$$d_T([X, f_1, X_1], [X, f_2, X_2]) \le C_2'(X) + d([X, f_1, X_1], [X, f_2, X_2]), \tag{17}$$

where $C_2' = \frac{1}{2}\log[2\pi|\chi(X_1)|C_2(X_1)]$ depends only on the Riemann surface X_1.

Proof. By Lemma 3 and Theorem 2, we have

$$\frac{1}{E_{X_2}\big(f_2 \circ f_1^{-1}(\alpha)\big)} \le \frac{2\pi|\chi(X_2)|}{l_{X_2}^2\big(f_2 \circ f_1^{-1}(\alpha)\big)}, \tag{18}$$

$$E_{X_1}(\alpha) \le C_2(X_1) l_{X_1}^2(\alpha), \tag{19}$$

where $\alpha \in \Sigma''_{X_1}$.

Then, by Theorem 1 and $\chi(X_1) = \chi(X_2)$ and the symmetrization of Teichmüller metric, we have

$$\begin{aligned} d_T([X, f_1, X_1], [X, f_2, X_2]) &= \frac{1}{2}\log \sup_{\alpha \in \Sigma''_{X_1}} \frac{E_{X_1}(\alpha)}{E_{X_2}\big(f_2 \circ f_1^{-1}(\alpha)\big)} \\ &\le \frac{1}{2}\log \sup_{\alpha \in \Sigma''_{X_1}} \frac{2\pi|\chi(X_2)|C_2(X_1) l_{X_1}^2(\alpha)}{l_{X_2}^2\big(f_2 \circ f_1^{-1}(\alpha)\big)} \\ &= \frac{1}{2}\log[2\pi|\chi(X_1)|C_2(X_1)] \\ &\quad + \frac{1}{2}\log \sup_{\alpha \in \Sigma''_{X_2}} \frac{l_{X_1}^2(\alpha)}{l_{X_2}^2\big(f_2 \circ f_1^{-1}(\alpha)\big)} \\ &\le C_2'(X_1) + d([X, f_1, X_1], [X, f_2, X_2]). \ \square \end{aligned} \tag{20}$$

Similarly to the proof of Theorem 1 in [5], we can prove the following theorem using Theorem 3. For the sake of simplicity, we omit the details.

Theorem 4. *Let X be a conformally finite-type Riemann surface. Then the metric d and the Teichmüller metric d_T are topologically equivalent.*

2. Topologically finite Riemann surfaces

The Riemann surfaces in this section are of type (g,m,n), where g, m, n are the number of genus, punctures and ideal boundaries, respectively, with $6g-6+m+3n>0$, $n>0$. The corresponding Teichmüller space is sometimes written as $T(g,m,n)$.

We know that any (g,m,n)-type Riemann surface X with $n>0$ is the Nielsen extension of a uniquely determined Riemann surface X_0. Further, X_0 is called the Nielsen kernel of X [2]. The above relation is unique and one-one. On the other hand, for any above Riemann surface X we can define its Schottky double X^d. This is a Riemann surface of type $(2g+n-1,2m,0)$ [1]. Because X is hyperbolic, there exists a Poincaré metric on X and there also exists a Poincaré metric on X^d. The metric on X induced by the Poincaré metric on X^d is called the intrinsic metric of X. The ideal boundaries are geodesic curves in their intrinsic metrics.

The topological types of the Riemann surface X and its Nielsen kernel X_0 are identical. The complement of the closure of X_0 in X is n funnels. Here each funnel is a ring domain with one of its boundaries is an ideal boundary of X. Moreover, Σ_X and Σ_{X_0} have the same corresponding relation. For the sake of convenience, we make no distinction between the corresponding elements in Σ_X and Σ_{X_0}.

The following result is due to Bers [2].

Lemma 4. *Let X_0 be the Nielsen kernel of X. Then the metric on X_0 induced by the Poincaré metric of X is the same as the intrinsic metric of X_0.*

Similarly to (7), we can define a metric on $T(X)$ using the intrinsic metric on X. For any $\alpha\in\Sigma_X$, denote the length of α in the intrinsic metric of X by $l^I_X(\alpha)$. For any two points $[X,f_1,X_1]$ and $[X,f_2,X_2]$ in $T(X)$, we define

$$d_I([X,f_1,X_1],[X,f_2,X_2])=\log\rho_I([X,f_1,X_1],[X,f_2,X_2]), \tag{21}$$

where

$$\rho_I([X,f_1,X_1],[X,f_2,X_2])=\sup_{\alpha\in\Sigma'_{X_1}}\left\{\frac{l^I_{X_2}\big(f_2\circ f_1^{-1}(\alpha)\big)}{l^I_{X_1}(\alpha)},\frac{l^I_{X_1}(\alpha)}{l^I_{X_2}\big(f_2\circ f_1^{-1}(\alpha)\big)}\right\}. \tag{22}$$

Next we will compare the Poincaré metric and the intrinsic metric. The following lemma is a consequence of Schwarz's lemma.

Lemma 5. *For any $\alpha\in\Sigma'_X$, we have*

$$l^I_X(\alpha)<l_X(\alpha). \tag{23}$$

Further, we have the following

Theorem 5. *There exist constants $M_3(X)$ and $M_4(X)$, which depend only on X, such that*

$$M_3(X) \leq \frac{l^I_X(\alpha)}{l_X(\alpha)} \leq M_4(X) \tag{24}$$

for any $\alpha \in \Sigma''_X$.

Proof. Consider the Schottky doubles X_0^d and X^d of X_0 and X, respectively. By Lemma 4, we know that $l_X(\alpha) = l^I_{X_0}(\alpha)$ for any $\alpha \in \Sigma_{X_0}$. Thus we know that, for any $\alpha \in \Sigma_{X_0}$,

$$l^I_X(\alpha) = l_{X^d}(\alpha),$$
$$l_X(\alpha) = l^d_{X_0}(\alpha).$$

Similarly to the proof of Theorem 2, $l_{X^d}(F)$ and $l_{X_0^d}(F)$ may be viewed as functions on $\mathrm{MF}(\widetilde{X}^d)$ and taking positive values on $\mathrm{MF}(\widetilde{X}^d) - \{0\}$. For any $r > 0$, we have

$$l_{X^d}(tF) = t l_{X^d}(F),$$
$$l_{X_0^d}(tF) = t l_{X_0^d}(F).$$

Therefore the function $l_{X^d}(F)/l_{X_0^d}(F)$ is a positive continuous function defined on a compact set $\mathrm{PMF}(\widetilde{X}^d)$. So it attains its maximum and minimum. Denote them by $M_3(X^d, X_0^d)$ and $M_4(X^d, X_0^d)$. But X^d and X_0^d are uniquely determined by X. Thus we can denote these two constants by $M_3(X)$ and $M_4(X)$. □

Similarly to the above discussion, we have:

Theorem 6. *There exist constants $M_5(X)$ and $M_6(X)$, which depend only on X, such that*

$$M_5(X) \leq \frac{E_{X^d}(\alpha)}{E_{X_0^d}(\alpha)} \leq M_6(X) \tag{25}$$

for any $\alpha \in \Sigma''_{X_1}$.

Using the pant decomposition of Riemann surface, we have:

Theorem 7. *Let d_I be the metric on $T(X)$ defined in (21) and let d be the metric on $T(X^d)$ defined in (7). Then the point sequence $\{[X, f_n, X_n]\}$, $n = 1, 2, \ldots$, in $T(X)$ converges to $[X, f_0, X_0]$ in the metric d_I if and only if the corresponding point sequence $\{[X^d, f_n^d, X_n^d]\}$, $n = 1, 2, \ldots$, in $T(X^d)$ converges to $[X^d, f_0^d, X_0^d]$ in the metric d.*

Proof. Because the intrinsic metric on a Riemann surface is induced by the Poincaré metric on its Schottky double, the sufficiency condition is obvious.

Next we prove the necessity condition. Consider the pant decomposition of a Riemann surface X. Suppose X is decomposed into disjoint pants $\{P_i\}$, $i = 1, 2, \ldots, k_1$. The decomposition curves are $\{L_k\}$, $k = 1, 2, \ldots, k_4$. Let $\{C_i\}$, $i = 1, 2, \ldots, k_2$, be the set of punctures and $\{l_i\}$, $i = 1, 2, \ldots, k_3$, be the set of ideal boundaries. Here the constants k_i, $i = 1, 2, 3, 4$, depend only on the topological type of X. All the boundary components of $\{P_i\}$ are $\{C_i\} \cup \{l_j\} \cup \{L_k\}$.

From above we can obtain the corresponding pant decomposition of X^d. The set of disjoint pants is $\{P_i\} \cup \{\overline{P}_i\}$, and the boundary components of $\{P_i\} \cup \{\overline{P}_i\}$ are $\{C_i\} \cup \{\overline{C}_i\} \cup \{l_j\} \cup \{L_k\} \cup \{\overline{L}_k\}$.

By the mappings f_n, f_n^d and f_0, we obtain the corresponding pant decompositions $\{P_i^{(n)}\}$, $\{P_i^{(n)}\} \cup \{\overline{P}_i^{(n)}\}$ and $\{P_i^{(0)}\}$ of X_n, X_n^d and X_0, respectively.

We can induce a metric on any pant P_i by the intrinsic metric of X, and on $\overline{P}_i$ such a metric is induced by the Poincaré metric of X^d. The above metric on a pant is determined by the lengths of its three boundary components. For every Riemann surface, its intrinsic metric is determined by the metric on all of its pants and all the twists of non-puncture and non-ideal boundary components of the pant decomposition. For X^d, the above claim remains valid. We also know that [1] the twist about a curve is determined by the lengths of some of curves which intersect this curve.

Now the sequence $\{[X, f_n, X_n]\}$, $n = 1, 2, \ldots$, converges to $[X, f_0, X_0]$ in the metric d_I. This means that $l^I_{X_n}\big(f_n(\alpha)\big)$ converges to $l^I_{X_0}\big(f_0(\alpha)\big)$ for every $\alpha \in \Sigma_X$. If we pick all the pant decomposition curves and sufficiently many curves which intersect the decomposition curves, we know that for $1 \le i \le k_1$ the metric on the pant $P_i^{(n)}$ converges to the metric on the pant $P_i^{(0)}$. And for any $1 \le i \le k_4$, the twist about the curve $f_n(L_i)$ converges to the twist about $f_0(L_i)$. For the Riemann surface X^d, its Poincaré metric is determined by the metrics on the pants $\{P_i\} \cup \{\overline{P}_i\}$ and the twists about $\{l_j\} \cup \{L_k\} \cup \{\bar{L}_k\}$. We know that the metrics on P_i and $\overline{P}_i$ are the same and that the twist about L_k and $\bar{L}_k$ are identical. The twists about the curves l_j are zero.

To prove the necessary condition, it is sufficient to prove that, for any $\alpha \in \Sigma'_{X^d}$, α intersects at least one of the curves in $\{l_j\}$, such that $l_{X_n^d}\big(f_n^d(\alpha)\big)$ converges to $l_{X_0^d}$, $f_0^d(X)$. Next we prove this claim.

As we know,

$$
\begin{aligned}
l_{X_n^d}\big(f_n^d(\alpha)\big) = {} & \sum_{i=1}^{k_1} \big[l_{P_I^{(n)}}\big(f_n^d(\alpha) \cap P_i^{(n)}\big) + l_{\overline{P}_i^{(n)}}\big(f_n^d(\alpha) \cap \overline{P}_i^{(n)}\big) \big] \\
& + \sum_{j=1}^{k_3} i\big(f_n^d(\alpha), f_n^d(\alpha)\big) t\big(f_n^d(l_j)\big) \\
& + \sum_{i=1}^{k_4} \big[i\big(f_n^d(\alpha), f_n^d(L_i)\big) t\big(f_n^d(L_i)\big) + i\big(f_n^d(\alpha), f_n^d(\bar{L}_i)\big) t\big(f_n^d(\bar{L}_i)\big) \big],
\end{aligned}
$$

where $i(\alpha,\beta)$ is the geometric intersection of α and β and $t(\alpha)$ is the twist about α.

From above we know that

$$\begin{aligned}
&l_{P_i^{(n)}}\big(f_n^d(\alpha)\cap P_i^{(n)}\big) \overset{n\to\infty}{\longrightarrow} l_{P_i^{(0)}}\big(f_0^d(\alpha)\cap P_i^{(0)}\big),\\
&l_{\overline{P}_i^{(n)}}\big(f_n^d(\alpha)\cap \overline{P}_i^{(n)}\big) \overset{n\to\infty}{\longrightarrow} l_{\overline{P}_i^{(0)}}\big(f_0^d(\alpha)\cap \overline{P}_i^{(0)}\big),\\
&i\big(f_n^d(\alpha), f_n^d(L_i)\big)t\big(f_n^d(L_i)\big) \overset{n\to\infty}{\longrightarrow} i\big(f_0^d(\alpha), f_0^d(L_i)\big)t\big(f_0^d(L_i)\big),\\
&i\big(f_n^d(\alpha), f_n^d(l_i)\big)t\big(f_n^d(l_i)\big) = 0,\\
&i\big(f_n^d(\alpha), f_n^d(\bar{L}_i)\big)t\big(f_n^d(\bar{L}_i)\big) \overset{n\to\infty}{\longrightarrow} i\big(f_0^d(\alpha), f_0^d(\bar{L}_i)\big)t\big(f_0^d(\bar{L}_i)\big).
\end{aligned}$$

Then we know that $l_{X_n^d}\big(f_n^d(\alpha)\big)$ converges to $l_{X_0^d}\big(f_0^d(\alpha)\big)$ as $n\to\infty$. This completes the proof of the theorem. □

For any two points $[X,f_1,X_1]$ and $[X,f_2,X_2]$ in $T(X)$ and the corresponding two points $[X^d,f_1^d,X_1^d]$ and $[X^d,f_2^d,X_2^d]$ in $T(X^d)$, let d_{T_1} and d_{T_2} be the Teichmüller metrics on $T(X)$ and $T(X^d)$, respectively. Then [1]

$$d_{T_1}([X,f_1,X_1],[X,f_2,X_2]) = d_{T_2}([X^d,f_1^d,X_1^d],[X^d,f_2^d,X_2^d]).$$

From Theorem 4, Theorem 7 and the above equation, we have

Corollary 1. *The metric d_I and the Teichmüller metric on $T(X)$ are topologically equivalent.*

Next we introduce several lemmas.

Lemma 6 [7]. *For $\alpha\in\Sigma'_X$, let $l_X(\alpha)$ and $E_X(\alpha)$ be its hyperbolic length and extremal length, respectively. Then we have*
(a) $l_X(\alpha)/\pi \le E_X(\alpha) \le \frac{1}{2}l_X(\alpha)e^{l_X(\alpha)/2}$,
(b) *$l_X(\alpha)$ and $E_X(\alpha)$ may be viewed as functions on $T(X)$, $l_X(\alpha)$ and $E_X(\alpha)$ go to zero together with*

$$\lim_{l_X(\alpha)\to 0}\frac{l_X(\alpha)}{E_X(\alpha)} = \pi.$$

Lemma 7 [3]. *For α_1, $\alpha_2\in\Sigma'_X$ and $i(\alpha_1,\alpha_2)\neq 0$. Then*
(a) *$l_X(\alpha_2)\to\infty$ if $l_X(\alpha_1)\to 0$,*
(b) *$l_X^I(\alpha_2)\to\infty$ if $l_X^I(\alpha_1)\to 0$.*

We prove that the metric d and the Teichmüller metric d_T on $T(X)$ are topologically equivalent. For this purpose, we first prove the following theorem.

Theorem 8. *Let X_0 be the Nielsen kernel of a Riemann surface X. If the point sequence $[X_0^d,f_{0n}^d,X_{0n}^d]$, $n=1,2,\ldots$, converges to $[X_0^d,f_{00}^d,X_{00}^d]$ in the Teichnüller metric d_{T_0} of a Teichmüller space $T(X_0^d)$, the corresponding point sequence $[X^d,f_n^d,X_n^d]$, $n=1,2,\ldots$, is bounded in the Teichmüller metric d_{T_2} of the Teichmüller space $T(X^d)$.*

Proof. To prove the conclusion, it is sufficient to prove that the number sequence

$$M_n = \sup_{\alpha\in\Sigma''_{X^d}} \frac{E_{X_1^d}\big(f_1^d(\alpha)\big)}{E_{X_n^d}\big(f_n^d(\alpha)\big)}, \qquad n = 1, 2, \dots, \tag{26}$$

is bounded.

Let $I = \{\beta_i\}$, $i = 1, 2, \dots, k$, be the set of the ideal boundaries of X. We assume that the sequence M_n, $n = 1, 2, \dots$, is unbounded. Rewrite M_n as

$$M_n = \max\left\{ \sup_{\alpha\in\Sigma''_{X^d},\, \alpha\notin I} \frac{E_{X_1^d}\big(f_1^d(\alpha)\big)}{E_{X_n^d}\big(f_n^d(\alpha)\big)}, \sup_{\alpha\in I} \frac{E_{X_1^d}\big(f_1^d(\alpha)\big)}{E_{X_n^d}\big(f_n^d(\alpha)\big)} \right\}.$$

From the definition of extremal length [4], we know that for any $\alpha \in \Sigma''_{X^d}$, $\alpha \notin I$,

$$E_{X_n^d}\big(f_n^d(\alpha)\big) \ge E_{X_{0n}^d}\big(f_{0n}^d(\alpha)\big).$$

From (25) in Theorem 6, we have

$$\begin{aligned} \sup_{\alpha\in\Sigma''_{X_d},\, \alpha\notin I} \frac{E_{X_1^d}\big(f_1^d(\alpha)\big)}{E_{X_n^d}\big(f_n^d(\alpha)\big)} &\le \sup_{\alpha\in\Sigma''_{X_0^d},\alpha\notin I} \frac{M_6(X_1) E^d_{X_01}\big(f_{01}^d(\alpha)\big)}{E_{X_{0n}^d}\big(f_{0n}^d(\alpha)\big)} \\ &\le M_6(X_1) d_{T_2}([X_0^d, f_{01}^d, X_{01}^d], [X_0^d, f_{0n}^d, X_{0n}^d]). \end{aligned} \tag{27}$$

From the conditions given in the theorem, we know that (27) is bounded for all $n \ge 1$. Therefore the assumption that $\{M_n\}$ is unbounded implies that the sequence

$$M'_n = \sup_{\alpha\in I} \frac{E_{X_1^d}\big(f_1^d(\alpha)\big)}{E_{X_n^d}\big(f_n^d(\alpha)\big)}, \qquad n = 1, 2, \dots,$$

is unbounded.

Without loss of generality, we may assume that $M'_n \to \infty$. Because there are only finite curves in I, we may pick some $\beta_1 \in I$ and a subsequence of $[X^d, f_n^d, X_n^d]$, still denoted by $[X^d, f_n^d, X_n^d]$, such that the sequence

$$M''_n = \frac{E^d_{X_1}\big(f_1^d(\beta_1)\big)}{E_{X_n^d}\big(f_n^d(\beta_1)\big)}$$

tends to ∞ as $n \to \infty$. This means that

$$E_{X_n^d}\big(f_n^d(\beta_1)\big) \to 0 \qquad \text{as } n \to \infty.$$

From Lemma 6, we have

$$\lim_{n\to\infty} l_{X_n^d}\big(f_n^d(\beta_1)\big) = 0.$$

By Lemma 7, we know that for any curve $\alpha \in \Sigma''_{X^d}$, $i(\alpha,\beta) \neq \emptyset$,

$$\lim_{n\to\infty} l_{X_n^d}\big(f_n^d(\alpha)\big) = \infty.$$

Then from Lemma 5 we have

$$\lim_{n\to\infty} l_{X_{0n}^d}\big(f_{0n}^d(\alpha)\big) = \infty,$$

where α is any closed curve that intersects β_1. From the convergence property of the sequence $\{[X_0^d, f_{0n}^d, X_{0n}^d]\}$, $n = 1,2,\ldots$, in the metric d_{T_0} and Theorem 4, we know that the above conclusion is impossible. So the assumption is false. This completes the proof of the theorem. □

Using Theorem 8, we obtain:

Theorem 9. *For any topologically finite-type Riemann surface X, the metric d and the Teichmüller metric d_T on a Teichmüller space $T(X) = T(g,m,n)$ are topologically equivalent.*

Proof. From Lemma 2 we know that, to prove the topological equivalence of the metrics d and d_T, it is sufficient to prove the following fact: if the sequence $[X, f_n, X_n]$ converges to $[X, f_0, X_0]$ in the metric d, the same sequence converges to $[X, f_0, X_0]$ in the metric d_T. From the proof of Theorem 1 in [5] we know that, to prove the above fact, it is sufficient to prove that if the sequence $[X, f_n, X_n]$ converges to $[X, f_0, X_0]$ in the metric d, the sequence is bounded in the metric d_T. Next we prove this claim.

Because the sequence $[X, f_n, X_n]$ converges to $[X, f_0, X_0]$ in the metric d, the metric d may be viewed as induced by the intrinsic metric of its Nielsen kernel. From Corollary 1 we know that the sequence $[X_0^d, f_{0n}^d, X_{0n}^d]$, $n = 1,2,\ldots$, converges to $[X_0^d, f_{00}^d, X_{00}^d]$ in the Teichmüller metric of a Teichmüller space $T(X_0^d)$. Then, by Theorem 8, we know that the sequence $[X^d, f_n^d, X_n^d]$, $n = 1,2,\ldots$, is bounded in the Teichmüller metric of the Teichmüller space $T(X^d)$. This means that the sequence $[X, f_n, X_n]$ is bounded in the Teichmüller metric of $T(X)$. By this conclusion, similarly to the proof of Theorem 1 in [5], we can prove that the metric d and d_T on $T(X)$ are topologically equivalent. We omit the details for the sake of simplicity. □

Remark. From Theorem 6, Theorem 7 and Corollary 1, we know that for any non-elementary topologically finite type Riemann surface X in the Teichmüller spaces $T(X)$ and $T(X^d)$, the metrics in (7), (21) and their Teichmüller metrics are topologically equivalent.

Remark. We do not know whether the conclusion in this paper also holds for Fuchsian groups with torsion or infinitely generated Fuchsian groups.

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