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CALORIC MEASURE ON DOMAINS BOUNDED BY WEIERSTRASS-TYPE GRAPHS

Thierry Bousch and Yanick Heurteaux

Université Paris-Sud, Département de Mathématique
Bât. 425-430, F-91405 Orsay Cedex, France
Thierry.Bousch@math.u-psud.fr, Yanick.Heurteaux@math.u-psud.fr

Abstract. Let ω be the caloric measure on a domain bounded by a Weierstrass-type graph. We prove that ω is a quasi-Bernoulli measure. Therefore, the multifractal formalism is available for such a measure. We can then give necessary and sufficient conditions which ensure that ω is supported by a set of small dimension and prove that this is the generic case. These results improve and generalize an earlier work due to R. Kaufman and J.M. Wu.

Let $f\colon \mathbf{R} \to \mathbf{R}$ be a real function of class $C^{1/2}$ (i.e., Hölder of exponent $1/2$) and $\Omega = \{(x,t) \in \mathbf{R}^2 : x > f(t)\}$. For every point $M_0 = (x_0, t_0) \in \Omega$, let ω^{M_0} be the harmonic measure at M_0 with respect to the heat operator $\mathscr{C} = \partial/\partial t - \partial^2/2\partial x^2$. This measure is called the caloric measure at M_0. It is supported by the set of points $(x,t) \in \partial\Omega$ such that $t \le t_0$ and may also be defined by

$$\omega^{M_0}(E) = \mathbf{P}_{x_0}\bigl(T < +\infty \text{ and } (B_T, t_0 - T) \in E\bigr),$$

where B_t is a standard Brownian motion and

$$T = \inf\bigl(\{t > 0 : (B_t, t_0 - t) \in \mathbf{R}^2 \setminus \Omega\}\bigr).$$

We are interested in the geometric properties of the measure ω^{M_0}. The domain Ω can be seen as a Lipschitz domain related to the parabolic distance in $\mathbf{R}^2$

$$\delta\bigl((x,t),(y,s)\bigr) = \sup(|x-y|, |t-s|^{1/2}).$$

Because of the homogeneity properties of the heat operator, this distance is adapted to our situation. It is then natural to ask whether the analogue of Dahlberg's theorem ([Da]) holds for the heat equation in Ω. Of course, the graph of a function of class $C^{1/2}$ is in general not rectifiable. Nevertheless, following Taylor and Watson ([TW]), we can introduce the Hausdorff measures related to the metric space $(\mathbf{R}^2, \delta)$. These are defined for $\alpha > 0$ by

$$\Lambda_\alpha(E) = \lim_{\varepsilon \to 0}\biggl(\inf\biggl(\sum_i r_i^\alpha,\ E \subset \bigcup_i D(a_i, r_i) \text{ and } r_i \le \varepsilon\biggr)\biggr),$$

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where $D(a_i,r_i)$ are balls for the distance δ. The related dimension is called the parabolic dimension. In this context, the parabolic dimension of the graph $\partial\Omega$ is equal to 2 and its Λ_2-measure is locally finite. Then, there is an analogy between the measure Λ_2 in graphs of $C^{1/2}$-functions and the length measure in Lipschitz graphs.

In [KW], Robert Kaufman and Jang-Mei Wu proved that the equivalent of Dahlberg's theorem is false for the heat equation and constructed a function f of class $C^{1/2}$ such that the caloric measure is supported by a set of parabolic dimension strictly less than 2. In other words, if $\tilde\omega^{M_0}$ denotes the image of the caloric measure ω^{M_0} under the projection

$$\Pi\colon \bigl(f(t),t\bigr)\in\partial\Omega\longmapsto t\in\mathbf{R}, \tag{1}$$

the measure $\tilde\omega^{M_0}$ is supported by a set of Hausdorff dimension strictly less than 1.

On the other hand, for slightly more regular functions f, the measure $\tilde\omega^{M_0}$ becomes equivalent to the Lebesgue measure dt. For example, in [LS], Lewis and Silver proved that this is the case when the modulus of continuity φ of the function f satisfies the Dini condition $\int_0^1 \varphi^2(t)/t^2\,dt<+\infty$. Moreover, when f is of class $C^{(1+\varepsilon)/2}$, we can prove that the relative densities between the measures $\tilde\omega^{M_0}$ and dt are locally bounded functions (see [H2, pp. 642–643]).

In this work, we want to give a more precise analysis of the caloric measure on domains bounded by Weierstrass-type graphs. Let g be a Lipschitz function, periodic with period 1, and l be an integer greater than 1. We are interested in the caloric measure on the domain Ω bounded by the graph of the function

$$f(t)=\sum_{k=0}^{+\infty} l^{-k/2}g(l^k t). \tag{2}$$

Such a curve $\partial\Omega$ will be called a Weierstrass-type curve (see [F]) and the domain Ω a Weierstrass domain. It is well known that when the function g is Lipschitz with Lipschitz constant $\widetilde K$, the function f is of class $C^{1/2}$ with Hölder constant $K=2\widetilde K/(1-l^{-1/2})$. In other words, the map $g\mapsto f$ is continous from the space of 1-periodic Lipschitz functions to the space of 1-periodic functions of class $C^{1/2}$ when we endow these spaces with their natural norms.

The first result of this work (Theorem 3.1) states that for a Weierstrass domain Ω and for suitable points M_0, the measure $\tilde\omega^{M_0}$ is a quasi-Bernoulli measure in $[0,1)$ (see (16)). To prove this theorem, we need a preliminary result which is developed in Section 2. When $\partial\Omega_2$ is a Lipschitz perturbation of $\partial\Omega_1$, we establish in Theorem 2.1 that the associated caloric measures (more precisely their projections on the real axis) are strongly equivalent.

Quasi-Bernoulli measures have been extensively studied in the literature ([C], [MV], [BMP], [H5], [BH]) and we can use known results to describe the geometric properties of the measure $\tilde{\omega}^{M_0}$. Let $\mathscr{F}_n$ be the family of l-adic intervals of $[0,1)$ of the n^{th} generation and denote by $I_n(t)$ the unique interval of $\mathscr{F}_n$ that contains t. It is well known (see for example [C] or [H5]) that there exists a real $d \le 1$ such that for almost every t,

$$\lim_{n\to+\infty} \frac{\log\bigl(\tilde{\omega}^{M_0}(I_n(t))\bigr)}{-n\log l} = d.$$

In particular, the measure $\tilde{\omega}^{M_0}$ is supported by a set of dimension d and every set of Hausdorff dimension $< d$ is negligible (in this case, we say that the measure is unidimensional and we call $d = \dim(\tilde{\omega}^{M_0})$ the dimension of the measure). In fact, the number d can be calculated in the following way. Let $\lambda = \tilde{\omega}^{M_0}/\tilde{\omega}^{M_0}\bigl([0,1)\bigr)$ be the normalized measure and define

$$\text{(3)} \qquad \tau_n(x) = \frac{1}{n\log l}\log\biggl(\sum_{I\in\mathscr{F}_n}\lambda(I)^x\biggr) \qquad \text{and} \qquad \tau(x) = \lim_{n\to+\infty}\tau_n(x)$$

(the limit exists in such a situation). It is proved in [H5] that the function τ is differentiable and satisfies

$$\text{(4)} \qquad \dim(\tilde{\omega}^{M_0}) = -\tau'(1) = \lim_{n\to+\infty}\frac{-1}{n\log l}\sum_{I\in\mathscr{F}_n}\lambda(I)\log\bigl(\lambda(I)\bigr).$$

Then we can conclude (see Corollary 3.2) that either the measure $\tilde{\omega}^{M_0}$ is equivalent to the Lebesgue measure on its support with locally bounded densities, or it has dimension $d < 1$.

Let us also remember that the multifractal formalism is available for such a measure. Using the result of Brown–Michon–Peyriere ([BMP]) and the differentiability of the function τ ([H5]), we have

$$\dim(E_\alpha) = \tau^*(\alpha) \qquad \text{for all } \alpha \in \bigl(-\tau'(+\infty), -\tau'(-\infty)\bigr),$$

where τ^* is the Legendre transform of the convex function τ and

$$E_\alpha = \biggl\{t \in [0,1) : \lim_{r\to 0}\frac{\log\bigl(\tilde{\omega}^{M_0}([t-r,t+r])\bigr)}{\log r} = \alpha\biggr\}.$$

In particular, in the case where $\dim(\tilde{\omega}^{M_0}) < 1$, we obtain that the measure $\tilde{\omega}^{M_0}$ is multifractal in the sense that the set E_α has a positive dimension for infinitely many values of α.

In Section 4, we characterize those functions g for which the dimension of the measure $\tilde{\omega}^{M_0}$ is strictly less than 1. Of course this is not the case for every Lipschitz function g. If there exists a Lipschitz function γ such that $g(t) =$

$\gamma(t) - l^{-1/2}\gamma(lt)$, then $f = \gamma$ is Lipschitz and the measure $\tilde{\omega}^{M_0}$ is equivalent to the Lebesgue measure. In fact, this is exceptional. A careful study of the functional equation $f(t) = g(t) + f(lt)/\sqrt{l}$ allows us to prove that $\dim(\tilde{\omega}^{M_0}) = 1$ if and only if f is Lipschitz. In particular, the set of Lipschitz functions g such that $\dim(\tilde{\omega}^{M_0}) < 1$ is an open dense subset of the space of Lipschitz functions. Moreover, in Proposition 4.7, we introduce an explicit criterion to test whether the function f is Lipschitz.

It is also interesting to give a lower bound for the dimension of the measure $\tilde{\omega}^{M_0}$. It seems difficult to obtain an optimal lower bound. Using Taylor–Watson results ([TW]), we know that, when f is of class $C^{1/2}$, every set of dimension strictly less than $1/2$ is negligible for $\tilde{\omega}^{M_0}$ (its projection on the graph is polar for the heat equation). In fact, in Section 5, we improve this result and establish that there exists a constant $\varepsilon = \varepsilon(K)$ such that every set of dimension strictly less than $\frac{1}{2}(1+\varepsilon)$ is negligible for the measure $\tilde{\omega}^{M_0}$. A reasonable conjecture would be that for every $\varepsilon > 0$, it is possible to construct a Weierstrass like function f for which the dimension d of the measure $\tilde{\omega}^{M_0}$ is lower than $\frac{1}{2}(1+\varepsilon)$. However, we do not know how to prove this result.

1. The boundary Harnack principle and some consequences

In this section, we introduce some notation and recall the boundary Harnack principle for the heat equation. We also state two important consequences of this principle. All of these results can be found in [H1] or in [H2] where they are proved for general parabolic operators. The domains we are working with are not necessarily of Weierstrass-type. Let $f\colon \mathbf{R} \to \mathbf{R}$ be a bounded function such that for some $K > 0$

$$|f(t) - f(t')| \le K|t-t'|^{1/2} \qquad \text{for all } (t,t') \in \mathbf{R}^2. \tag{5}$$

Put $\Omega = \{(x,t) \in \mathbf{R}^2 : x > f(t)\}$. For every $Q = (y,s) \in \partial\Omega$ and $r > 0$ define

$$\begin{cases} T(Q,r) = \{(x,t) \in \Omega : |t-s| < r,\ |x-y| < 10K\sqrt{r}\,\}, \\ \Delta(Q,r) = \{(x,t) \in \mathbf{R}^2 : |t-s| < r,\ x = f(t)\}. \end{cases} \tag{6}$$

The following comparison theorem is due to J.T. Kemper ([K]). A generalization is given for parabolic operators in [H1] and [H2].

Theorem 1.1. *There exists a strictly positive constant $C = C(K)$ such that for every non-negative caloric function u in Ω (i.e., satisfying $\mathscr{C}u = 0$) which converges to 0 on $\partial\Omega \setminus \Delta(Q, \frac{1}{2}r)$ and which is dominated by a potential in a neighbourhood of infinity, we have*

$$u(P) \le Cu(M_r) \qquad \text{for all } P \in \Omega \setminus T(Q,r), \tag{7}$$

where $M_r = Q + (10K\sqrt{r}\,, r)$.

Remarks. 1. Since the domain Ω is unbounded, we need a domination hypothesis. This hypothesis allows us to use the maximum principle. It is automatically satisfied when the function u can be written $u(P) = \int \psi(M)\, d\omega^P(M)$ with ψ a bounded function supported by $\Delta(Q, \frac{1}{2}r)$.

2. The conclusions of Theorem 1.1 remain true when u is caloric in $\Omega \setminus T(Q, \frac{1}{2}r)$.

The two following results describe comparison principles near the boundary between two non-negative solutions of the heat equation and can be found in [H1] and [H2].

Theorem 1.2 (Weak boundary Harnack principle). *There exists a strictly positive constant $C = C(K)$ such that if u and v are two non-negative caloric functions in $T(Q, 4r)$ which are equal to 0 at every point of $\Delta(Q, 2r)$, then*

$$\frac{u(P)}{u(P_r)} \le C \frac{v(P)}{v(P_r^*)} \qquad \text{for all } P \in T(Q, r), \tag{8}$$

where $P_r = Q + (10K\sqrt{r}\,, 2r)$ and $P_r^* = Q + (10K\sqrt{r}\,, -2r)$.

Theorem 1.3 (Strong boundary Harnack principle). *There exists a strictly positive constant $C = C(K)$ such that if u and v are two non-negative caloric functions in $\Omega \setminus T(Q, r)$ which are equal to 0 at every point of $\partial\Omega \setminus \Delta(Q, r)$ and which are dominated by a potential in a neighbourhood of infinity, then*

$$\frac{u(P_s)}{u(P_r)} \le C \frac{v(P_s)}{v(P_r)} \qquad \text{for all } s > r. \tag{9}$$

Remark. In [H1] and [H2], inequalities (8) and (9) are established for a large class of parabolic operators and require respectively $r \le 1$ and $s \le 1$. Because of the homogeneity of the heat operator $\mathscr{C}$, these assumptions are not needed in the present situation. In [H2], inequality (9) is established for bounded domains. The general case can be found in [H1] and does not need additional ideas. Let us also recall that an inequality similar to (9) is established in [Ny] for bounded domains and for slightly more general operators, the constant C depending on $\operatorname{diam}(\Omega)$.

An important consequence of the boundary Harnack principle is the doubling property satisfied by the caloric measure. This principle was observed by many authors (see for example [W], [FGS], [H1], [H4] or [Ny]). Let us remember what it means.

Theorem 1.4 (Doubling property for the caloric measure). *Let $Q_0 \in \partial\Omega$, $r_0 > 0$ and put $M_0 = Q_0 + (10K\sqrt{r_0}\,, 2r_0)$. There exists a strictly positive constant $C = C(K)$ such that if $\Delta(Q, 2r) \subset \Delta(Q_0, \frac{1}{2}r_0)$, then,*

$$\omega^{M_0}\big(\Delta(Q, 2r)\big) \le C\omega^{M_0}\big(\Delta(Q, r)\big). \tag{10}$$

We will end this section with another important consequence of the boundary Harnack principle which gives comparisons between the caloric measure and the Green function on Ω. These estimates were proved by Wu in [W] for the heat equation and extended to parabolic operators in [H2].

Theorem 1.5. *Let $Q_0 \in \partial\Omega$, $r_0 > 0$ and put $M_0 = Q_0 + (10K\sqrt{r_0}\,, 2r_0)$. Denote by $G(A, \cdot\,)$ the Green function of Ω related to the heat equation with singularity at A. There exists a strictly positive constant $C = C(K)$ such that if $\Delta(Q, s) \subset \Delta(Q_0, \frac{1}{2}r_0)$, then*

$$\frac{1}{C}\omega^{M_0}\big(\Delta(Q,s)\big) \le \sqrt{s}\, G(Q_s, M_0) \le C\omega^{M_0}\big(\Delta(Q,s)\big), \tag{11}$$

where $Q_s = Q + (\sqrt{s}\,, 0)$.

Notation. The important fact in (7), (8), (9), (10) and (11) is that the constants C are independent of r and Q. We write $A \lesssim B$ when there exists such a constant C for which the quantities A and B satisfy $A \le CB$. The notation $A \approx B$ means that the relations $A \lesssim B$ and $B \lesssim A$ are true.

2. Behavior of the caloric measure under perturbations of $\partial\Omega$

In this section, we establish a preliminary result which will be the key of the proof of Theorem 3.1. Let f_1 and f_2 be two bounded functions satisfying (5), a time $t_0 \in \mathbf{R}$ and a real $r > 0$. For $i = 1, 2$, denote by $Q_i(t) = \big(f_i(t), t\big)$, $M_i = Q_i(t_0) + (20K\sqrt{r}\,, 8r)$ and $\omega_i^{M_i}$ the caloric measure related to the domain $\Omega_i = \{x > f_i(t)\}$ and to the point M_i. We have the following result:

Theorem 2.1. *Let $h = f_2 - f_1$ and suppose that for some $K' > 0$*

$$|h(t) - h(t')| \le \frac{K'}{\sqrt{r}}|t - t'| \qquad \text{for all } t,\ t' \in [t_0 - r, t_0 + r]. \tag{12}$$

In other words, suppose that f_2 is a Lipschitz perturbation of f_1 in $[t_0 - r, t_0 + r]$. Let $\tilde\omega_i$ be the image of the measure $\omega_i^{M_i}$ under the projection $\big(f_i(t), t\big) \mapsto t$. There exists a strictly positive constant $C = C(K, K')$ such that for every Borel set $E \subset [t_0 - r, t_0 + r]$, we have

$$\frac{1}{C}\tilde\omega_2(E) \le \tilde\omega_1(E) \le C\tilde\omega_2(E). \tag{13}$$

Proof. Let us begin with some reductions of the problem. Because of the regularity properties of the measures $\tilde\omega_i$, we may assume that E is an open set and even an interval $E = [s_0 - s, s_0 + s)$ of small length. Using the translation invariance of the heat equation, we may also assume that $s_0 = 0$ and $f_1(s_0) = f_2(s_0) = 0$. Let

$$U_i = \Omega_i \cap \big(\{(x,t) \in \mathbf{R}^2 : |x| < 10K\sqrt{r} \text{ and } |t| < r\}\big)$$

and

$$\mathscr{C}^* = -\frac{\partial}{\partial t} - \frac{\partial^2}{2\partial x^2}$$

the adjoint heat operator. Denote by φ_i^* the $\mathscr{C}^*$-harmonic measure of $\partial U_i \setminus \partial\Omega_i$ in U_i. If Δ_i is the projection of E on $\partial\Omega_i$ and if $G_i(A,\cdot)$ is the Green function of Ω_i with singularity at A, we deduce from Theorem 1.5 and from the weak boundary Harnack principle related to the adjoint heat operator that for small values of s,

$$\begin{aligned}\omega_i^{P_r}(\Delta_i) &\approx \sqrt{s}\, G_i\big((\sqrt{s}\,,0), P_r\big)\\ &\lesssim \sqrt{s}\,\frac{\varphi_i^*\big((\sqrt{s}\,,0)\big)}{\varphi_i^*(P_{r/4})}\, G_i(P^*_{r/4}, P_r) \approx \sqrt{\frac{s}{r}}\,\varphi_i^*\big((\sqrt{s}\,,0)\big)\end{aligned}$$

where $P_r = (10K\sqrt{r}\,, 2r)$ and $P_r^* = (10K\sqrt{r}\,, -2r)$. Conversely, we have

$$\sqrt{s}\, G_i\big((\sqrt{s}\,,0), P_r\big) \gtrsim \sqrt{s}\,\frac{\varphi_i^*\big((\sqrt{s}\,,0)\big)}{\varphi_i^*(P^*_{r/4})}\, G_i(P_{r/4}, P_r) \approx \sqrt{\frac{s}{r}}\,\varphi_i^*\big((\sqrt{s}\,,0)\big)$$

and we can conclude that

$$\omega_i^{P_r}(\Delta_i) \approx \sqrt{\frac{s}{r}}\,\varphi_i^*\big((\sqrt{s},0)\big).$$

On the other hand, Harnack inequalities and Theorem 1.1 give respectively

$$\begin{cases}\omega_i^{P_r}(\Delta_i) \lesssim \omega_i^{M_i}(\Delta_i),\\ \omega_i^{M_i}(\Delta_i) \lesssim \omega_i^{P_r}(\Delta_i)\end{cases}$$

and we obtain

$$\tilde{\omega}_i(E) = \omega_i^{M_i}(\Delta_i) \approx \sqrt{\frac{s}{r}}\,\varphi_i^*\big((\sqrt{s}\,,0)\big). \tag{14}$$

Finally, the proof of Theorem 2.1 will be complete if we establish that

$$\varphi_1^*\big((\sqrt{s},0)\big) \approx \varphi_2^*\big((\sqrt{s}\,,0)\big) \tag{15}$$

for small values of s. This is an easy consequence of a result proved in [H2]. Let us introduce the map H

$$H\colon (x,t) \in \mathbf{R}^2 \longmapsto (\sqrt{r}\,x, rt) \in \mathbf{R}^2.$$

The homogeneity properties of the operator $\mathscr{C}^*$ ensure that the function $\varphi_i^* \circ H$ is the $\mathscr{C}^*$-harmonic measure of $H^{-1}(\partial U_i \setminus \partial\Omega_i)$ in $H^{-1}(U_i)$. This domain is delimited by the function $\tilde{f}_i(t) = f_i(rt)/r^{1/2}$ and we can deduce from (12) that

$$|\tilde{f}_1(t) - \tilde{f}_2(t)| \le 2K'|t|.$$

Then, we can make use of [H2, Théorème 2.2] which is also true for the operator $\mathscr{C}^*$ and obtain (15) for small values of s. This gives the conclusion of Theorem 2.1.

3. The quasi-Bernoulli property for the caloric measure

In this section, we choose a Lipschitz function g with Lipschitz constant $\widetilde{K}$, periodic with period 1 and we define f using (2). As recalled in the introduction, f is $C^{1/2}$ with Hölder constant $K = 2\widetilde{K}/(1-l^{-1/2})$. Denote by $\mathscr{M}$ the set of finite words constructed with the alphabet $\{0,\dots,l-1\}$. If $a=\varepsilon_1\cdots\varepsilon_n\in\mathscr{M}$, define

$$t_a=\sum_{i=1}^{n}\varepsilon_i l^{-i},\qquad I_a=[t_a,t_a+l^{-n}),\qquad M_a=\big(f(t_a),t_a\big)+(20Kl^{-n/2},8l^{-n}),$$

and let $|a|=n$ denote the length of a. Take the convention that if $a=\emptyset$, then $t_\emptyset=0$ and $|\emptyset|=0$. Finally, let us write ab the concatenation of the words a and b.

It is now possible to state the quasi-Bernoulli property for the caloric measure.

Theorem 3.1. *Let $\omega^{M_\emptyset}$ be the caloric measure in Ω at $M_\emptyset$ and ω be the image of $\omega^{M_\emptyset}$ under the projection $\big(f(t),t\big)\mapsto t$. The measure ω is a quasi-Bernoulli measure in $[0,1)$. In other words, there exists a strictly positive constant $C=C(\widetilde{K})$, such that*

$$\frac{1}{C}\omega(I_a)\omega(I_b)\le\omega(I_{ab})\le C\omega(I_a)\omega(I_b)\qquad\text{for all } a,b\in\mathscr{M}. \tag{16}$$

We can deduce the following corollary.

Corollary 3.2. *There are only two possible cases, mutually exclusive.*

(i) *There exists a constant $\kappa>0$ such that $\sigma/\kappa\le\omega\le\kappa\sigma$ where σ is the Lebesgue measure on $[0,1)$. In particular $\dim(\omega)=1$;*

or

(ii) *there exists a real $d<1$ such that ω is supported by a subset of $[0,1)$ with dimension d and such that every set of dimension $<d$ is negligible for ω; the measure ω is unidimensional with dimension*

$$\dim(\omega)=d<1.$$

Moreover, in this second case, ω is a multifractal measure. That is, $\dim(E_\alpha)>0$ for infinitely many $\alpha\in\mathbf{R}$, where

$$E_\alpha=\Big\{t\in[0,1):\lim_{r\to 0}\frac{\log\omega([t-r,t+r])}{\log r}=\alpha\Big\}. \tag{17}$$

Remarks. 1. In the next section, we will prove that case (ii) is satisfied if and only if the function f is not Lipschitz; this is the generic situation.

2. We observe that self-similar curves for the caloric measure play the same role as self-similar Cantor sets for the harmonic measure. Carleson proved that the harmonic measure of a self-similar Cantor set is a quasi-Bernoulli measure ([C]). In 1986, Makarov and Volberg stated that it is supported by a set of dimension strictly less than the dimension of its support ([MV]). Let us also remember that Batakis generalized Makarov–Volberg's result to a large class of Cantor sets ([Ba]).

3.1. Proof of Theorem 3.1. Using the invariance of the heat equation under translations, we can suppose that $g(0)=0$. For every $a\in\mathscr{M}$, let Δ_a be the projection of I_a on the graph $\partial\Omega$. Fix $a\in\mathscr{M}$ and let $\widetilde{M}_\emptyset=\big(f(t_a),t_a\big)+(20K,8)$. Using Harnack inequalities and Theorem 1.1, we first remark that

$$\omega^{\widetilde{M}_\emptyset}(\Delta_{ab})\approx\omega^{M_\emptyset}(\Delta_{ab})\qquad\text{for all } b\in\mathscr{M}.$$

Then, by the strong boundary Harnack principle, we obtain

$$\frac{\omega(I_{ab})}{\omega(I_a)}=\frac{\omega^{M_\emptyset}(\Delta_{ab})}{\omega^{M_\emptyset}(\Delta_a)}\approx\frac{\omega^{\widetilde{M}_\emptyset}(\Delta_{ab})}{\omega^{\widetilde{M}_\emptyset}(\Delta_a)}\approx\frac{\omega^{M_a}(\Delta_{ab})}{\omega^{M_a}(\Delta_a)}\approx\omega^{M_a}(\Delta_{ab}).$$

On the other hand, if $n=|a|$, we have

$$f(t)=\sum_{k=0}^{n-1}l^{-k/2}g(l^kt)+l^{-n/2}f(l^nt)=h(t)+l^{-n/2}f(l^nt),$$

where h is a Lipschitz function with Lipschitz constant

$$c\approx l^{n/2}\approx\frac{1}{\sqrt{\sigma(I_a)}}.$$

Let us write with a hat all quantities related to the function $\hat f(t)=l^{-n/2}f(l^nt)$ and the domain $\widehat\Omega=\{(x,t):x>\hat f(t)\}$. Theorem 2.1 ensures that

$$\omega^{M_a}(\Delta_{ab})\approx\hat\omega^{\widehat{M}_a}(\hat\Delta_{ab}).$$

If we observe that the domain Ω is the image of $\widehat\Omega$ under the application

$$H\colon (x,t)\longmapsto(l^{n/2}x,l^nt),$$

the invariance of the heat operator under such a transformation ensures that

$$\hat\omega^{\widehat{M}_a}(\hat\Delta_{ab})=\omega^{H(\widehat{M}_a)}\big(H(\hat\Delta_{ab})\big).$$

Finally, observing that $n_0=l^nt_a$ is an integer such that

$$H(\widehat{M}_a)=(0,n_0)+M_\emptyset\qquad\text{and}\qquad H(\hat\Delta_{ab})=(0,n_0)+\Delta_b,$$

and using the periodicity of the function f, we conclude that

$$\omega^{H(\widehat{M}_a)}\big(H(\hat\Delta_{ab})\big)=\omega^{M_\emptyset}(\Delta_b)=\omega(I_b).$$

This completes the proof of Theorem 3.1.

3.2. Proof of Corollary 3.2. Corollary 3.2 is always true for quasi-Bernoulli measures. Let us begin with some notation. Let $\mathscr{F}_n$ be the set of l-adic intervals of the n^{th} generation. If $I = I_a$ and $J = I_b$ are two intervals of $\bigcup_n \mathscr{F}_n$, let $IJ = I_{ab}$. Suppose first that there exists $\kappa > 0$ such that

$$\frac{1}{\kappa}\sigma(I) \le \omega(I) \le \kappa\sigma(I) \qquad \text{for all } I \in \bigcup_{n\ge 0} \mathscr{F}_n. \tag{18}$$

Since every open set of $[0,1)$ is a countable disjoint union of intervals of $\bigcup_n \mathscr{F}_n$, this inequality can be extended to any open set. By regularity of the measures σ and ω, it is also true for every Borel set. Thus, (i) is satisfied.

If (18) is not satisfied, we can for example suppose that there exists an integer $n_0 \ge 0$ and an interval $I_0 \in \mathscr{F}_{n_0}$ such that

$$\omega(I_0) < \frac{1}{lC}\sigma(I_0)$$

where C is the constant which appears in (16). It follows that for every $I \in \bigcup_n \mathscr{F}_n$,

$$\frac{\omega(II_0)}{\omega(I)} \le \frac{1}{l}\sigma(I_0) = l^{-(n_0+1)}.$$

The related mass of the subinterval II_0 is then smaller than expected and we can use the same idea as in [H3] and [H5] to prove that the dimension of the measure ω is strictly less than 1. This idea was also previously used by Bourgain in [Bg] and Batakis in [Ba]. For every $I \in \bigcup_n \mathscr{F}_n$ and for every $x \in [0,1)$, we have

$$\begin{aligned}\sum_{J\in\mathscr{F}_{n_0}} \omega(IJ)^x &\le \omega(II_0)^x + (l^{n_0}-1)\left[\frac{\omega(I)-\omega(II_0)}{l^{n_0}-1}\right]^x \\ &\le \left(l^{-(n_0+1)x} + (l^{n_0}-1)\left[\frac{1-l^{-(n_0+1)}}{l^{n_0}-1}\right]^x\right)\omega(I)^x = \beta(x)\omega(I)^x.\end{aligned}$$

If we sum over all intervals I of the same generation and iterate this inequality, we obtain

$$\sum_{I\in\mathscr{F}_{pn_0}} \omega(I)^x \le \bigl(\beta(x)\bigr)^p \omega\bigl([0,1)\bigr)^x \qquad \text{for all } p \ge 0.$$

Using definition (3), we conclude that

$$\tau_{pn_0}(x) \le \frac{1}{n_0 \log l}\log\beta(x) \qquad \text{and} \qquad \tau(x) \le \frac{1}{n_0\log l}\log\beta(x).$$

As recalled in the introduction, the function τ is differentiable and the dimension of the measure is equal to $-\tau'(1)$. Observing that $\beta(1) = 1$ and using the strict convexity of the logarithm, it is easy to establish that

$$\dim(\omega) = -\tau'(1) \le -\frac{1}{n_0\log l}\beta'(1) < 1.$$

The case where $\omega(I_0) > lC\sigma(I_0)$ can be treated in a similar way.

The end of the proof of Corollary 3.2 is easy. The function τ satisfies

$$\tau(0) = 1, \qquad \tau(1) = 0 \qquad \text{and} \qquad -\tau'(1) = d < 1.$$

Using the convexity of τ we deduce that

$$-\tau'(0) = \delta > d.$$

Let τ^* be the Legendre transform of τ defined by

$$\tau^*(\alpha) = \inf\{\alpha x + \tau(x) : x \in \mathbf{R}\}.$$

It is well known that the function τ^* is increasing in $\big(-\tau'(+\infty), -\tau'(0)\big]$ and then decreasing in $\big[-\tau'(0), -\tau'(-\infty)\big)$. On the other hand, if $I_n(t)$ is the unique interval $I \in \mathscr{F}_n$ such that $t \in I$ and if E_α is defined as in (17), the doubling property ensures that

$$E_\alpha = \left\{ t \in [0,1) : \lim_{n\to+\infty} \frac{\omega\big(I_n(t)\big)}{-n \log l} = \alpha \right\}.$$

Using [BMP] and the differentiability of the function τ, we get

$$\dim(E_\alpha) = \tau^*(\alpha) \geq \tau^*(d) = d \qquad \text{for all } \alpha \in [d, \delta].$$

The doubling property clearly implies that $d > 0$ (in fact we will see in Section 5 that $d \geq \frac{1}{2}(1+\varepsilon)$). Then, we can conclude that $\dim(E_\alpha)$ is positive for infinitely many values of the parameter α.

4. Characterization of the functions g such that $\dim(\omega) < 1$

In this section, we shall describe the set of Lipschitz functions g such that $\dim(\omega) < 1$. In particular, we prove that its complement is closed and nowhere dense in the set of Lipschitz functions. With the same notation as in Section 3, we obtain the two following results.

Theorem 4.1. *There are only two possible cases, mutually exclusive.*

(i) *f is Lipschitz with $\|f'\|_\infty \leq \|g'\|_\infty / \big(\sqrt{l} - 1\big)$. In this case, ω is strongly equivalent to the Lebesgue measure and* $\dim(\omega) = 1$;

or

(ii) *there exist two positive numbers a and b such that for every interval I of length $|I| \leq 1$,*

$$a|I|^{1/2} \leq \operatorname{osc}(f, I) \leq b|I|^{1/2}, \tag{19}$$

where $\operatorname{osc}(f, I) = \max_I f - \min_I f$. In this case, ω is singular with respect to the Lebesgue measure and $\dim(\omega) < 1$.

Theorem 4.2. *The set of Lipschitz functions g such that f is Lipschitz is a closed vector subspace of infinite codimension of the space of 1-periodic Lipschitz functions (endowed with its natural topology). Consequently, its complement is an open dense subset of the space of 1-periodic Lipschitz functions.*

Remarks. 1. Theorems 4.1 and 4.2 ensure that the case where $\dim(\omega) < 1$ is generic.

2. Conclusion (19) is often present in the literature (note that the right-hand part of (19) simply means that f is of class $C^{1/2}$). Falconer states that (19) is sufficient to prove that the box-counting dimension of the graph $\partial\Omega$ is equal to $\frac{3}{2}$ (see [F]). In [PU], the authors find sufficient conditions on g which ensure that (19) is satisfied. Finally, assuming some stronger regularity properties on g, Kaplan et al. [KMPY] prove a theorem similar to ours about the behaviour of f.

This section is organized as follows. In the first subsection, we obtain a geometric condition (on the function f) which ensures that $\dim(\omega) < 1$. In Section 4.2, we describe the behaviour of the function f and prove Theorem 4.1. In Section 4.3, we establish a necessary and sufficient condition for the functional equation $f(t) = g(t) + f(lt)/\sqrt{l}$ to have a Lipschitz solution and then prove Theorem 4.2. Finally, in the last subsection, we give an example where f is not Lipschitz and therefore $\dim(\omega) < 1$.

4.1. A geometric condition on f which implies that $\dim(\omega) < 1$

Proposition 4.3. *Suppose that f is $C^{1/2}$ (not necessarily of Weierstrass-type). If we can find $t_0 \in [0,1)$ and $r > 0$ such that*

$$(20) \quad f(t_0+h) \geq f(t_0) \quad \text{for all } h \in [0,r) \quad \text{and} \quad \limsup_{h\to 0^+} \frac{f(t_0+h)-f(t_0)}{h^{1/2}} > 0,$$

then

$$\lim_{s\to 0} \frac{\omega^{M_\emptyset}(\Delta_s)}{s} = 0,$$

where $Q_0 = \big(f(t_0), t_0\big)$ and $\Delta_s = \Delta(Q_0, s)$.

Proposition 4.3 is not specific to Weierstrass-type curves; it is true for any function of class $C^{1/2}$, but does not imply $\dim(\omega) < 1$ in general. However, in the case where f is a Weierstrass-type function, the existence of one point satisfying (20) implies the existence of many other such points where the behaviour of the measure is similar. The rigorous property behind this heuristic argument is the quasi-Bernoulli property for the measure ω and, as a consequence of Corollary 3.2, we have

Corollary 4.4. *Suppose that f is a Weierstrass-type function. Under condition* (20), *we have*

$$\dim(\omega) < 1.$$

Remark. The conclusion of Corollary 4.4 would also hold if we could find $t_0 \in [0,1)$ and $r > 0$ such that

$$f(t_0+h) \le f(t_0) \quad \text{for all } h \in [0,r) \quad \text{and} \quad \liminf_{h \to 0^+} \frac{f(t_0+h) - f(t_0)}{h^{1/2}} < 0. \tag{21}$$

In that case, we could prove that

$$\lim_{s \to 0} \frac{\omega^{M_\emptyset}(\Delta_s)}{s} = +\infty.$$

Proof of Proposition 4.3. Let δ_s be the projection on the graph of $[t_0, t_0+s)$. We know that $\omega^{M_\emptyset}(\Delta_s) \approx \omega^{M_\emptyset}(\delta_s)$. The maximum principle ensures that $\omega^{M_\emptyset}(\delta_s)$ does not depend on the values of $f(t)$ for $t < t_0$. So, we can suppose that $f(t) = f(t_0)$ for every $t < t_0$. On the other hand, if $\widetilde{M}_\emptyset = Q_0 + (10Kr^{1/2}, r)$, we have previously remarked that $\omega^{M_\emptyset}(\Delta_s) \approx \omega^{\widetilde{M}_\emptyset}(\Delta_s)$ when $s \to 0$ (here, the constants may depend on r). Since $\omega^{\widetilde{M}_\emptyset}(\Delta_s)$ does not depend on the values of $f(t)$ for $t \ge t_0 + r$, we can also suppose that $f(t) = f(t_0 + r)$ for every $t \ge t_0 + r$. In other words, these two remarks say that we can suppose that t_0 is a global minimum for the function f.

Let $\widehat{\Omega} = \{(x,t) : x > f(t_0)\}$. We have $\Omega \subset \widehat{\Omega}$. Therefore, if G and $\widehat{G}$ denote the Green functions on Ω and $\widehat{\Omega}$, we know that

$$G(M,P) = \widehat{G}(M,P) - \widehat{R}^{\widehat{\Omega}\setminus\Omega}_{\widehat{G}(M,\cdot)}(P), \tag{22}$$

where $\widehat{R}^{\widehat{\Omega}\setminus\Omega}_{\widehat{G}(M,\cdot)}$ is the smooth reduction of $\widehat{G}(M,\cdot)$ on $\widehat{\Omega} \setminus \Omega$ (for more details about reduction, we can refer to [Do]). Let $\hat{\omega}^{M_\emptyset}$ be the caloric measure at $M_\emptyset$ related to $\widehat{\Omega}$ and $\hat{\Delta}_s = \{(f(t_0), t) : t \in (t_0 - s, t_0 + s)\}$. Using Theorem 1.5 we get

$$\frac{\omega^{M_\emptyset}(\Delta_s)}{s} \approx \frac{\omega^{M_\emptyset}(\Delta_s)}{\hat{\omega}^{M_\emptyset}(\hat{\Delta}_s)} \approx \frac{G(Q_s, M_\emptyset)}{\widehat{G}(Q_s, M_\emptyset)}$$

where $Q_s = Q_0 + (\sqrt{s}, 0)$.

To complete the proof of the proposition, we adapt to our situation an old idea due to Naïm ([Na]). We must be careful, because the heat operator is not self-adjoint. According to [K] and [H2], let χ be the unique caloric minimal function in $\widehat{\Omega}$ which tends to 0 at every point of $\partial\widehat{\Omega} \setminus \{Q_0\}$ and satisfies $\chi(M_\emptyset) = 1$. Following [H2, Part 4], we know that

$$\chi(P) = \lim_{s\to 0} \frac{\widehat{G}(Q_s, P)}{\widehat{G}(Q_s, M_\emptyset)}.$$

On the other hand, the additivity and positivity of the map $s \mapsto \widehat{R}_s^{\widehat{\Omega}\setminus\Omega}(M_\emptyset)$ ensure the existence of a non-negative Radon measure μ such that for every non-negative super-caloric function,

$$\widehat{R}_s^{\widehat{\Omega}\setminus\Omega}(M_\emptyset) = \int s(\xi)\, d\mu(\xi).$$

Using (22) and Fatou's lemma, we get

$$\begin{aligned}\limsup_{s\to 0} \frac{G(Q_s, M_\emptyset)}{\widehat{G}(Q_s, M_\emptyset)} &= 1 - \liminf_{s\to 0} \int \frac{\widehat{G}(Q_s, \xi)}{\widehat{G}(Q_s, M_\emptyset)}\, d\mu(\xi) \\ &\le 1 - \int \chi(\xi)\, d\mu(\xi) = 1 - \widehat{R}_\chi^{\widehat{\Omega}\setminus\Omega}(M_\emptyset).\end{aligned}$$

According to [H2, Proposition 4.2], hypothesis (20) implies that the set $\widehat{\Omega} \setminus \Omega$ is not thin in the minimal function χ. So,

$$\widehat{R}_\chi^{\widehat{\Omega}\setminus\Omega}(M_\emptyset) = \chi(M_\emptyset) = 1.$$

This completes the proof.

4.2. Proof of Theorem 4.1. In this section, we identify 1-periodic functions with functions defined on the torus $\mathbf{R}/\mathbf{Z}$. The usual distance on $\mathbf{R}/\mathbf{Z}$ is noted d. We begin with the following elementary lemma.

Lemma 4.5. *Suppose that* $\|g'\|_\infty \le 1$. *Let* $s, t \in \mathbf{R}/\mathbf{Z}$ *such that* $d(t,s) \le 1/2l$ (*this is equivalent to* $d(lt, ls) = l\, d(t,s)$) *and suppose*

$$\frac{|f(lt) - f(ls)|}{d(lt, ls)} \ge \frac{1}{\sqrt{l} - 1}(1 + u) \qquad \text{for some } u \ge 0.$$

Then

$$\frac{|f(t) - f(s)|}{d(t,s)} \ge \frac{1}{\sqrt{l} - 1}(1 + u\sqrt{l}\,).$$

The above lemma is an easy consequence of the functional equation between f and g. We have

$$f(t) - f(s) = g(t) - g(s) + \frac{1}{\sqrt{l}}\big(f(lt) - f(ls)\big).$$

Thus,

$$\frac{|f(t)-f(s)|}{d(t,s)} \geq \sqrt{l}\,\frac{|f(lt)-f(ls)|}{d(lt,ls)} - \frac{|g(t)-g(s)|}{d(t,s)} \geq \sqrt{l}\,\frac{|f(lt)-f(ls)|}{d(lt,ls)} - 1$$

which can be rewritten as

$$\frac{|f(t)-f(s)|}{d(t,s)} - \frac{1}{\sqrt{l}-1} \geq \sqrt{l}\left[\frac{|f(lt)-f(ls)|}{d(lt,ls)} - \frac{1}{\sqrt{l}-1}\right]$$

and the lemma is proved.

Iterating Lemma 4.5, we obtain the following corollary.

Corollary 4.6. *Suppose once again that* $\|g'\|_\infty \leq 1$. *Let* $s,t \in \mathbf{R}/\mathbf{Z}$ *such that* $d(t,s) \leq 1/2l^n$ *(this is equivalent to* $d(l^n t, l^n s) = l^n\, d(t,s)$*) and suppose*

$$\frac{|f(l^n t)-f(l^n s)|}{d(l^n t, l^n s)} \geq \frac{1}{\sqrt{l}-1}(1+u) \qquad \text{for some } u \geq 0.$$

Then

$$\frac{|f(t)-f(s)|}{d(t,s)} \geq \frac{1}{\sqrt{l}-1}(1+u l^{n/2}).$$

Now we can prove Theorem 4.1. Let us assume, without loss of generality, that the Lipschitz constant of g is equal to 1. If f is Lipschitz, it is known that the measure ω is strongly equivalent to the Lebesgue measure (see [H2, pp. 642–643]).

On the other hand, suppose that f is not $(\sqrt{l}-1)^{-1}$-Lipschitz. This means that we can find two points $s_0, t_0 \in \mathbf{R}/\mathbf{Z}$ such that

$$\begin{cases} \dfrac{|f(t_0)-f(s_0)|}{d(t_0,s_0)} = \dfrac{1}{\sqrt{l}-1}(1+v), & v > 0, \\ t_0 = s_0 + \alpha, & 0 < \alpha \leq \frac{1}{2}. \end{cases}$$

Let I be a closed interval on the circle $\mathbf{R}/\mathbf{Z}$ and let n be the unique integer ≥ 1 such that

$$l^{-n} \leq \tfrac{1}{2}|I| < l^{-n+1}.$$

We can choose s_n such that $s_n \in I$, $s_n + l^{-n} \in I$ and $l^n s_n = s_0$. Define $t_n = s_n + l^{-n}\alpha$; we have $l^n t_n = t_0$ and $t_n \in I$. Using Corollary 4.6 we obtain

$$\frac{|f(t_n)-f(s_n)|}{d(t_n,s_n)} \geq \frac{1}{\sqrt{l}-1}(1+v l^{n/2}) \geq \frac{v l^{n/2}}{\sqrt{l}-1}.$$

We conclude that

$$\operatorname{osc}(f,I) \geq \frac{v\, d(t_0,s_0)}{\sqrt{l}-1} l^{-n/2} \geq a|I|^{1/2}$$

with $a = v\, d(t_0,s_0)/(\sqrt{l}-1)(2l)^{1/2}$.

This inequality ensures that (20) is satisfied if t_0 is the minimum of f. Then, Corollary 4.4 allows us to conclude that $\dim(\omega) < 1$.

4.3. Proof of Theorem 4.2. We know that, when g is Lipschitz, the following functional equation

$$f(t) = g(t) + \frac{1}{\sqrt{l}} f(lt) \qquad \text{for all } t \in \mathbf{R}/\mathbf{Z}, \tag{23}$$

(qua equation in f) has a unique solution in $C^0(\mathbf{R}/\mathbf{Z}, \mathbf{R})$, defined by

$$f(t) = \sum_{n=0}^{\infty} l^{-n/2} g(l^n t).$$

We would like to know if f is Lipschitz or, equivalently, if (23) has a solution in $\mathrm{Lip}\,(\mathbf{R}/\mathbf{Z}, \mathbf{R})$.

Proposition 4.7. *Let $g \in \mathrm{Lip}\,(\mathbf{R}/\mathbf{Z}, \mathbf{R})$ and $f \in C^0(\mathbf{R}/\mathbf{Z}, \mathbf{R})$ satisfying (23). Define the linear operator A_l from $\mathrm{Lip}\,(\mathbf{R}/\mathbf{Z}, \mathbf{R})$ into itself by*

$$A_l(\varphi) = \psi \qquad \text{with } \psi(t) = \frac{1}{l} \sum_{ls=t} \varphi(s)$$

and put $U_l = -(\mathrm{Id} - \sqrt{l}\, A_l)^{-1}(\sqrt{l}\, A_l)$ (the operator $\mathrm{Id} - \sqrt{l}\, A_l$ is invertible). Then, f is Lipschitz if and only if $f = U_l(g)$. Equivalently, f is Lipschitz if and only if

$$U_l(g)(t) = g(t) + \frac{1}{\sqrt{l}} U_l(g)(lt) \qquad \text{for all } t \in \mathbf{R}/\mathbf{Z}. \tag{24}$$

This is a closed, linear condition, of infinite codimension in $\mathrm{Lip}\,(\mathbf{R}/\mathbf{Z}, \mathbf{R})$.

Proof. Note that equation (23) implies the following (a priori weaker) condition

$$A_l(f - g) = \frac{1}{\sqrt{l}}\, f. \tag{25}$$

We claim that (25), qua equation in f, has a unique solution in $\mathrm{Lip}\,(\mathbf{R}/\mathbf{Z}, \mathbf{R})$. Therefore, (23) will have a solution in $\mathrm{Lip}\,(\mathbf{R}/\mathbf{Z}, \mathbf{R})$ if and only if the Lipschitz solution of (25) also verifies (23).

Equation (25) can be rewritten as

$$(\mathrm{Id} - \sqrt{l}\, A_l) f = -\sqrt{l}\, A_l g.$$

To prove our claim, we have to establish that the operator $\mathrm{Id} - \sqrt{l}\, A_l$ is invertible on $\mathrm{Lip}\,(\mathbf{R}/\mathbf{Z}, \mathbf{R})$. This will imply that (25) has a unique solution, namely

$$U_l(g) = -(\mathrm{Id} - \sqrt{l}\, A_l)^{-1}(\sqrt{l}\, A_l)(g)$$

and will give the first part of Proposition 4.7.

To prove that $\mathrm{Id} - \sqrt{l}\, A_l$ is invertible, let us denote by $\mathbf{1}$ the constant function equal to 1 and let

$$\mathrm{Lip}^0(\mathbf{R}/\mathbf{Z}, \mathbf{R}) = \Big\{ g \in \mathrm{Lip}\,(\mathbf{R}/\mathbf{Z}, \mathbf{R}) : \int_{\mathbf{R}/\mathbf{Z}} g(t)\, dt = 0 \Big\}.$$

We have the decomposition

$$\mathrm{Lip}\,(\mathbf{R}/\mathbf{Z}, \mathbf{R}) = \mathrm{Lip}^0(\mathbf{R}/\mathbf{Z}, \mathbf{R}) \oplus \mathbf{R}\mathbf{1}$$

which is preserved by $\sqrt{l}\, A_l$. The operator $\mathrm{Id} - \sqrt{l}\, A_l$ is obviously invertible on $\mathbf{R}\mathbf{1}$, so we simply have to prove invertibility on $\mathrm{Lip}^0(\mathbf{R}/\mathbf{Z}, \mathbf{R})$. But, $\sqrt{l}\, A_l$ is a contraction on $\mathrm{Lip}^0(\mathbf{R}/\mathbf{Z}, \mathbf{R})$ (endowed with norm $\|g'\|_\infty$). So, $\mathrm{Id} - \sqrt{l}\, A_l$ is invertible in this space and its inverse is given by $\sum_{k \geq 0} (\sqrt{l}\, A_l)^k$. Moreover, we have $\|(\mathrm{Id} - \sqrt{l}\, A_l)^{-1}\| \leq 1/(1 - l^{-1/2})$ on $\mathrm{Lip}^0(\mathbf{R}/\mathbf{Z}, \mathbf{R})$. Therefore, $\mathrm{Id} - \sqrt{l}\, A_l$ is indeed invertible on the whole space $\mathrm{Lip}\,(\mathbf{R}/\mathbf{Z}, \mathbf{R})$. This proves the claim.

To finish the proof of Proposition 4.7, we observe that condition (24), which states that $U_l(g)$ verifies (23), is closed and linear. Moreover, it is easy to construct a lot of functions g which do not verify (24). An elementary calculation gives

$$U_l(\mathbf{1}) = \frac{1}{1 - l^{-1/2}} \quad \text{and} \quad U_l(e^{2i\pi n t}) = - \sum_{k \geq 1,\, l^k | n} l^{k/2} e^{2i\pi (n/l^k) t} \quad \text{if } n \neq 0.$$

In particular, when g is a nonzero trigonometric polynomial whose frequencies are in $\mathbf{Z} \setminus l\mathbf{Z}$, condition (24) is not verified and the Weierstrass-like function f is not Lipschitz.

More generally, when g is Lipschitz, we can compute the Fourier coefficients of $U_l(g)$ and we obtain that g satisfies (24) if and only if, for every $n \in \mathbf{Z}$,

$$l \not| n \implies \sum_{k=0}^{+\infty} l^{k/2} \hat{g}(l^k n) = 0. \tag{26}$$

These formulas have been already established by Kaplan et al. ([KMPY]) under more restrictive hypotheses on g.

All these remarks allow us to construct an infinite-dimensional space of functions g which do not verify condition (24), except the null function. In other words, the space of functions g such that f is Lipschitz has infinite codimension.

To conclude, note that the estimate

$$\|(\mathrm{Id} - \sqrt{l}\, A_l)^{-1}\| \leq \frac{1}{1 - l^{-1/2}} \qquad \text{on } \mathrm{Lip}^0(\mathbf{R}/\mathbf{Z}, \mathbf{R})$$

implies that for every $g \in \mathrm{Lip}\,(\mathbf{R}/\mathbf{Z}, \mathbf{R})$,

$$\|U_l(g)'\|_\infty \le \frac{1}{\sqrt{l}-1}\|g'\|_\infty.$$

If f is Lipschitz (that is if $f = U_l(g)$), this gives another proof of the inequality

$$\|f'\|_\infty \le \frac{1}{\sqrt{l}-1}\|g'\|_\infty.$$

4.4. An example. To finish this section, we give a simple sufficient condition on g and l which guarantees $\dim(\omega) < 1$.

Proposition 4.8. *Suppose that g is not constant. If l is sufficiently large, then*

$$\dim(\omega) < 1.$$

In fact, it happens as soon as

$$\sqrt{l}\,\frac{\sqrt{l}-1}{\sqrt{l}+1} > \frac{\|g'\|_\infty}{2\mathrm{osc}\,(g, \mathbf{R})}.$$

Proof. Assume f and g are both Lipschitz. Then,

$$\begin{cases} f(t) = g(t) + f(lt)/\sqrt{l}, \\ f'(t) = g'(t) + \sqrt{l} f'(lt) \quad \text{almost everywhere.} \end{cases} \tag{27}$$

We can easily deduce that

$$\begin{cases} \mathrm{osc}\,(g, \mathbf{R}) \le \left[1 + \dfrac{1}{\sqrt{l}}\right] \mathrm{osc}\,(f, \mathbf{R}), \\ \|f'\|_\infty \le \dfrac{\|g'\|_\infty}{\sqrt{l}-1} \end{cases} \tag{28}$$

(the second inequality is not new; we have already proved it, by two different methods in the previous sections).

Observing that $\mathrm{osc}\,(f, \mathbf{R}) \le \frac{1}{2}\|f'\|_\infty$ (since f is 1-periodic), we get

$$\mathrm{osc}\,(g, \mathbf{R}) \le \frac{\sqrt{l}+1}{2\sqrt{l}\,(\sqrt{l}-1)}\|g'\|_\infty$$

and the proof of Proposition 4.8 is complete.

5. A minoration for the lower dimension of caloric measure

In this section, f is a bounded function of class $C^{1/2}$ (not necessarily of Weierstrass-type) with Hölder constant K. Denote by Ω the corresponding domain and choose an arbitrary point M_0 in Ω. We want to give estimates for the lower dimension of $\tilde{\omega}^{M_0}$, which is defined by

$$\dim_*(\tilde{\omega}^{M_0}) = \inf\bigl(\dim(E) : \tilde{\omega}^{M_0}(E) > 0\bigr).$$

Using Taylor–Watson results about the description of polar sets related to the heat equation ([TW]), we know that $\dim_*(\tilde{\omega}^{M_0})$ is not less than $\frac{1}{2}$. In the next theorem, we improve this result.

Theorem 5.1. *There exists $\varepsilon = \varepsilon(K) > 0$ such that for every $M_0 \in \Omega$,*

$$\dim_*(\tilde{\omega}^{M_0}) \geq \tfrac{1}{2}(1+\varepsilon).$$

Remark. If $\dim^*(\tilde{\omega}^{M_0}) = \inf\bigl(\dim(E) : \tilde{\omega}^{M_0}(\mathbf{R} \setminus E) = 0\bigr)$, we obviously conclude that

$$\dim^*(\tilde{\omega}^{M_0}) \geq \dim_*(\tilde{\omega}^{M_0}) \geq \tfrac{1}{2}(1+\varepsilon).$$

In fact, when f is a Weierstrass-type function, the quasi-Bernoulli property, proved in Section 3, ensures that

$$\dim^*(\tilde{\omega}^{M_0}) = \dim_*(\tilde{\omega}^{M_0}) = \dim(\tilde{\omega}^{M_0}).$$

However, for a general function f, we can have

$$\dim^*(\tilde{\omega}^{M_0}) > \dim_*(\tilde{\omega}^{M_0}).$$

Proof of Theorem 5.1. We first remark that if $M_0 = (x_0, t_0)$ and $M_1 = (x_1, t_1)$ are two points in Ω with $t_0 \leq t_1$, then, the measures ω^{M_0} and ω^{M_1} have the same null sets in $\partial\Omega \cap \{t \leq t_0\}$. In fact, Harnack inequalities ensure that ω^{M_0} is absolutely continous with respect to ω^{M_1} and Theorem 1.1 ensures that the measure ω^{M_1} is absolutely continous with respect to ω^{M_0} in $\partial\Omega \cap \{t \leq t_0\}$. To prove Theorem 5.1, it is then sufficient to estimate the Hausdorff dimension of non negligible sets E related to $\tilde{\omega}^{M_0}$ when $E \subset \Pi\bigl(\Delta(Q_0, \frac{1}{2}r)\bigr)$, $M_0 = Q_0 + (20K\sqrt{r}, 8r)$ and r arbitrarily small (see (1) and (6) for the notation). We will use the following lemma.

Lemma 5.2. *There are two positive numbers $C = C(K)$ and $\varepsilon = \varepsilon(K)$ such that for every $Q_0 = \bigl(f(t_0), t_0\bigr) \in \partial\Omega$ and $r > 0$,*

$$\tilde{\omega}^{M_0}([s_0 - s, s_0 + s]) \leq C\left(\frac{s}{r}\right)^{(1+\varepsilon)/2},$$

where $M_0 = Q_0 + (20K\sqrt{r}, 8r)$ and $[s_0 - s, s_0 + s] \subset [t_0 - r, t_0 + r]$.

It is elementary to deduce from Lemma 5.2 that there exists $C = C(r,K) > 0$ such that for every Borel set $E \subset [t_0 - r, t_0 + r]$,

$$\tilde{\omega}^{M_0}(E) \le C\mathscr{H}^{(1+\varepsilon)/2}(E)$$

where $\mathscr{H}^{(1+\varepsilon)/2}$ is the Hausdorff measure in dimension $\frac{1}{2}(1+\varepsilon)$. Then, the lower dimension of the measure $\tilde{\omega}^{M_0}$ restricted to $[t_0 - r, t_0 + r]$ is not less than $\frac{1}{2}(1+\varepsilon)$ and the theorem follows.

Proof of Lemma 5.2. Using (14) in the present situation, we have

$$\tilde{\omega}^{M_0}([t-s,t+s]) \lesssim \sqrt{\frac{s}{r}}\,\varphi^*\big(Q + (\sqrt{s},0)\big)$$

where $Q = \big(f(s_0), s_0\big)$ and φ^* is the $\mathscr{C}^*$-harmonic measure of $\partial T(Q,r) \setminus \Delta(Q,r)$ in $T(Q,r)$. Let us introduce

$$U = \big\{(x,t) \in \mathbf{R}^2 : |t| < 1,\ |x| < 10K \text{ and } x > -10K\sqrt{|t|}\big\}$$

and denote by ψ^* the $\mathscr{C}^*$ harmonic measure of $\partial U \setminus \{(x,t) : x = -10K\sqrt{|t|}\,\}$ in U. Using the maximum principle and the homogeneity of the operator $\mathscr{C}^*$, we have

$$\varphi^*\big(Q + (\sqrt{s},0)\big) \le \psi^*\bigg(\sqrt{\frac{s}{r}},0\bigg).$$

Therefore, Lemma 5.2 is a consequence of the following

Lemma 5.3. *There are two positive numbers $C = C(K)$ and $\varepsilon = \varepsilon(K)$ such that*

$$\psi^*\big((x,0)\big) \le Cx^{\varepsilon} \qquad \textit{for all } x \in (0,5K].$$

Proof. Similar ideas can be found in [H2]. Let us sketch the proof to be self-contained. Put

$$V = \{(x,t) \in \mathbf{R}^2 : |t| < 1 \text{ and } |x| < 10K\}$$

and denote by $G(A,\cdot)$ the Green function of V with singularity at A. Then, $G(\,\cdot\,,B)$ is the Green function of V related to the adjoint operator with singularity at B. Let $B = (-8K, \frac{1}{2})$ and observe that $B \in V \setminus U$. If

$$\widetilde{V} = \{(x,t) \in \mathbf{R}^2 : |t| < \tfrac{1}{4} \text{ and } |x| < 5K\},$$

there exists a constant $C_1 = C_1(K) > 0$ such that

$$G(M,B) \ge C_1 \qquad \text{for all } M \in \widetilde{V}.$$

On the other hand, if $\Gamma(\,\cdot\,,\cdot\,)$ is the heat kernel, and if

$$W = B + \{(x,t) \in \mathbf{R}^2 : |t| < 1/100 \text{ and } |x| < K/10\},$$

there exists a constant $C_2 = C_2(K) > 0$ such that

$$G(M,B) \le \Gamma(M,B) \le C_2 \qquad \text{for all } M \in \partial W. \tag{29}$$

Using the maximum principle, we extend (29) to $V \setminus W$. In particular, it is true in U. Then, by the maximum principle,

$$1 - \psi^*(M) \ge \frac{1}{C_2} G(M,B) \qquad \text{for all } M \in U.$$

We can conclude that

$$\psi^*(M) \le 1 - \frac{C_1}{C_2} \qquad \text{for all } M \in \widetilde{V} \cap U.$$

Iterating this inequality, we have

$$\psi^*(M) \le \left(1 - \frac{C_1}{C_2}\right)^k$$

where $k \ge 1$ and $M \in U \cap \{(x,t) : |t| \le 4^{-k},\ |x| \le 10K2^{-k}\}$. We can easily deduce that Lemma 5.3 is true with

$$\varepsilon = \frac{-\log(1 - C_1/C_2)}{\log 2}.$$

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