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POINCARÉ INEQUALITIES FOR POWERS AND PRODUCTS OF ADMISSIBLE WEIGHTS

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Abstract. Our main result is that if μ is an s-admissible measure in $\mathbf{R}^n$ and $v \in A_p(d\mu)$, then the measure $d\nu = v\,d\mu$ is ps-admissible. A two-weighted version of this result is also proved. It is further shown that every strong A_∞-weight w in $\mathbf{R}^n$, $n \geq 2$, is $n/(n-1)$-admissible, that its power $w^{1-1/n}$ is 1-admissible and that the weights $w^{1-p/n}$ with $1 < p < n$ are q-admissible for some $q < p$. A counterexample showing that we cannot take $q = 1$ in general is also given. Finally, a new class of p-admissible weights is described.

1. Introduction

In Fabes–Kenig–Serapioni [4] four conditions sufficient for extending Moser's iteration technique to weighted degenerate equations were singled out. Later, in Heinonen–Kilpeläinen–Martio [12] such weights were called "p-admissible" and a rich potential theory was developed for them. Recently, Hajłasz, Heinonen, Koskela and Semmes showed that the conditions described in Fabes–Kenig–Serapioni [4] can be reduced to only two, see Theorem 2 in Hajłasz–Koskela [10] and Theorem 5.2 in Heinonen–Koskela [13]. Thus, a non-negative locally integrable function w in $\mathbf{R}^n$ is a *p-admissible weight* with $1 \leq p < \infty$ if and only if the measure μ associated with w through $d\mu = w\,dx$, where dx denotes integration with respect to the Lebesgue measure, satisfies the following two conditions:

Doubling condition: $0 < w < \infty$ a.e. in $\mathbf{R}^n$ and there is a constant $C > 0$ such that

$$\mu(2B) < C\mu(B)$$

for all balls $B \subset \mathbf{R}^n$, where $2B$ denotes the ball concentric with B and with twice the radius.

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Weak $(1,p)$*-Poincaré inequality*: There exist constants $C>0$ and $\lambda\ge 1$ such that

$$\fint_B |u-u_{B,\mu}|\,d\mu \le Cr\biggl(\fint_{\lambda B} |\nabla u|^p\,d\mu\biggr)^{1/p}$$

holds whenever B is a ball with radius r and u is, say, a locally Lipschitz function on λB. Here and in what follows, $u_{B,\mu}=\fint_B u\,d\mu$ and the symbol $\fint$ stands for the mean-value integral

$$\fint_B f\,d\mu = \frac{1}{\mu(B)}\int_B f\,d\mu.$$

A measure μ satisfying the above conditions will also be called p-admissible. The Hölder inequality implies that every p-admissible measure is also p'-admissible for all $p'>p$. Note also that by Theorem 1 in Hajłasz–Koskela [10], a weak $(1,p)$-Poincaré inequality for a doubling measure in $\mathbf{R}^n$ implies a strong $(1,p)$-Poincaré inequality with $\lambda=1$.

Many interesting examples of p-admissible weights are provided by weights from the Muckenhoupt class A_p, see e.g. Chapter 15 in Heinonen–Kilpeläinen–Martio [12]. Non-A_p examples of p-admissible weights have been given in e.g. Chanillo–Wheeden [2] and Franchi–Gutiérrez–Wheeden [5]. In Chapter 15 in Heinonen–Kilpeläinen–Martio [12] it is shown for $1<p<n$ that $(1-p/n)$-powers of Jacobians of quasiconformal mappings in $\mathbf{R}^n$, $n\ge 2$, are p-admissible. This result was extended to strong A_∞-weights by Heinonen and Koskela [13], viz. they prove the following theorem in the case $1<p<n$. We will provide a new proof of this theorem which covers also the case $p=1$.

Theorem 1. *Let w be a strong A_∞-weight in $\mathbf{R}^n$, $n\ge 2$, and $1\le p<n$. Then the weight $w^{1-p/n}$ is p-admissible.*

Strong A_∞-weights were introduced in David–Semmes [3] and further studied in e.g. Semmes [17] and [18]. For a doubling measure μ consider the function $\delta(x,y)=\mu(B_{xy})^{1/n}$, where B_{xy} denotes the smallest closed ball containing both x and y. The doubling condition of μ implies that δ is a quasi-metric, i.e. it is symmetric, vanishes only if $x=y$ and satisfies the weak triangle inequality $\delta(x,y)\le C\bigl(\delta(x,z)+\delta(z,y)\bigr)$. As mentioned in David–Semmes [3], an argument by Gehring [9] shows that if there exists a metric d in $\mathbf{R}^n$ such that

$$C^{-1}d(x,y)\le \delta(x,y)\le C\,d(x,y) \tag{1}$$

for some $C>0$ and all $x,y\in\mathbf{R}^n$, then the measure μ is absolutely continuous with respect to the Lebesgue measure and its Radon–Nikodym derivative w satisfies the reverse Hölder inequality

$$\biggl(\fint_B w(x)^r\,dx\biggr)^{1/r} \le C\fint_B w(x)\,dx \tag{2}$$

for some constants $C>0$, $r>1$ and every ball $B\subset\mathbf{R}^n$.

Definition 2. Weights satisfying the reverse Hölder inequality (2) are called A_∞*-weights* and those satisfying (1) for some metric d in $\mathbf{R}^n$ are *strong* A_∞*-weights.*

The above mentioned argument shows that every strong A_∞-weight is an A_∞-weight. Note also that every A_1-weight is a strong A_∞-weight and that Jacobians of quasiconformal mappings in $\mathbf{R}^n$, $n \geq 2$, are strong A_∞-weights, see e.g. David–Semmes [3].

We shall use the notation $v \in A_p(d\mu)$, $1 \leq p < \infty$, if for some $C > 0$ and all balls $B \subset \mathbf{R}^n$,

$$\fint_B v\, d\mu < \begin{cases} C\biggl(\displaystyle\fint_B v^{1/(1-p)}\, d\mu\biggr)^{1-p} & \text{for } p > 1, \\ C \operatorname*{ess\,inf}_B v & \text{for } p = 1. \end{cases}$$

Equivalently, we can consider all cubes in $\mathbf{R}^n$ with sides parallel to the coordinate axes. If μ is the Lebesgue measure, then we write A_p rather than $A_p(dx)$. For various properties of A_p-weights see e.g. García-Cuerva–Rubio de Francia [8] and Torchinsky [21]. We shall need the fact that w is an A_∞-weight if and only if $w \in A_p$ for some $p < \infty$.

Heinonen–Koskela's proof of Theorem 1 is based on the following result due to Franchi–Gutiérrez–Wheeden [5] (here stated in terms of the usual gradient rather than the λ-gradient considered in [5]). For a similar result on spaces of homogeneous type see Corollary 3.2 in Franchi–Pérez–Wheeden [7]. From now on, the open ball in $\mathbf{R}^n$ with centre x and radius r will be denoted $B(x,r)$.

Theorem 3. *Let w be a strong A_∞-weight in $\mathbf{R}^n$, $n \geq 2$, $1 \leq p < q < \infty$ and $v \in A_p(w^{1-1/n}\, dx)$. Put $d\mu = vw^{1-1/n}\, dx$ and let ν be a doubling measure absolutely continuous with respect to the Lebesgue measure, satisfying the condition*

$$\frac{r'}{r}\biggl(\frac{\nu(B')}{\nu(B)}\biggr)^{1/q} \leq C\biggl(\frac{\mu(B')}{\mu(B)}\biggr)^{1/p} \tag{3}$$

for all balls $B = B(x,r)$ and $B' = B(x',r')$ in $\mathbf{R}^n$ such that $B' \subset cB$ with some fixed $c > 1$. Then the pair (ν,μ) admits the two-weighted (q,p)-Poincaré inequality

$$\biggl(\fint_B |u - u_{B,\nu}|^q\, d\nu\biggr)^{1/q} \leq Cr\biggl(\fint_B |\nabla u|^p\, d\mu\biggr)^{1/p}. \tag{4}$$

Remark. Note that by Chanillo–Wheeden [1] the condition (3) is essentially necessary for the Poincaré inequality (4) to be valid.

In this paper we give a short proof of the following analogue of Theorem 3 in the setting of admissible measures. Theorem 3 itself follows from Proposition 5 and Theorem 7 below.

Theorem 4. *Let μ be an s-admissible measure, $1 \le s < \infty$, and $v \in A_p(d\mu)$, $1 \le p < \infty$. Then the measure ν given by $d\nu = v\,d\mu$ is ps-admissible.*

As $(1-1/n)$-powers of strong A_∞-weights are 1-admissible by e.g. Theorem 1, Theorem 4 provides a simple proof of the following one-weighted ($\nu = \mu$) special case of Theorem 3.

Proposition 5. *Let w be a strong A_∞-weight in $\mathbf{R}^n$, $n \ge 2$, and $v \in A_p(w^{1-1/n}\,dx)$, $1 \le p < \infty$. Then the weight $vw^{1-1/n}$ is p-admissible.*

Contrary to Theorem 3, Theorem 4 allows us to consider weights which are not strong A_∞-weights. For example, we can prove the following generalization of a result due to Chanillo–Wheeden [2].

Proposition 6. *Let $2 \le k \le n$ and write the points in $\mathbf{R}^n$ as $x = (x', x'')$, where $x' \in \mathbf{R}^k$ and $x'' \in \mathbf{R}^{n-k}$. Let $1 \le p < \infty$, $v \in A_p$, $a_1, a_2, \ldots, a_m \in \mathbf{R}^k$ and $\gamma_j \ge 0$, $j = 0, 1, \ldots, m$. Then the weight*

$$v(x)(1+|x'|)^{\gamma_0} \prod_{j=1}^{m} \left(\frac{|x'-a_j|}{1+|x'-a_j|} \right)^{\gamma_j}$$

is p-admissible.

Two-weighted versions of Theorem 4 and Propositions 5 and 6 follow immediately from the following theorem.

Theorem 7. *Let μ be a p-admissible measure. Let $1 \le p < q < \infty$ and let ν be a doubling measure satisfying the condition (3) for all balls $B = B(x, r)$ and $B' = B(x', r')$ such that $B' \subset B$. Then the pair (ν, μ) admits the two-weighted (q, p)-Poincaré inequality (4).*

Remark. A reader familiar with Poincaré inequalities on metric spaces easily verifies that Theorem 4 (with the same proof) and Theorem 7 (with a weak two-weighted (q, p)-Poincaré inequality in the conclusion and a slightly modified proof) are valid in the setting of doubling metric measure spaces. We will not dwell on this generalization in this paper.

Let us also mention some consequences of Proposition 5 and the reverse Hölder inequality (2), which improve Theorem 1.

Corollary 8. *Let $w \in A_q$ be a strong A_∞-weight in $\mathbf{R}^n$, $n \ge 2$, satisfying the reverse Hölder inequality (2) with $r > 1$ and let $1 - 1/n \le \sigma \le r$. Then the weight w^σ is p-admissible for all*

$$p \ge \frac{n\bigl(\sigma(q-1)+1\bigr)}{q(n-1)+1}.$$

In particular, every strong A_∞-weight in $\mathbf{R}^n$, $n \ge 2$, is $n/(n-1)$-admissible.

Corollary 9. *Let w be a strong A_∞-weight in $\mathbf{R}^n$, $n \geq 2$, and $1 < p < n$. Then the weight $w^{1-p/n}$ is q-admissible for all*

$$q \geq \frac{n(r-1)+p}{n(r-1)+1},$$

where r is the exponent from the reverse Hölder inequality (2). *In particular, $w^{1-p/n}$ is q-admissible for some $q < p$.*

If moreover $w^{-t} \in A_1$ for some $t > 0$, then it can be derived from Lemma 3.17 in Semmes [17] that $w^s = w^{s-1}w$ is a strong A_∞-weight for all $0 < s < 1$, and hence the weight $w^{1-p/n} = (w^{(n-p)/(n-1)})^{1-1/n}$ with $1 \leq p \leq n$ is 1-admissible, by e.g. Theorem 1. On the other hand it is shown in the counterexample to Question 4.1 in Semmes [17], that there are strong A_∞-weights (even Jacobians of quasiconformal mappings) such that w^s is not a strong A_∞-weight for any $0 < s < 1$ and consequently the above argument cannot be applied. In fact, the following occurs.

Proposition 10. *There exists a strong A_∞-weight w in $\mathbf{R}^2$ such that none of the weights $w^{1-p/2}$, $1 < p < 2$, is 1-admissible.*

Finally, note that the well-known p-admissibility of A_p-weights is a direct consequence of Theorem 4 and the 1-admissibility of the Lebesgue measure.

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2. The proofs

Let us first recall a simple consequence of the reverse Hölder inequality (2). If $0 < s < 1$, the Hölder inequality with $w = w^\theta w^{1-\theta}$ and $\theta = s(r-1)/(r-s) < 1$ yields

$$\left(\fint_B w^r\,dx\right)^{1/r} \leq C \fint_B w\,dx \leq C\left(\fint_B w^s\,dx\right)^{\theta/s} \left(\fint_B w^r\,dx\right)^{1-\theta/s}$$

and division by the last factor on the right-hand side gives

$$\left(\fint_B w^r\,dx\right)^{1/r} \leq C^{1/\theta}\left(\fint_B w^s\,dx\right)^{1/s}. \tag{5}$$

From now on, the letter C will denote a positive constant whose exact value is unimportant and may change even within a line. We shall also use the notation $a \simeq b$ if $a/C \leq b \leq Ca$ holds for some C.

Proof of Theorem 1. It suffices to consider the case $p=1$. If $p>1$, then (5) and the Hölder inequality show that $w^{(1-p)/n}\in A_p(w^{1-1/n}\,dx)$ and Theorem 4 then implies that the weight $w^{1-p/n}=w^{(1-p)/n}w^{1-1/n}$ is p-admissible.

First, note that by (5), $w^{1-1/n}$ is an A_∞-weight and hence the measure $d\mu=w^{1-1/n}\,dx$ is doubling. Let also $d\nu=w\,dx$. The $(1,1)$-Poincaré inequality for $w^{1-1/n}$ is a consequence of the following Poincaré type inequality for strong A_∞-weights by David and Semmes [3],

$$\text{(6)}\qquad \fint_B\fint_B |u(\xi)-u(\eta)|\,d\nu(\xi)\,d\nu(\eta)\le C\frac{\mu(2B)}{\nu(B)^{1-1/n}}\fint_{2B}|\nabla u(\xi)|\,d\mu(\xi),$$

which holds for all balls B in $\mathbf{R}^n$ and all locally Lipschitz functions u. In order to prove that $w^{1-1/n}$ is 1-admissible we have to show that the measure ν on the left-hand side can be replaced by μ and that the factor in front of the integral on the right-hand side is comparable to r. This is done using an argument as in Franchi–Hajłasz [6]:

Let $B=B(x_0,r)$ be a ball in $\mathbf{R}^n$, $u_{B,\nu}=\fint_B u\,d\nu$ and let for $x\in B$ and $k=0,1,\dots$,

$$B_k(x)=B(x,2^{1-k}r)\qquad\text{and}\qquad u_k=\fint_{B_k(x)}u\,d\nu.$$

Then the doubling property of ν and the fact that $u_k\to u(x)$, as $k\to\infty$, imply

$$\begin{aligned}|u(x)-u_{B,\nu}|&\le|u_0-u_{B,\nu}|+\sum_{k=0}^\infty|u_{k+1}-u_k|\\&\le|u_0-u_{B,\nu}|+C\sum_{k=0}^\infty\fint_{B_k(x)}\fint_{B_k(x)}|u(\xi)-u(\eta)|\,d\nu(\xi)\,d\nu(\eta).\end{aligned}$$

Note that $B\subset B_0(x)$ and hence $|u_{B,\nu}-u_0|\le C\fint_{B_0(x)}|u-u_0|\,d\nu$, which can be included in the above sum. The Poincaré type inequality (6) then implies

$$|u(x)-u_{B,\nu}|\le C\sum_{k=0}^\infty\frac{\mu\bigl(2B_k(x)\bigr)}{\nu\bigl(B_k(x)\bigr)^{1-1/n}}\fint_{2B_k(x)}|\nabla u(\xi)|\,d\mu(\xi).$$

The quotient $\mu\bigl(2B_k(x)\bigr)/\nu\bigl(B_k(x)\bigr)^{1-1/n}$ does not exceed $C|B_k(x)|^{1/n}\simeq 2^{-k}r$, by the doubling property of μ and the Hölder inequality, and we obtain

$$|u(x)-u_{B,\nu}|\le Cr\sum_{k=0}^\infty 2^{-k}\fint_{2B_k(x)}|\nabla u(\xi)|\,d\mu(\xi).$$

Averaging both sides over the ball B with respect to μ gives

$$\fint_B |u - u_{B,\nu}|\,d\mu \le Cr \sum_{k=0}^{\infty} 2^{-k} \fint_B \int_{\mathbf{R}^n} \frac{\chi_{2B_k(x)}(\xi)}{\mu\bigl(2B_k(x)\bigr)} |\nabla u(\xi)|\,d\mu(\xi)\,d\mu(x), \tag{7}$$

where χ denotes the characteristic function of a set. Note that $2B_0(x) \subset 5B$, $\chi_{2B_k(x)}(\xi) = \chi_{2B_k(\xi)}(x)$ and that the doubling property of μ implies $\mu\bigl(2B_k(x)\bigr) \simeq \mu\bigl(2B_k(\xi)\bigr)$ whenever $\xi \in 2B_k(x)$. It follows that

$$\frac{\chi_{2B_k(x)}(\xi)}{\mu\bigl(2B_k(x)\bigr)} \le C\chi_{5B}(\xi) \frac{\chi_{2B_k(\xi)}(x)}{\mu\bigl(2B_k(\xi)\bigr)},$$

which inserted into (7) together with the Fubini theorem and the doubling property of μ yields

$$\fint_B |u - u_{B,\nu}|\,d\mu \le Cr \fint_{5B} |\nabla u|\,d\mu.$$

The required $(1,1)$-Poincaré inequality for μ then follows from the inequality

$$\fint_B |u - u_{B,\nu}|\,d\mu \le 2 \fint_B |u - u_{B,\mu}|\,d\mu. \ \square \tag{8}$$

Proof of Theorem 4. We shall assume $p > 1$, the case $p = 1$ is treated similarly. If B is a ball then the Hölder inequality and the fact that $v \in A_p(d\mu)$ yield

$$\begin{aligned} \mu(B) = \int_B v^{-1/p} v^{1/p}\,d\mu &\le \biggl(\int_{2B} v^{1/(1-p)}\,d\mu\biggr)^{1-1/p} \biggl(\int_B v\,d\mu\biggr)^{1/p} \\ &\le C\mu(2B)\,\nu(2B)^{-1/p}\,\nu(B)^{1/p}. \end{aligned}$$

The doubling condition $\mu(2B) \le C\mu(B)$ then implies $\nu(2B) \le C\nu(B)$, i.e. the measure ν is doubling.

By Theorems 3.2 and 3.3 in Hajłasz–Koskela [11] or Lemma 5.15 in Heinonen–Koskela [14], the weak $(1,s)$-Poincaré inequality for μ is equivalent to the validity of the inequality

$$|u(x) - u(y)| \le C|x-y| \bigl(M_{\mu,\lambda|x-y|}\,|\nabla u|^s(x) + M_{\mu,\lambda|x-y|}\,|\nabla u|^s(y)\bigr)^{1/s}$$

for some $C > 0$, $\lambda \ge 1$, μ-a.e. $x, y \in \mathbf{R}^n$ and all locally Lipschitz functions u. Here $M_{\mu,\lambda|x-y|}\,|\nabla u|^s(x)$ is the maximal function defined for $g \in L^1_{\mathrm{loc}}(\mathbf{R}^n,\mu)$ by

$$M_{\mu,R}\,g(x) = \sup_{0<\varrho<R} \fint_{B(x,\varrho)} g\,d\mu.$$

If we can show that for some $C>0$,

$$M_{\mu,R}\, g(x) \le C\bigl(M_{\nu,R}\, g^p(x)\bigr)^{1/p}, \tag{9}$$

then another application of Lemma 5.15 in Heinonen–Koskela [14] or Theorem 3.3 in Hajłasz–Koskela [11] implies that ν admits the weak $(1,ps)$-Poincaré inequality and the theorem follows. In order to prove (9), let B be a ball. The Hölder inequality gives

$$\begin{aligned}\fint_B g\,d\mu &\le \biggl(\fint_B g^p v\,d\mu\biggr)^{1/p}\biggl(\fint_B v^{1/(1-p)}\,d\mu\biggr)^{1-1/p}\\ &= \biggl(\fint_B g^p\,d\nu\biggr)^{1/p}\biggl(\fint_B v\,d\mu\biggr)^{1/p}\biggl(\fint_B v^{1/(1-p)}\,d\mu\biggr)^{1-1/p}.\end{aligned}$$

As $v\in A_p(d\mu)$, the product of the last two factors is bounded by a constant independent of B and (9) follows by taking supremum over all balls B with radius $0<\varrho<R$ and centre x. □

Proof of Proposition 6. We can assume that the points $a_0=0,a_1,a_2,\dots,a_m$ are distinct and that $p>1$. The case $p=1$ is treated similarly. Let

$$w(x)=\widetilde{w}(x')=(1+|x'|)^{\gamma_0}\prod_{j=1}^{m}\biggl(\frac{|x'-a_j|}{1+|x'-a_j|}\biggr)^{\gamma_j}.$$

First, we claim that $v\in A_p(w\,dx)$. Indeed, let $Q=Q'\times Q''$ be a cube in $\mathbf{R}^k\times\mathbf{R}^{n-k}$ with sides parallel to the coordinate axes. Partitioning the cube Q' into $\bigl(6(m+1)\bigr)^k$ equally sized cubes we find a subcube $\widetilde{Q}$ of Q' with sidelength comparable to that of Q', such that the sidelength h of $\widetilde{Q}$ satisfies $2h\le d_j=\operatorname{dist}(\widetilde{Q},a_j)$ for all $j=0,1,\dots,m$. Consider the weights defined on $\mathbf{R}^k$ by

$$\tilde v_1(x')=\int_{Q''} v(x',x'')\,dx'' \qquad\text{and}\qquad \tilde v_2(x')=\int_{Q''} v(x',x'')^{1/(1-p)}\,dx''.$$

One easily verifies that both $\tilde v_1$ and $\tilde v_2$ satisfy the doubling condition on $\mathbf{R}^k$ with the same doubling constants as v and $v^{1/(1-p)}$, respectively. Lemma 6.3 in Strömberg–Wheeden [20] then implies that both $\tilde v_1\widetilde{w}$ and $\tilde v_2\widetilde{w}$ are doubling weights in $\mathbf{R}^k$. Consequently, we have

$$\begin{aligned}&\int_Q v(x)w(x)\,dx\,\biggl(\int_Q v(x)^{1/(1-p)}w(x)\,dx\biggr)^{p-1}\\ &\qquad\qquad= \int_{Q'}\tilde v_1(x')\widetilde{w}(x')\,dx'\,\biggl(\int_{Q'}\tilde v_2(x')\widetilde{w}(x')\,dx'\biggr)^{p-1}\\ &\qquad\qquad\le C\int_{\widetilde{Q}}\tilde v_1(x')\widetilde{w}(x')\,dx'\,\biggl(\int_{\widetilde{Q}}\tilde v_2(x')\widetilde{w}(x')\,dx'\biggr)^{p-1}\end{aligned}$$

and as $d_j \le |x' - a_j| \le \bigl(1 + \sqrt{k}\,/2\bigr) d_j$ for all $x' \in \widetilde{Q}$ and $j = 0, 1, \dots, m$, this is comparable to

$$\int_{\widetilde{Q}} \tilde{v}_1(x')\,dx' \left(\int_{\widetilde{Q}} \tilde{v}_2(x')\,dx' \right)^{p-1} \left[(1 + d_0)^{\gamma_0} \prod_{j=1}^{m} \left(\frac{d_j}{1 + d_j} \right)^{\gamma_j} \right]^p .$$

The A_p-condition for v implies

$$\begin{aligned} \int_{\widetilde{Q}} \tilde{v}_1(x')\,dx' \left(\int_{\widetilde{Q}} \tilde{v}_2(x')\,dx' \right)^{p-1} &\le \int_Q v(x)\,dx \left(\int_Q v(x)^{1/(1-p)}\,dx \right)^{p-1} \\ &\le C|Q|^p \le C|\widetilde{Q} \times Q''|^p . \end{aligned}$$

Altogether, using once again $|x' - a_j| \simeq d_j$ for $x' \in \widetilde{Q}$, we obtain

$$\begin{aligned} \int_Q & v(x) w(x)\,dx \left(\int_Q v(x)^{1/(1-p)} w(x)\,dx \right)^{p-1} \\ &\le C \left[(1 + d_0)^{\gamma_0} \prod_{j=1}^{m} \left(\frac{d_j}{1 + d_j} \right)^{\gamma_j} \right]^p |\widetilde{Q} \times Q''|^p \\ &\le C \left(\int_{Q''} \int_{\widetilde{Q}} \widetilde{w}(x')\,dx'\,dx'' \right)^p \\ &\le C \left(\int_Q w(x)\,dx \right)^p , \end{aligned}$$

i.e. $v \in A_p(w\,dx)$.

The proposition now follows from Theorem 4 if we show that the weight w is 1-admissible. To this end, it is easily verified that the product

$$\prod_{j=1}^{m} \left(\frac{|x' - a_j|}{1 + |x' - a_j|} \right)^{k\alpha_j} ,$$

with $\alpha_j > 0$, is comparable to the Jacobian of the k-dimensional quasiconformal mapping

$$f(x') = \begin{cases} a_j + \left(\dfrac{|x' - a_j|}{\varrho} \right)^{\alpha_j} (x' - a_j) & \text{if } |x' - a_j| \le \varrho, \\ x' & \text{otherwise,} \end{cases}$$

where $\varrho > 0$ is fixed so that the balls $\{x' \in \mathbf{R}^k : |x' - a_j| \le \varrho\}$, $j = 0, 1, \dots, m$, are pairwise disjoint. Similarly, the factor $(1 + |x'|)^{k\alpha_0}$ with $\alpha_0 > 0$ is comparable to the Jacobian of the quasiconformal mapping

$$g(x') = \begin{cases} x' & \text{if } |x'| \le 1, \\ |x'|^{\alpha_0} x' & \text{otherwise.} \end{cases}$$

As $|f(x')| \simeq |x'|$ for all $x' \in \mathbf{R}^k$, we obtain by choosing $\alpha_j = \gamma_j/(k-1)$ that

$$\widetilde{w}(x') \simeq J(x')^{1-1/k},$$

where J denotes the Jacobian of the quasiconformal mapping $g \circ f$. By Theorem 1 and the fact that Jacobians of quasiconformal mappings in $\mathbf{R}^k$, $k \geq 2$, are strong A_∞-weights, $\widetilde{w}$ is 1-admissible in $\mathbf{R}^k$. The 1-admissibility of w in $\mathbf{R}^n$ then follows from the following lemma which is easily proved using the Fubini theorem, cf. Lemma 2 in Lu–Wheeden [15]. □

Lemma 11. *Let μ_1 and μ_2 be doubling measures on $\mathbf{R}^{n_1}$ and $\mathbf{R}^{n_2}$, respectively, admitting the $(1,p)$-Poincaré inequality with $p \geq 1$. Then the product measure $\mu = \mu_1 \times \mu_2$ on $\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$ is doubling and admits the $(1,p)$-Poincaré inequality.*

Proof of Theorem 7. The argument is similar to the proof of Theorem 5.3 in Hajłasz–Koskela [11]. Let $B = B(x_0, r)$ be a ball in $\mathbf{R}^n$ and let u be a Lipschitz function on B. We can assume that $u_{B,\mu} = 0$ and that $\lambda = 1$ in the $(1,p)$-Poincaré inequality for μ. The $(1,p)$-Poincaré inequality for μ then implies as in the proof of Theorem 1 that for every $x \in B$,

$$|u(x)| \leq C \sum_{j=0}^{\infty} r_j \left(\fint_{B_j} |\nabla u|^p \, d\mu \right)^{1/p},$$

where $B_j = B(x_j, r_j)$, $r_j = 2^{-j} r$ and each x_j lies on the segment $[x_0, x]$ at the distance $2^{-j}|x - x_0|$ from x. The condition (3) applied to the balls B_j and B then yields

$$|u(x)| \leq \frac{C r \nu(B)^{1/q}}{\mu(B)^{1/p}} \sum_{j=0}^{\infty} \frac{1}{\nu(B_j)^{1/q}} \left(\int_{B_j} |\nabla u|^p \, d\mu \right)^{1/p}.$$

Next, we write the above sum as $\Sigma' + \Sigma''$, where the summation in Σ' and Σ'' is over $j < j_0$ and $j \geq j_0$, respectively (j_0 will be chosen later). Note also that as $B_{j+1} \subset B_j$ and the set $B_j \setminus B_{j+1}$ contains a ball B'_j such that $B_j \subset 7B'_j$, the doubling property of ν implies $\nu(B_{j+1}) \leq \gamma \nu(B_j)$ for some $\gamma < 1$ independent of j. It follows that $\nu(B_j) \geq \gamma^{j-j_0} \nu(B_{j_0})$ for $j < j_0$ and $\nu(B_j) \leq \gamma^{j-j_0} \nu(B_{j_0})$ for $j \geq j_0$. Hence,

$$\Sigma' = \sum_{j=0}^{j_0-1} \frac{1}{\nu(B_j)^{1/q}} \left(\int_{B_j} |\nabla u|^p \, d\mu \right)^{1/p} \leq \frac{C}{\nu(B_{j_0})^{1/q}} \left(\int_B |\nabla u|^p \, d\mu \right)^{1/p}$$

and

$$\Sigma'' = \sum_{j=j_0}^{\infty} \frac{1}{\nu(B_j)^{1/q}} \left(\int_{B_j} |\nabla u|^p \, d\mu \right)^{1/p} \leq C \nu(B_{j_0})^{1/p-1/q} M(x)^{1/p},$$

where

$$M(x) = \sup_{B'} \frac{1}{\nu(B')} \int_{B'} |\nabla u|^p \, d\mu$$

and the supremum is taken over all balls $B' \subset B$ containing x. Next, as $\nu(B)^{-1} \int_B |\nabla u|^p \, d\mu \le M(x)$, we can find j_0 such that

$$\nu(B_{j_0}) \simeq \frac{1}{M(x)} \int_B |\nabla u|^p \, d\mu$$

and inserting this into the above estimates of Σ' and Σ'' yields

$$|u(x)| \le \frac{Cr\nu(B)^{1/q}}{\mu(B)^{1/p}} (\Sigma' + \Sigma'') \le \frac{Cr\nu(B)^{1/q}}{\mu(B)^{1/p}} \biggl(\int_B |\nabla u|^p \, d\mu \biggr)^{1/p-1/q} M(x)^{1/q}.$$

A standard argument using a Vitali type covering lemma (e.g. Lemma 5.5 in Heinonen–Koskela [14]) and the doubling property of ν shows that

$$\nu\bigl(\{x \in B : M(x) \ge \tau\}\bigr) \le \frac{C}{\tau} \int_B |\nabla u|^p \, d\mu,$$

cf. e.g. Chapter 1 in Stein [19]. Hence

$$\nu\bigl(\{x \in B : |u(x)| \ge t\}\bigr) \le \frac{Cr^q \nu(B)}{t^q \mu(B)^{q/p}} \biggl(\int_B |\nabla u|^p \, d\mu \biggr)^{q/p}.$$

The rest of the proof is by Maz'ya's truncation method [16] as in the proof of Lemma 5.15 in Heinonen–Koskela [14] or Theorem 2.1 in Hajłasz–Koskela [11]: We apply the above argument to the truncation of u given by

$$v(x) = \min\bigl\{2^j, \max\{u(x) - 2^j, 0\}\bigr\},$$

and conclude

$$2^{jq} \nu\bigl(\{x \in B : u(x) \ge 2^{j+1}\}\bigr) \le \frac{Cr^q \nu(B)}{\mu(B)^{q/p}} \biggl(\int_{\{x \in B : 2^j < u(x) < 2^{j+1}\}} |\nabla u|^p \, d\mu \biggr)^{q/p}.$$

Summing up over all integers j then gives

$$\begin{aligned}
\int_{\{x \in B : u(x) > 0\}} |u|^q \, d\nu &\le \sum_{j=-\infty}^{\infty} 2^{(j+2)q} \nu\bigl(\{x \in B : u(x) \ge 2^{j+1}\}\bigr) \\
&\le \frac{Cr^q \nu(B)}{\mu(B)^{q/p}} \sum_{j=-\infty}^{\infty} \biggl(\int_{\{x \in B : 2^j < u(x) < 2^{j+1}\}} |\nabla u|^p \, d\mu \biggr)^{q/p} \\
&\le Cr^q \nu(B) \biggl(\fint_B |\nabla u|^p \, d\mu \biggr)^{q/p}.
\end{aligned}$$

The integral over $\{x \in B : u(x) < 0\}$ is estimated similarly and the inequality (8) finishes the proof. $\square$

Proof of Corollary 8. Apply Proposition 5 to $v = w^{\sigma-(1-1/n)}$ and

$$p = \frac{n\bigl(\sigma(q-1)+1\bigr)}{q(n-1)+1}.$$

The inequality (5) with $s = 1-1/n$, $v^{1/(1-p)} w^{1-1/n} = w^{1/(1-q)}$ and $w \in A_q$ imply that $v \in A_p(w^{1-1/n}\,dx)$ and the claim follows. ▫

Proof of Corollary 9. By Proposition 5 it suffices to show that $w^{(1-p)/n} \in A_q(w^{1-1/n}\,dx)$, where

$$q = \frac{n(r-1)+p}{n(r-1)+1}.$$

This follows from the inequality (5) with $s = 1-1/n$ and the Hölder inequality. ▫

Proof of Proposition 10. Let $w_1\colon \mathbf{R}^2 \to \mathbf{R}$ be the weight from the counterexample to Question 4.1 in Semmes [17] and let

$$w(x_1,x_2) = w_1(x_1,x_2) + w_1(x_1,-x_2),$$

so that $w(x_1,x_2) = w(x_1,-x_2)$. It is easily verified using the definition (1) that w is also a strong A_∞-weight. Let $1 < p < 2$ and $d\mu = w^{1-p/2}\,dx$. Define the functions $u_\varepsilon\colon \mathbf{R}^2 \to \mathbf{R}$, $\varepsilon > 0$, by

$$u_\varepsilon(x_1,x_2) = \begin{cases} \dfrac{x_2}{\varepsilon} & \text{if } |x_2| < \varepsilon, \\ \operatorname{sgn} x_2 & \text{otherwise,} \end{cases}$$

and let B be the ball in $\mathbf{R}^2$ with center $\bigl(\frac{1}{2},0\bigr)$ and radius $\frac{1}{2}$. Then $u_{\varepsilon,B} = \fint_B u_\varepsilon\,d\mu = 0$ and

$$\fint_B |u_\varepsilon - u_{\varepsilon,B}|\,d\mu = \fint_B |u_\varepsilon|\,d\mu \to 1, \qquad \text{as } \varepsilon \to 0.$$

On the other hand, for $0 < \varepsilon < \frac{1}{2}$,

$$\begin{aligned} \int_B |\nabla u_\varepsilon|\,d\mu &\le \frac{1}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_0^1 w(x_1,x_2)^{1-p/2}\,dx_1\,dx_2 \\ &\le \frac{2}{\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_0^1 w_1(x_1,x_2)^{1-p/2}\,dx_1\,dx_2. \end{aligned}$$

It is shown in Semmes [17, p. 224], that for $0 < s < 1$,

$$\lim_{x_2\to 0}\int_0^1 w_1(x_1,x_2)^{s/2}\,dx_1 = 0.$$

Taking $s = 2-p$ now yields that

$$\fint_B |\nabla u_\varepsilon|\,d\mu \to 0, \qquad \text{as } \varepsilon \to 0,$$

i.e. the weight $w^{1-p/2}$, $1 < p < 2$, does not admit the 1-Poincaré inequality. ▫

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