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# USE OF LOGARITHMIC SUMS TO ESTIMATE POLYNOMIALS

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**Abstract.** Given any simple closed curve $\Gamma$ with enough smoothness we take any large number, say $N$, of points $\zeta_n$ on $\Gamma$, equispaced thereon with respect to harmonic measure, seen from $\infty$, for the exterior of $\Gamma$. Then, if $P$ is any polynomial of degree $M < N$, the values $|P(z)|$ can, for $z$ inside $\Gamma$, be estimated in terms of the logarithmic average $(1/N)\sum_{n=1}^{N} \log^+ |P(\zeta_n)|$. When $M$ and $N$ both tend to $\infty$ the estimate holds uniformly for each fixed $z$ inside $\Gamma$ as long as the ratio $M/N$ remains bounded away from $1$, and that requirement cannot be lightened.

The least superharmonic majorant and its properties play an important rôle in the proof of this result; other tools used are Jensen's formula and the Koebe $\frac{1}{4}$-theorem.

## Introduction

The set of polynomials $P(z)$ for which the sum $\sum_{-\infty}^{\infty} (\log^+ |P(n)|)/(1+n^2)$ comes out sufficiently small is a normal family in the complex plane. This result, to be found on p. 522 of [4], was proved again, in a different way from that of [4], by Pedersen in [11], and a simplified version of his argument has been given in [8]. It was shown in [7] that the conclusion holds for the entire functions $P(z)$ of any exponential type $A < \pi$. This limitation on the exponential type is sharp.

Taking the sums involving $\log^+ |P|$ over *the integers* is not essential; in [12] Pedersen has obtained a result (for polynomials) analogous to the above one, in which the sum over $\mathbf{Z}$ is replaced by one over a real sequence $\{\lambda_n\}$ with the property that every real interval of some given length $L$ contains at least one point $\lambda_n$. Here, it is not necessary that the points $\lambda_n$ be disposed symmetrically about $0$, and the conclusion still holds for entire functions $P(z)$ of sufficiently small exponential type depending on $L$.

In all of these results, *a sum* stands in place of the familiar Poisson integral $(1/\pi)\int_{-\infty}^{\infty} \bigl((\log^+ |P(x)|)/(1+x^2)\bigr)\,dx$, and it is well known that the boundedness of *the latter* for any set of functions $P(z)$ figuring in it (whether polynomials or entire and of bounded exponential type) makes that set a normal family in $\mathbf{C}$. Statements involving *sums* are in fact stronger than those phrased in terms of the integral, and seem therefore to hold out some promise for applications. The results initially cited are indeed strong enough to imply the multiplier theorem of Beurling and Malliavin; see [6] or especially Section 3 of [7].

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This motivates us to seek analogous statements for other configurations of the nodes over which the sums involving $\log^+|P|$ are to be taken; one desires in particular to see what can be deduced when those nodes *do not* lie on a straight line.

Some configurations can be reduced to the original one by direct transformation. Consider, for instance, the *parabola* $x = 1 - \frac{1}{4}y^2$ in the $z$-plane, image of the line $\Re w = 1$ under the mapping $w \to z = w^2$. If $P(z)$ is a polynomial, so is $Q(w) = P(w^2)$; it therefore follows from the result initially cited that the polynomials $P(z)$ with

$$\sum_{-\infty}^{\infty} \frac{\log^+\bigl|P\bigl((1+ni)^2\bigr)\bigr|}{1+n^2} = \sum_{-\infty}^{\infty} \frac{\log^+|Q(1+ni)|}{1+n^2}$$

sufficiently small form a normal family in $\mathbf{C}$; here the nodes of the left-hand sum lie on the parabola. Taking into account the more general result from [7], we see that the last conclusion also holds for the entire functions $P(z)$ of order $\frac{1}{2}$ growing at most like $\exp(A|z|^{1/2})$, where $A < \pi$. The limitation on $A$ is sharp.

Such generalizations could be multiplied, but the scope of the (rather trivial) procedure yielding them is limited, and it seems worthwhile to look for a direct way of extending [8]'s approach to various configurations of the nodes. For that one needs first of all to recognize the approach's essential ideas, distinguishing between them and the technical particularities accompanying their use when considering nodes located on the real line. Among those particularities must be counted the extensive use of Schwarz reflection made in [8], the formation there of functions like $1+f(z)\overline{f(\bar{z})}$, subsequent application of the Fejér–Riesz "square root" theorem, and so forth. These techniques are no longer available for general configurations of the nodes, and have to be abandoned.

Certainly the main feature of [8] lies in its systematic use of the *least superharmonic majorant*, the service rendered there by this object being largely due to the *smoothness of its Riesz mass* under the given circumstances. *Positivity* of a certain quadratic form involving the least superharmonic majorant and its Riesz mass also played a crucial rôle in [11], in [7] (the continuation of [8]), and again in [12]. It seems likely that these are the ideas that would have to be kept in adapting the method of [8] to more general situations.

Here we have, *as a first step*, carried out the adaptation for a somewhat special configuration: *all the nodes lie on a given simple closed curve* $\Gamma$. This curve is required to be *rather smooth*, but is otherwise arbitrary. Only sums involving *polynomials* are considered.

This is what we will prove. Fix any number $\lambda$, $0 < \lambda < 1$, and take any large number, say $N$, of nodes $\zeta_n$ on $\Gamma$, equally spaced thereon with respect to harmonic measure, as seen from $\infty$, for the exterior of $\Gamma$. *Then, if* $P(z)$ *is any* *polynomial of degree* $\le \lambda N$ *and* $z_0$ *is any point inside* $\Gamma$, *we have*

$$\log|P(z_0)| \le \frac{A(z_0)}{N}\sum_{n=1}^{N}\log^+|P(\zeta_n)| + B(z_0),$$

*where* $A(z_0)$ *and* $B(z_0)$ *depend on* $z_0$, *on* $\lambda$, *and on the curve* $\Gamma$, *but are independent of* $N$.

The result is new even when $\Gamma$ is a circle. In that situation it is reminiscent of an old one due to S. Bernstein, in which the average of $\log^+|P(\zeta_n)|$ is replaced by $\sup_n|P(\zeta_n)|$. The latter follows in turn from (and is essentially equivalent to) a well-known theorem of Miss Cartwright about entire functions of exponential type; see [1, p. 180].

**1.** Throughout the remainder of this article we consider a *given* simple closed curve $\Gamma$, supposed to have a certain amount of smoothness. Such a curve divides the complex plane into two domains: its *inside* $\mathscr{D}$ and its *exterior* $\mathscr{E}$, this usually supposed to (also) include the point at $\infty$.

We shall require a little more than $\mathscr{C}_1$ smoothness of $\Gamma$. If $f_{\mathscr{D}}(z, z_0)$ and $f_{\mathscr{E}}(z)$ denote conformal mappings of $\mathscr{D}$ and $\mathscr{E}$ respectively onto the unit disk, the first sending any given $z_0 \in \mathscr{D}$ to $0$ and the second taking $\infty$ to $0$, we shall need to know that $f'_{\mathscr{D}}(z, z_0)$ and $f'_{\mathscr{E}}(z)$ are bounded away from zero near $\Gamma$ and extend continuously up to it. This property is guaranteed by *Kellogg's theorem* provided that the direction of the tangent to $\Gamma$ at a point $\zeta$ depends in $\operatorname{Lip}\alpha$ fashion on $\zeta$ for some $\alpha > 0$. See, e.g., pp. 103–105 of [5].

Let us agree from now on to say that the curve $\Gamma$ *has Kellogg smoothness* when the geometric condition just enunciated holds for it. In that event, the continuous extensions of $f'_{\mathscr{D}}(z, z_0)$ and $f'_{\mathscr{E}}(z)$ to $\Gamma$ are designated by $f'_{\mathscr{D}}(\zeta, z_0)$ and $f'_{\mathscr{E}}(\zeta)$ respectively, and then the quantities $|f'_{\mathscr{D}}(\zeta, z_0)|$, $|f'_{\mathscr{E}}(\zeta)|$ are bounded above and below on $\Gamma$ by two strictly positive constants.

Harmonic measure for the domain $\mathscr{D}$ is, as customary, denoted by $\omega_{\mathscr{D}}(\,\cdot\,, z_0)$ when seen from $z_0 \in \mathscr{D}$; that measure for $\mathscr{E}$ (seen from $z_0 \in \mathscr{E}$) is similarly denoted by $\omega_{\mathscr{E}}(\,\cdot\,, z_0)$. When $\Gamma$ has Kellogg smoothness, we have, for $\zeta$ running along that curve,

$$d\omega_{\mathscr{D}}(\zeta, z_0) = (1/2\pi)|f'_{\mathscr{D}}(\zeta, z_0)\,d\zeta|$$

and

$$d\omega_{\mathscr{E}}(\zeta, \infty) = (1/2\pi)|f'_{\mathscr{E}}(\zeta)\,d\zeta|;$$

*both* left-hand members therefore lie between two strictly positive constant multiples of $|d\zeta|$, the element of arc length along $\Gamma$. $\omega_{\mathscr{D}}(\,\cdot\,, z_0)$ and $\omega_{\mathscr{E}}(\,\cdot\,, \infty)$ are, in other words, both *equivalent* to arc-length measure on $\Gamma$, and therefore *equivalent to each other*. The strictly positive constants describing this equivalence depend of course on the choice of $z_0 \in \mathscr{D}$ as well as on the curve $\Gamma$.

Green's function for the domain $\mathscr{E}$ is denoted by $G_{\mathscr{E}}(z, z_0)$; we shall make particular use of $G_{\mathscr{E}}(z,\infty)$, for which see, e.g., pp. 124–125 of [10] or pp. 106–110 and 132 of [13]. $G_{\mathscr{E}}(z,\infty)$ is harmonic in $\mathscr{E}$ (save at $\infty$) and continuous down to $\Gamma$, where it equals *zero*. For $z \to \infty$,

$$G_{\mathscr{E}}(z,\infty) = \log|z| + \gamma_{\mathscr{E}} + O(1/|z|), \tag{1}$$

where $\gamma_{\mathscr{E}}$ is a number called the *Robin constant* for $\mathscr{E}$.

With the help of $G_{\mathscr{E}}(z,\infty)$ and $\omega_{\mathscr{E}}(\,\cdot\,,z)$ we can represent any function $U(z)$ harmonic in $\mathscr{E}$ (save at $\infty$), continuous down to $\Gamma$, and asymptotic to $M\log|z|$ for $z \to \infty$. It suffices to look at the two functions $U(z) - (M+\delta)G_{\mathscr{E}}(z,\infty)$, $U(z) - (M-\delta)G_{\mathscr{E}}(z,\infty)$, where $\delta > 0$. Both are harmonic in $\mathscr{E}$ (save at $\infty$), continuous down to $\Gamma$, and equal to $U(z)$ thereon, but one goes to $-\infty$ and the other to $\infty$ when $z \to \infty$. They must therefore *straddle* $\int_\Gamma U(\zeta)\,d\omega_{\mathscr{E}}(\zeta,z)$, harmonic and *bounded* in $\mathscr{E}$, and equal to $U(z)$ on $\Gamma$. On making $\delta \to 0$ we find from this that

$$U(z) = \int_\Gamma U(\zeta)\,d\omega_{\mathscr{E}}(\zeta,z) + MG_{\mathscr{E}}(z,\infty), \qquad z \in \mathscr{E}. \tag{2}$$

This relation will be used when $U(z)$ is *the smallest harmonic majorant*, in $\mathscr{E}$, of the subharmonic function $\log^+|P(z)|$, $P(z)$ being a polynomial of *precise* degree $M$. In that case we have

$$U(z) = \int_\Gamma \log^+|P(\zeta)|\,d\omega_{\mathscr{E}}(\zeta,z) + MG_{\mathscr{E}}(z,\infty) \tag{3}$$

for $z \in \mathscr{E}$; making the usual interpretation of the integral as $\log^+|P(z)|$ for $z \in \Gamma$, we extend the formula down to $\Gamma$.

A second expression for this particular function $U(z)$, based on the Riesz representation formula (see [3, pp. 311–328] or [13, pp. 71–78]) will be very useful. The *Riesz mass* $\mu$ corresponding to the subharmonic function $\log^+|P(z)|$ is *positive, finite* and also of *compact support*, being carried on certain closed curves surrounding the zeros of $P(z)$. We can therefore write

$$\log^+|P(z)| = \int_{\mathbf{C}} \log|z-w|\,d\mu(w) + a \tag{4}$$

with a certain constant $a$; this is the Riesz representation of $\log^+|P(z)|$.

Now when $z \in \Gamma$,

$$\log|z-w| = \int_\Gamma \log|z-\zeta|\,d\omega_{\mathscr{E}}(\zeta,w) + G_{\mathscr{E}}(w,\infty) \qquad \text{for } w \in \mathscr{E}$$

and

$$\log|z-w| = \int_\Gamma \log|z-\zeta|\, d\omega_{\mathscr{D}}(\zeta, w) \qquad \text{for } w \in \mathscr{D}.$$

For $z \in \Gamma$, we thus have, by (4),

$$\begin{aligned}\log^+|P(z)| = &\int_{\mathscr{E}}\int_\Gamma \log|z-\zeta|\, d\omega_{\mathscr{E}}(\zeta,w)\, d\mu(w) + \int_{\mathscr{E}} G_{\mathscr{E}}(w,\infty)\, d\mu(w)\\ &+ \int_{\mathscr{D}}\int_\Gamma \log|z-\zeta|\, d\omega_{\mathscr{D}}(\zeta,w)\, d\mu(w) + \int_\Gamma \log|z-\zeta|\, d\mu(\zeta) + a,\end{aligned}$$

as $\mu$ may well put some charge on $\Gamma$. The right side of this relation is of the form $\int_\Gamma \log|z-\zeta|\, d\nu(\zeta) + a'$, where $\nu$ is a certain *positive* measure on $\Gamma$, called the *balayage* of $\mu$ to $\Gamma$. Since $P(z)$ is a polynomial of precise degree $M$, $\mu(\mathbf{C}) = M$ and therefore

$$\nu(\Gamma) = \mu(\mathbf{C}) = M.$$

Thus,

$$\log^+|P(z)| = \int_\Gamma \log|z-\zeta|\, d\nu(\zeta) + a' \qquad \text{for } z \in \Gamma. \tag{5}$$

The right side of (5) is harmonic for $z \in \mathscr{E}$ (save at $\infty$), coincides with $\log^+|P(z)|$ on $\Gamma$, and is asymptotic to $\nu(\Gamma)\log|z| = M\log|z|$ for $z \to \infty$. It is, moreover, continuous down to $\Gamma$. That follows, for instance, from the Evans–Vasilesco theorem ([3, p. 335]; [13, p. 54]); the right side, equal to $\log^+|P(z)|$ on $\Gamma$, *the support* of $\nu$, is certainly continuous *there*. The right-hand members of (3) and (5) therefore have the same boundary behaviour at $\Gamma$ and the same behaviour at $\infty$. Being both harmonic in $\mathscr{E}$, they are thus *equal* there, and we have

$$\int_\Gamma \log|z-\zeta|\, d\nu(\zeta) + a' = \int_\Gamma \log^+|P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,z) + MG_{\mathscr{E}}(z,\infty)$$

for $z \in \mathscr{E}$, as well as for $z$ on $\Gamma$.

A similar argument can be made for the domain $\mathscr{D}$ and for it an analogous relation is found. We thus arrive at

**Lemma 1.** *If P(z) is a polynomial of precise degree $M$,*

$$\int_\Gamma \log^+|P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,z) + MG_{\mathscr{E}}(z,\infty) = \int_\Gamma \log|z-\zeta|\, d\nu(\zeta) + a' \qquad \textit{for } z \in \mathscr{E} \tag{6}$$

*and*

$$\int_\Gamma \log^+|P(\zeta)|\, d\omega_{\mathscr{D}}(\zeta,z) = \int_\Gamma \log|z-\zeta|\, d\nu(\zeta) + a' \qquad \textit{for } z \in \mathscr{D}, \tag{7}$$

*where $\nu$ is a positive measure on $\Gamma$ with $\nu(\Gamma) = M$ and $a'$ is a constant depending on $P$. Both relations hold for $z$ on $\Gamma$ provided that the integrals standing on their left sides are there given the usual interpretation.*

What is the value of the constant $a'$ figuring in this lemma? It is given by the relation

$$a' = M\gamma_{\mathscr{E}} + \int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,\infty). \tag{8}$$

It is easy to deduce this formula—which in any event will not be needed here—from (1) and (6); cf. the discussion of the constant $c$ in the next section.

**2.** We now fix a small constant $\kappa > 0$; later on we shall see how it is to be chosen. Then, given any polynomial $P(z)$ of precise degree $M$, we form a new function $F(z)$ by putting

$$F(z) = \int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,z) - \kappa M G_{\mathscr{E}}(z,\infty) \qquad \text{for } z \in \mathscr{E}, \tag{9}$$

$$F(z) = \int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{D}}(\zeta,z) \qquad \text{for } z \in \mathscr{D}, \tag{10}$$

and

$$F(z) = \log^+ |P(z)|, \qquad z \in \Gamma. \tag{11}$$

This function $F(z)$ is continuous everywhere, and harmonic both in $\mathscr{D}$ and in $\mathscr{E}$ (save at $\infty$).

It will be convenient for us to *extend the Green's function $G_{\mathscr{E}}(z,\infty)$ to $\mathscr{D}$ by putting it equal to zero there.* Then $G_{\mathscr{E}}(z,\infty)$ becomes subharmonic in $\mathbf{C}$, and for the function $F$ we clearly have

$$F(z) \le C - \kappa M G_{\mathscr{E}}(z,\infty) \tag{12}$$

(everywhere), with $C$ some constant depending on $P$. The right side is *superharmonic* in $\mathbf{C}$, so $F(z)$ has a (finite) *superharmonic majorant* in $\mathbf{C}$, and therefore a *least* one, which we denote by $(\mathfrak{M}F)(z)$.

The main properties of $(\mathfrak{M}F)(z)$ are described in [3, pp. 363–371]. This function is *continuous* and everywhere $\ge F(z)$; it is *harmonic* wherever it is $> F(z)$ and also wherever the latter function is harmonic. In the present circumstances, that makes $(\mathfrak{M}F)(z)$ *harmonic both in $\mathscr{D}$ and in $\mathscr{E}$; its Riesz mass is therefore carried on $\Gamma$ and indeed on the closed subset $E$ of that curve where $(\mathfrak{M}F)(\zeta) = F(\zeta)$.*

Referring to (9) and (12) we find that

$$\int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,z) - \kappa M G_{\mathscr{E}}(z,\infty) \le (\mathfrak{M}F)(z) \le C - \kappa M G_{\mathscr{E}}(z,\infty)$$

for $z \in \mathscr{E}$, and on making $z \to \infty$ we see from this that the total Riesz mass corresponding to $\mathfrak{M}F$ (carried on $E \subseteq \Gamma$ and taken, with some inconsistency of language, as *positive*) *is equal to $\kappa M$*. We can thus write the Riesz representation

$$(\mathfrak{M}F)(z) = \int_\Gamma \log \frac{1}{|z-\zeta|}\, d\varrho(\zeta) + c, \tag{13}$$

where $c$ is a constant and $\varrho$, the Riesz mass, is a positive measure of total mass $\kappa M$ carried, as already noted and by (11), on

$$E = \{\zeta \in \Gamma;\ (\mathfrak{M}F)(\zeta) = \log^+ |P(\zeta)|\}. \tag{14}$$

Later on we shall need a *lower bound* for $c$. From [10, pp. 124–125] or [13, pp. 106–110] we have the formula

$$G_{\mathscr{E}}(z,\infty) = \int_\Gamma \log|z-\zeta|\, d\omega_{\mathscr{E}}(\zeta,\infty) + \gamma_{\mathscr{E}} \tag{15}$$

where $\gamma_{\mathscr{E}}$ is Robin's constant for $\mathscr{E}$; with the above specification of $G_{\mathscr{E}}(z,\infty)$ in $\mathscr{D}$ and the usual interpretation of the integral for $z \in \Gamma$, *this holds everywhere.* Substituting (15) into (9) and then using (13) and the value $\varrho(\Gamma) = \kappa M$ in the relation $F(z) \le (\mathfrak{M}F)(z)$, we find that

$$\int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,z) - \kappa M(\gamma_{\mathscr{E}} + \log|z|) + O(1/|z|) \le -\kappa M \log|z| + O(1/|z|) + c$$

for $|z| \to \infty$. Thence,

$$c \ge \int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,\infty) - \kappa M \gamma_{\mathscr{E}}. \tag{16}$$

**3.** A remarkable *smoothness relation* for the Riesz mass $\varrho$, due to that measure's being supported on the set $E$ given by (14), will be very important for us. From Lemma 1 (end of Section 1), (9), (10), (11) and (13) we have

$$F(z) - (\mathfrak{M}F)(z) = \int_\Gamma \log|z-\zeta|\,(d\nu(\zeta) + d\varrho(\zeta)) - (1+\kappa)MG_{\mathscr{E}}(z,\infty) - b \tag{17}$$

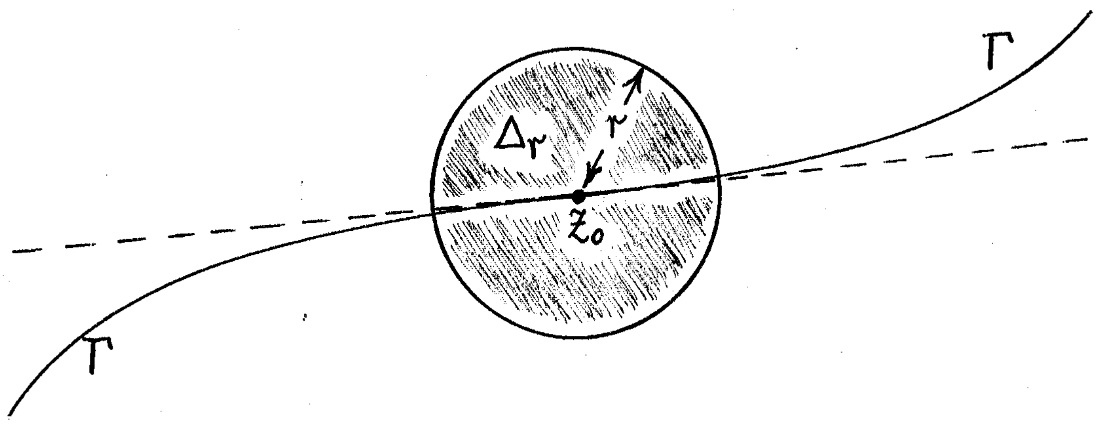
where $b$ is a certain constant; with $G_{\mathscr{E}}(z,\infty)$ interpreted as zero for $z \in \mathscr{D}$, this relation holds *everywhere.*

**Lemma 2.** *Provided that $\Gamma$ has Kellogg smoothness, the measure $\varrho$ is absolutely continuous with respect to arc length on $\Gamma$, and we have*

$$d\varrho(\zeta) \le (1+\kappa)M d\omega_{\mathscr{E}}(\zeta,\infty). \tag{18}$$

*Proof.* The left side of (17) is everywhere $\le 0$ and equal to zero precisely at the points of $E$. Fixing any $z_0 \in E$, we therefore have

$$\int_\Gamma \log|z_0-\zeta|\,\big(d\nu(\zeta) + d\varrho(\zeta)\big) = (1+\kappa)MG_{\mathscr{E}}(z_0,\infty) + b = b \tag{19}$$




and

$$\int_\Gamma \log|z-\zeta|\,(d\nu(\zeta)+d\varrho(\zeta)) \le (1+\kappa)MG_{\mathscr{E}}(z,\infty)+b \qquad \text{for all } z, \tag{20}$$

the Green's function being zero on $\Gamma \supseteq E$.

Let us denote by $\Delta_r$ the disk $|z-z_0|<r$. Then, by (19), (20) and Jensen's formula we have, for $r_0>0$,

$$\int_0^{r_0} \frac{\nu(\Delta_r)+\varrho(\Delta_r)}{r}\,dr \le \frac{(1+\kappa)M}{2\pi r_0}\int_{\partial\Delta_{r_0}} G_{\mathscr{E}}(\zeta,\infty)\,|d\zeta|.$$

By Jensen's formula and (15) the right side of this last relation equals

$$(1+\kappa)M\int_0^{r_0}\frac{\omega_{\mathscr{E}}(\Delta_r,\infty)}{r}\,dr$$

since $G_{\mathscr{E}}(z_0,\infty)=0$; the measure $\nu$ being positive, we thence get

$$\int_0^{r_0}\frac{\varrho(\Delta_r)}{r}\,dr \le (1+\kappa)M\int_0^{r_0}\frac{\omega_{\mathscr{E}}(\Delta_r,\infty)}{r}\,dr. \tag{21}$$

Now under our hypothesis the density

$$\frac{d\omega_{\mathscr{E}}(\zeta,\infty)}{|d\zeta|}=\omega'_{\mathscr{E}}(\zeta,\infty)$$

exists everywhere on $\Gamma$ and is continuous (see beginning of Section 1). Therefore, and since $\Gamma$ has, in particular, a tangent at $z_0$, the right side of (21) is of the form

$$2(1+\kappa)Mr_0\,\omega'_{\mathscr{E}}(z_0,\infty)+o(r_0) \tag{22}$$

for $r_0\to 0$.

Concerning

$$\frac{d\varrho(\zeta)}{|d\zeta|}=\varrho'(\zeta),$$

all we know to begin with is that it *exists a.e.* (with respect to arc length) on $\Gamma$. If it exists and is *finite* at $z_0$, the left side of (21) will be

$$2r_0\,\varrho'(z_0) + o(r_0) \tag{23}$$

when $r_0 \to 0$, but if $\varrho'(z_0)$ exists and is *infinite*, the left side of (21) eventually becomes larger than *any* multiple $Lr_0$ when $r_0 \to 0$. Referring to (22) and (21) and remembering that $\omega'_{\mathscr{E}}(\zeta,\infty)$ is *bounded*, we see first of all that this *last* possibility cannot occur. Therefore $\varrho$ must be *absolutely continuous* with respect to arc length; otherwise there would certainly be points $z_0$ in its support $E$ with $\varrho'(z_0) = \infty$.

*Outside* the closed set $E$ we have everywhere $\varrho'(\zeta) = 0$, and for almost every $z_0 \in E$ we get, from (21), (22) and (23),

$$\varrho'(z_0) \le (1+\kappa)M\omega'_{\mathscr{E}}(z_0,\infty).$$

Therefore (18) holds since $\varrho$ is absolutely continuous with respect to arc length.

As we observed in the course of the proof, $\omega'_{\mathscr{E}}(\zeta,\infty)$ is bounded on $\Gamma$. We thus have the useful

**Corollary.** *Under the hypothesis of the lemma,*

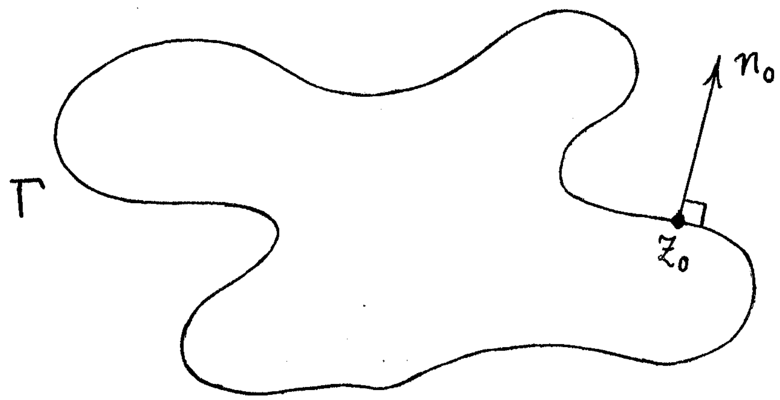
$$d\varrho(\zeta) \le (1+\kappa)MC\,|d\zeta|,$$

*where $C$ is a constant depending on the curve $\Gamma$.*

**Remark.** Kellogg smoothness is not really needed for the conclusion of Lemma 2, which holds even for non-rectifiable curves $\Gamma$. That follows from a general measure-theoretic result, due to Grishin [2] and recently extended by Sodin in [15]. The version given here—actually a consequence of Grishin's result—is sufficient for our purposes, and its proof, which replaces an earlier longer argument, has been included for the reader's convenience. The idea of using Jensen's formula in that proof comes from Sodin's paper and goes back to Grishin.

**4.** From Lemma 2 and its corollary we can deduce a useful limitation on the variation of $(\mathfrak{M}F)(\zeta)$ along $\Gamma$, but for that we will need a preliminary result.[1]

[1] The geometric considerations in this and the next sections can be largely avoided, for the simple inequality in Remark 2 near the end of Section 11 easily yields a serviceable version of Theorem 1, to be given in the next section. But the result obtained by the present method is capable of *further refinement* than can be achieved using the alternative approach. See the remark at the end of the next section.




Given $z_0$ on $\Gamma$, *we denote by* $n_0$ *the unit outward normal to* $\Gamma$ *at* $z_0$.

**Lemma 3.** *If* $\Gamma$ *has Kellogg smoothness,*

$$\limsup_{t\to 0+}\left|\frac{d(\mathfrak{M}F)(z_0+n_0t)}{dt}\right| \;\le\; (1+\kappa)MC'$$

*for each* $z_0\in\Gamma$, *where* $C'$ *is a constant depending only on* $\Gamma$.

*Proof.* In order to simplify the notation, we may just as well take $z_0$ to be at the origin and the axis of abscissae tangent to $\Gamma$ there, making $n_0 = i$. Writing $\zeta = \xi + i\eta$, we then have, from (13),

$$(24)\qquad -\frac{d(\mathfrak{M}F)(iy)}{dy} = -\Im\int_\Gamma \frac{d\varrho(\zeta)}{iy-\zeta} = \int_\Gamma \frac{y}{|iy-\zeta|^2}\,d\varrho(\zeta) - \int_\Gamma \frac{\eta}{|iy-\zeta|^2}\,d\varrho(\zeta).$$

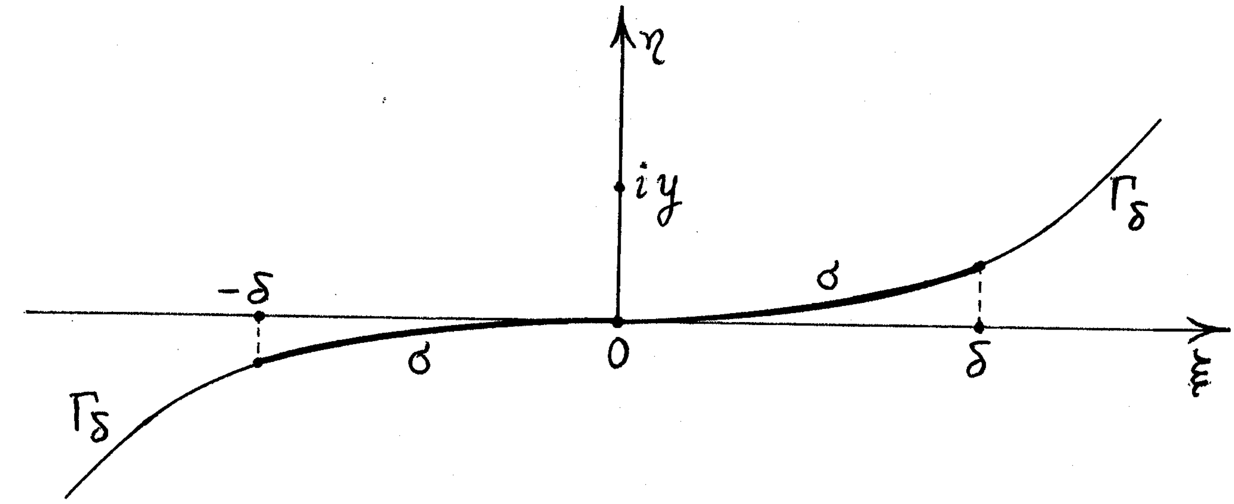
The *first* of the two integrals on the right is *positive* for $y>0$. In order to examine its behaviour for $y\to 0+$, we fix a small quantity $\delta>0$, denote by $\sigma$ the arc of $\Gamma$ through $0$ whose abscissae $\xi$ lie between $-\delta$ and $\delta$ (see figure), and by $\Gamma_\delta$ the complement $\Gamma\sim\sigma$. We have

$$(25)\qquad \int_\Gamma \frac{y}{|iy-\zeta|^2}\,d\varrho(\zeta) = \int_\sigma \frac{y}{|iy-\zeta|^2}\,d\varrho(\zeta) + \int_{\Gamma_\delta} \frac{y}{|iy-\zeta|^2}\,d\varrho(\zeta),$$

and the second integral on the right clearly goes to $0$ as $y\to 0$.

In the first integral on the right we can put $\zeta = \xi + i\eta(\xi)$ with $\xi$ ranging from $-\delta$ to $\delta$ where, for some $\alpha>0$, $\eta(\xi) = O(|\xi|^{1+\alpha})$ and $\eta'(\xi) = O(|\xi|^\alpha)$ in view of the Kellogg smoothness of $\Gamma$. Along $\sigma$ we also have

$$d\varrho(\zeta) = \frac{d\varrho(\zeta)}{|d\zeta|}\sqrt{1+(\eta'(\xi))^2}\;d\xi \le (1+\kappa)MC\sqrt{1+O(|\xi|^{2\alpha})}\;d\xi$$




by the corollary to Lemma 2, and besides,

$$|iy-\zeta|^2 = \xi^2+\big(y-\eta(\xi)\big)^2 = \xi^2+y^2-2y\xi O(|\xi|^\alpha)+O(|\xi|^{2+2\alpha}) \geq \big(1-O(\delta^\alpha)\big)(y^2+\xi^2).$$

Substituting these estimates into the first right-hand integral of (25), we find that quantity to be

$$\leq \big(1+O(\delta^\alpha)\big)(1+\kappa)MC\int_{-\delta}^{\delta}\frac{y\,d\xi}{y^2+\xi^2} < \pi\big(1+O(\delta^\alpha)\big)(1+\kappa)MC;$$

*this*, then, is an *upper bound* on the *left side* of (25) for sufficiently small $y>0$. (By taking ever smaller values of $\delta$ one can show by a refinement of the argument just made that the left side of (25) *tends to* $\pi\varrho'(0)$ for $y\to 0+$; that result will not be needed here.)

Consider now the *second* integral on the right in (24). With our fixed $\delta>0$ (which can be taken to be the same no matter to *which* $z_0$ on $\Gamma$ our origin corresponds in the present discussion), we write

$$(26)\qquad \int_\Gamma \frac{\eta}{|iy-\zeta|^2}\,d\varrho(\zeta) = \int_\sigma \frac{\eta}{|iy-\zeta|^2}\,d\varrho(\zeta) + \int_{\Gamma_\delta}\frac{\eta}{|iy-\zeta|^2}\,d\varrho(\zeta).$$

Here the *first* right-hand term is in absolute value

$$\leq (1+\kappa)MC\int_{-\delta}^{\delta}\frac{|\eta(\xi)|}{\xi^2}\sqrt{1+\big(\eta'(\xi)\big)^2}\,d\xi$$

$$\leq (1+\kappa)MC\int_{-\delta}^{\delta}O\bigg(\frac{1}{|\xi|^{1-\alpha}}\bigg)\,d\xi = (1+\kappa)MC\cdot O(\delta^\alpha),$$

thanks again to the corollary of Lemma 2 and to the Kellogg smoothness of $\Gamma$. (Essential use of the second property is being made at this point; $\mathscr{C}_1$ smoothness of $\Gamma$ would have been enough for the examination of (25).)

In the *second* right-hand integral of (26) we have

$$|iy - \zeta| \geq \text{const.} > 0$$

for $\zeta$ on $\Gamma_\delta$, *provided that* $y > 0$ *is sufficiently small*—$\Gamma$ is *simple*! At the same time, $|\eta|$ is bounded, so the integral in question is

$$\leq \text{const.}\, \varrho(\Gamma) = \text{const.}\, \kappa M$$

for small enough $y > 0$.

Putting these results together we see that *the left side of* (26) *is in absolute value*

$$\leq \text{const.}\, (1+\kappa)M$$

for sufficiently small $y > 0$, with a constant depending only on $\Gamma$.

The estimates just found for the left sides of (25) and (26) are now substituted into (24), and we find that for small enough $y > 0$,

$$\left| \frac{d(\mathfrak{M}F)(iy)}{dy} \right| \leq (1+\kappa)MC'$$

with a constant $C'$ depending only on $\Gamma$. The lemma is thus proved.

Using the fact, noted in Section 1, that $d\omega_{\mathscr{E}}(\zeta,\infty)/|d\zeta| = \omega'_{\mathscr{E}}(\zeta,\infty)$ is bounded away from zero, the result just obtained can be rephrased as a

**Corollary.** *Under the conditions of the lemma, we have*

$$\limsup_{t\to 0+} \left| \frac{d(\mathfrak{M}F)(z_0 + n_0 t)}{dt} \right| \leq (1+\kappa)MC''\omega'_{\mathscr{E}}(z_0,\infty) \tag{27}$$

*at each* $z_0$ *on* $\Gamma$, *with a constant* $C''$ *depending only on* $\Gamma$.

**Remark.** The hypothesis of Kellogg smoothness has again been used here, after having already been called on in Section 1. In both instances, this condition on $\Gamma$ could be weakened *slightly*, but that curve would still be required to have more than $\mathscr{C}_1$ regularity. Such further refinement hardly seems worthwhile in the present context.

**5.** We can now proceed with the comparison of $(\mathfrak{M}F)(z_1)$ and $(\mathfrak{M}F)(z_2)$ for two neighbouring points, $z_1$, $z_2$, on $\Gamma$. For this purpose it is best to bring in the function

$$U(z) = (\mathfrak{M}F)(z) + \kappa M G_{\mathscr{E}}(z,\infty); \tag{28}$$

according to (9) and the relation $(\mathfrak{M}F)(z) \geq F(z)$, $U(z)$ is *positive* in $\mathscr{E}$. It is also *harmonic* there, including at $\infty$ by (13) and the relation $\varrho(\Gamma) = \kappa M$, and continuous down to $\Gamma$.

**Theorem 1.** *If* $\Gamma$ *has Kellogg smoothness, we have*

$$(\mathfrak{M}F)(z_2) \leq 3(\mathfrak{M}F)(z_1) + (1+\kappa)MB\omega_{\mathscr{E}}(\widehat{z_1, z_2}, \infty) \tag{29}$$

*for any two points* $z_1$, $z_2$ *on* $\Gamma$, *where* $\widehat{z_1, z_2}$ *denotes either arc of* $\Gamma$ *from* $z_1$ *to* $z_2$ *and* $B$ *is a constant depending only on* $\Gamma$.[2]

*Proof.* Take a conformal mapping $z \to w = \phi(z)$ of $\mathscr{E}$ onto $\{|w| > 1\}$ which sends $\infty$ to $\infty$, and for $|w| \geq 1$, put

$$V(w) = U(z) \text{ when } w = \phi(z). \tag{30}$$

Defined in this way, the function $V(w)$ is positive and harmonic in $\{|w| > 1\}$, including at $\infty$, and continuous down to the unit circle. Introduce polar coordinates in the $w$-plane, writing $w = re^{i\vartheta}$ when $w = \phi(z)$ with $z \in \mathscr{E} \cup \Gamma$. Then, since $G_{\mathscr{E}}(z, \infty) = \log|\phi(z)|$, we have, from (28) and (30),

$$V(re^{i\vartheta}) = (\mathfrak{M}F)(z) + \kappa M \log r. \tag{31}$$

Consider any point $z_0$ on $\Gamma$. If $\phi(z_0) = e^{i\vartheta_0}$, say, the unit normal $n_0$ to $\Gamma$ at $z_0$ corresponds, under the conformal mapping $\phi$, to a *radial* vector at $e^{i\vartheta_0}$, of length $|\phi'(z_0)|$. Thence, and by (31),

$$\limsup_{r\to 1+} \left| \frac{\partial V(re^{i\vartheta_0})}{\partial r} \right| = \frac{1}{|\phi'(z_0)|} \limsup_{t \to 0+} \left| \frac{d(\mathfrak{M}F)(z_0 + n_0 t)}{dt} \right| + \kappa M.$$

Referring now to (27) and keeping in mind that

$$\omega'_{\mathscr{E}}(\zeta, \infty) = \frac{d\omega_{\mathscr{E}}(\zeta,\infty)}{|d\zeta|} = \frac{|\phi'(\zeta)|}{2\pi}$$

for $\zeta \in \Gamma$, we see from this that

$$\limsup_{r\to 1+} \left| \frac{\partial V(re^{i\vartheta_0})}{\partial r} \right| \leq \frac{(1+\kappa)MC''}{2\pi} + \kappa M. \tag{32}$$

This holds at all points $e^{i\vartheta_0}$ of the unit circle.

Now the function

$$r\,\frac{\partial V(re^{i\vartheta})}{\partial r}$$

is *also* harmonic for $r > 1$ including at $\infty$; that can, for instance, be seen by looking at the expansion of $V(re^{i\vartheta})$ in negative powers of $r$. From (32) and the principle of maximum we therefore have

$$\left| \frac{\partial V(re^{i\vartheta})}{\partial r} \right| \leq \frac{(1+\kappa)ML}{r} \qquad \text{for } r > 1, \tag{33}$$

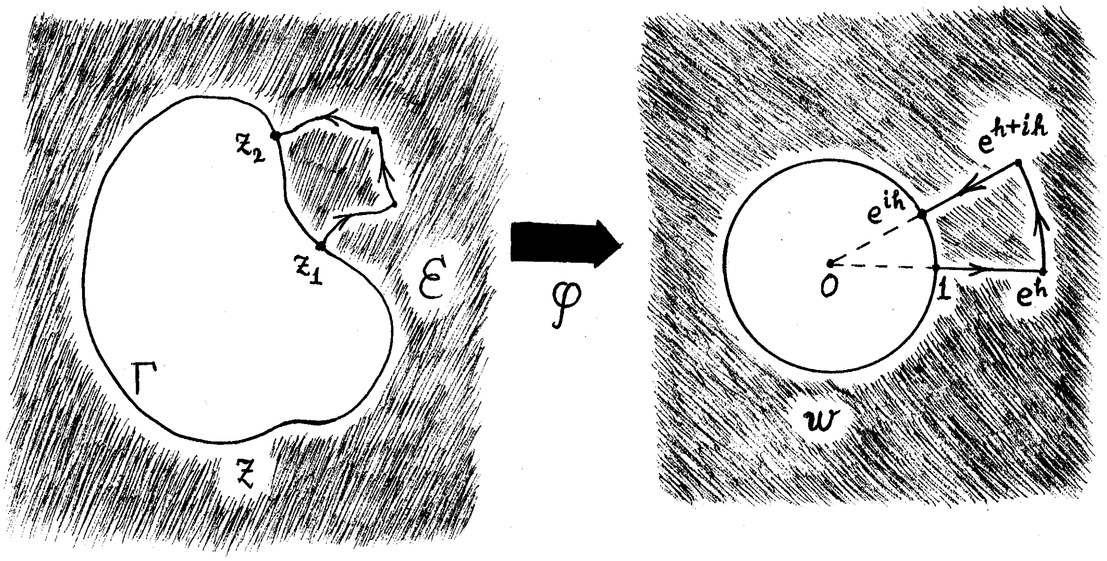
with a constant $L$ depending only on $\Gamma$.

[2] See footnote at beginning of Section 4.

Suppose now that we are given two points $z_1$, $z_2$ on $\Gamma$; without loss of generality $\phi(z_1) = 1$ and $\phi(z_2) = e^{ih}$ with $h > 0$. Then

$$(34) \qquad (\mathfrak{M}F)(z_1) = U(z_1) = V(1), \qquad (\mathfrak{M}F)(z_2) = U(z_2) = V(e^{ih}),$$

and our task has boiled down to the comparison of $V(e^{ih})$ with $V(1)$. The procedure is suggested by the figure.




We have, by (33),

$$V(e^h) \le V(1) + (1+\kappa)MLh.$$

Again, since $V(w)$ is harmonic and $> 0$ for $|w| > 1$,

$$V(e^{h+ih}) \le 3V(e^h)$$

by Harnack. Finally,

$$V(e^{ih}) \le V(e^{h+ih}) + (1+\kappa)MLh,$$

again by (33).

Putting these estimates together, we get

$$V(e^{ih}) \le 3V(1) + 4(1+\kappa)MLh$$

with, here, $h = 2\pi\omega_{\mathscr{E}}(\widehat{z_1, z_2}, \infty)$. From this and (34) we immediately obtain (29), proving the theorem.

**Remark.** The result obtained will suffice for our purposes. Relation (29) could, however, be much improved by going out, in the above argument, to a circle $|w| = r_0$ with $r_0 > 1$ suitably chosen rather than to the circle $|w| = e^h$.

**6.** Let us carry on with the main plan of this paper. Fixing a number $N > (1+\kappa)M$, we take $N$ points $\zeta_1, \dots \zeta_N$ on $\Gamma$, equidistant thereon with respect to the measure $d\omega_{\mathscr{E}}(\zeta, \infty)$, and seek to estimate $\int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,\infty)$ in terms of the *average* $(1/N)\sum_{n=1}^N \log^+ |P(\zeta_n)|$ for polynomials $P(z)$ of degree $M$. From an observation made at the beginning of Section 1 it follows that the preceding integral can, in the case of Kellogg smoothness, be used to estimate $\int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{D}}(\zeta, z_0)$ for each $z_0 \in \mathscr{D}$; our program, if successful, will thus enable us to give a bound on $|P(z_0)|$ *in terms of the average* for such $z_0$.

On the closed subset $E$ of $\Gamma$ we have $(\mathfrak{M}F)(\zeta) = \log^+ |P(\zeta)|$. At the same time, if a point $\zeta_n$ figuring in the average is *not too far* from a $\zeta \in E$, $(\mathfrak{M}F)(\zeta_n)$ and $(\mathfrak{M}F)(\zeta)$ are roughly comparable by Theorem 1. This suggests that we try to use the values $(\mathfrak{M}F)(\zeta_n)$ to get an upper bound on $\int_\Gamma (\mathfrak{M}F)(\zeta)\, d\varrho(\zeta)$, taking advantage of $\varrho$'s being carried on $E$. Further support for this idea comes from

**Lemma 4.** *We have*

$$\int_\Gamma (\mathfrak{M}F)(\zeta)\, d\varrho(\zeta) \geq \kappa M \int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,\infty). \tag{35}$$

*Proof.* By (13), (16) and the relation $\varrho(\Gamma) = \kappa M$, we have

$$\begin{aligned} \int_\Gamma (\mathfrak{M}F)(\zeta)\, d\varrho(\zeta) \geq & \int_\Gamma \int_\Gamma \log \frac{1}{|z-\zeta|}\, d\varrho(\zeta)\, d\varrho(z) \\ & + \kappa M \int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,\infty) - (\kappa M)^2 \gamma_{\mathscr{E}}. \end{aligned} \tag{36}$$

Now $G_{\mathscr{E}}(z,\infty)$ vanishes on $\Gamma$, so by (15),

$$\gamma_{\mathscr{E}} = \int_\Gamma \log \frac{1}{|z-\zeta|}\, d\omega_{\mathscr{E}}(\zeta,\infty) \qquad \text{for } z \in \Gamma.$$

It is, moreover, known from Frostman's theorem ([13, p. 59]) that for positive measures $\mu$ on $\Gamma$ with $\mu(\Gamma) = 1$,

$$\int_\Gamma \int_\Gamma \log \frac{1}{|z-\zeta|}\, d\mu(\zeta)\, d\mu(z)$$

*assumes its minimum* for $\mu = \omega_{\mathscr{E}}(\,\cdot\,,\infty)$; that minimum is therefore equal to $\gamma_{\mathscr{E}}$.

We see from this that the double integral in (36) is $\geq \big(\varrho(\Gamma)\big)^2 \gamma_{\mathscr{E}} = (\kappa M)^2 \gamma_{\mathscr{E}}$. The whole right side of (36) is therefore

$$\geq \kappa M \int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,\infty),$$

and we have (35), as required.

Theorem 1 now enables us, as mentioned above, to give an upper bound for the *left* side of (35) in terms of the values $(\mathfrak{M}F)(\zeta_n)$, but if that bound is to be of any use to us we must know that those values are not much larger than $\log^+|P(\zeta_n)|$ for at least *some* of the $\zeta_n$. There is hope of our being able to verify this for the points $\zeta_n$ near the set $E$ on which $(\mathfrak{M}F)(\zeta)$ and $\log^+|P(\zeta)|$ are equal. It does not, however, seem possible to do that by working *solely* in $\mathscr{D}$ or in $\mathscr{E}$, and we will have to make use of $P(z)$'s analyticity *across* the curve $\Gamma$.

**7.** According to the idea just put forth, $P(z)$ should at least *not vanish* at too many of the points $\zeta_n$ adjacent to any $\zeta_0 \in E$. We are able to show this for such points $\zeta_0$ where $|P(\zeta_0)| \geq 1$, and shall indeed need to know a bit *more* than that.

In order to make things explicit, let us specify once and for all that the $N$ points $\zeta_1, \zeta_2, \ldots \zeta_N$ are taken to be arranged in that order around the curve $\Gamma$, and in such fashion as to have

$$\omega_{\mathscr{E}}(\widehat{\zeta_n, \zeta_{n+1}}, \infty) = \omega_{\mathscr{E}}(\widehat{\zeta_N, \zeta_1}, \infty) = \frac{1}{N} \tag{37}$$

for each of the arcs $\widehat{\zeta_1, \zeta_2}$, $\widehat{\zeta_2, \zeta_3}$, $\ldots$ $\widehat{\zeta_N, \zeta_1}$ of $\Gamma$ between two neighbouring points $\zeta_n$. *We shall have to take $N$ quite large.* That will entail *no restriction* in our *final results* because, for *bounded* $N$, any limitation on the *average* $(1/N)\sum_{n=1}^{N} \log^+|P(\zeta_n)|$ implies a corresponding one (perhaps enormous, but no matter!) on the *individual values* $P(\zeta_n)$, and thereby (e.g., via Lagrange's interpolation formula) a limitation on $|P(z_0)|$ for each $z_0 \in \mathscr{D}$, *as long as the degree* $M$ of $P$ is $< N$.

It will in fact be necessary to have the ratio $M/N$ bounded away from $1$. With that in mind, *we first fix* a number $\lambda < 1$ and *then* require the quantity $\kappa > 0$ introduced at the beginning of Section 2 to be so small as to still have

$$(1+\kappa)\frac{M}{N} < \lambda \tag{38}$$

for the degree $M$ of any polynomial $P(z)$ under consideration. The estimates we obtain will depend on this number $\lambda$.

From now on, it will *always be assumed that the curve* $\Gamma$ *has Kellogg smoothness*, usually without further mention in the various hypotheses.

If $|P(\zeta_0)| \geq 1$ for a point $\zeta_0 \in E$, it will turn out that $P(\zeta_n)$ cannot, as stated above, vanish for too many of the $\zeta_n$ next to $\zeta_0$, and that will depend on the analyticity of $P(z)$ across an arc $\sigma$ of $\Gamma$ on which the $\zeta_n$ in question are supposed to lie. Now $P(z)$ is, of course, analytic *everywhere*, but we shall need the corresponding result for certain functions $H(z)$ analytic in $(\mathscr{D} \cup \mathscr{E}) \cap \mathbf{C}$ and across $\sigma$, but not necessarily elsewhere. The behaviour of these functions imitates that of $P(z)$ in its relation to $(\mathfrak{M}F)(z)$.

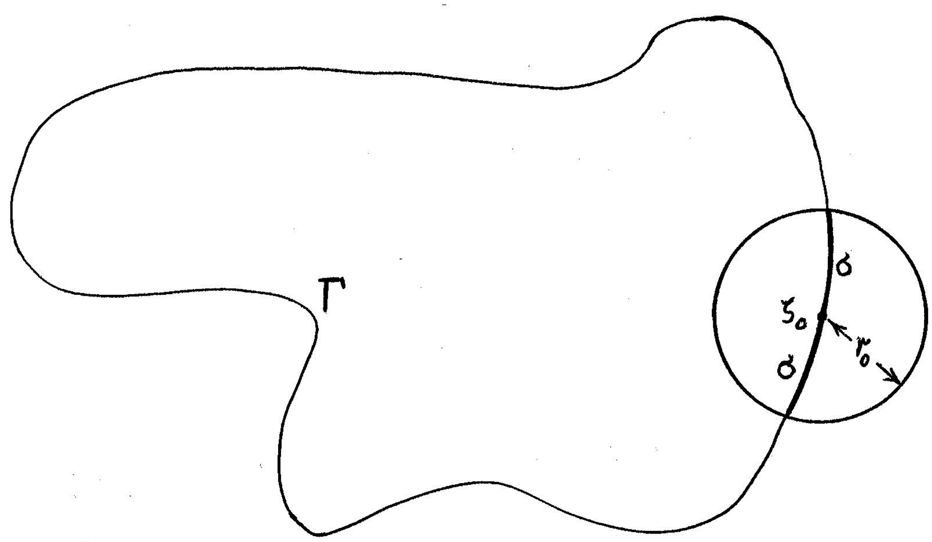
The difference $\log|P(z)| - (\mathfrak{M}F)(z)$ is subharmonic everywhere (save at $\infty$), so its magnitude in $\mathbf{C}$ is governed by how big it is on $\Gamma$ and its growth at $\infty$. On $\Gamma$, we have

$$\log|P(z)| - (\mathfrak{M}F)(z) \le \log^+|P(z)| - (\mathfrak{M}F)(z) \le 0,$$

and when $z \to \infty$ both differences go to $\infty$ like $(1+\kappa)M\log|z|$. Therefore

$$\log|P(z)| - (\mathfrak{M}F)(z) \le (1+\kappa)MG_{\mathscr{E}}(z,\infty)$$

(everywhere) by the principle of maximum (cf. the derivation of (2) in Section 1). *The functions $H(z)$ to be considered here will also satisfy this condition.*




The arc $\sigma$ over which analytic continuation of one of our functions $H(z)$ is to be assumed possible can be given the following description. Fixing any $\zeta_0$ on $\Gamma$, we describe a circle of (small) radius $r_0$, say, about $\zeta_0$, and take for $\sigma$ the arc of $\Gamma$ included within that circle. The radius $r_0$ is furthermore to be so chosen as to make

$$\omega_{\mathscr{E}}(\sigma,\infty) = \frac{2k+1}{N}, \tag{39}$$

where $k$ is some fixed and reasonably large integer, much smaller, however, than $N$.

Under these circumstances, we have

**Lemma 5.** *Suppose that for $H(z)$, analytic in $(\mathscr{D}\cup\mathscr{E})\cap\mathbf{C}$ and across the arc $\sigma$ of $\Gamma$, we have*

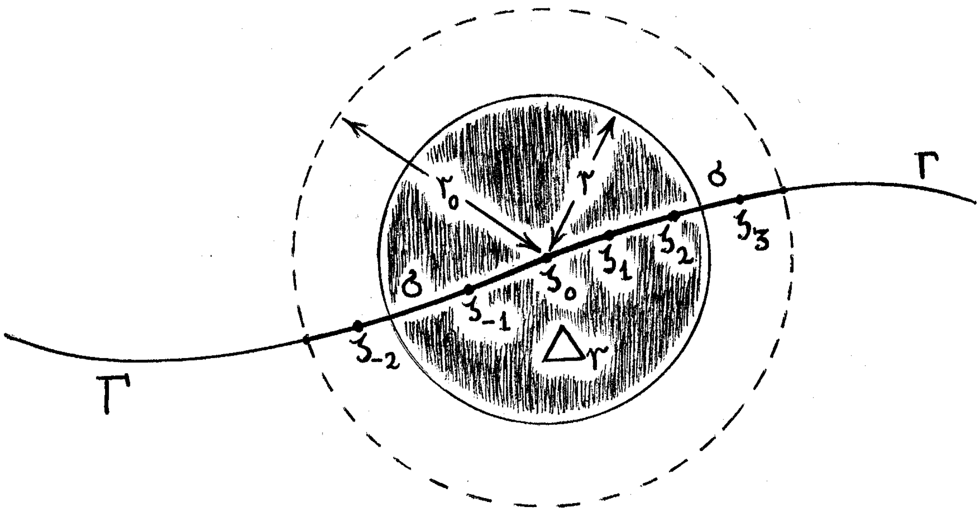
$$\log|H(z)| - (\mathfrak{M}F)(z) \le (1+\kappa)MG_{\mathscr{E}}(z,\infty) \tag{40}$$

(*wherever* $H(z)$ *is defined*), *with* $M$ *and* $\kappa > 0$ *satisfying* (38) *for some fixed* $\lambda < 1$. *Suppose also that*

$$\log |H(\zeta_0)| - (\mathfrak{M}F)(\zeta_0) \geq -c_0 > -\infty, \tag{41}$$

*where* $\zeta_0 \in \Gamma$ *is the centre of the circle cutting off the arc* $\sigma$ *on* $\Gamma$. *Then, if the integer* $k$ *in* (39) *is taken large enough* (*depending on* $\lambda$ *and* $c_0$), *the function* $H(z)$ *cannot, for sufficiently large* $N$ (*depending on* $\lambda$ *and on the curve* $\Gamma$), *vanish at all of the points* $\zeta_n$ *on* $\sigma$.

*Proof.* By contradiction, following the idea used in proving Lemma 2. Suppose, then, that $H(z)$ *does* vanish at each of the points $\zeta_n$ on $\sigma$. It is convenient for the following discussion to *re-index those points*, denoting those to *one* side of $\zeta_0$ by $\zeta_1, \zeta_2, \ldots$ etc. (in that order), and those to the *other* side by $\zeta_{-1}, \zeta_{-2}, \ldots$ successively. (According to (41), $\zeta_0$ cannot *coincide* with any of the points $\zeta_n$ on $\sigma$.) The disk $\{z\,;\,|z-\zeta_0| < r\}$ is designated by $\Delta_r$.




Let $m(\Delta_r)$ be *the number of zeros of* $H(z)$ in $\Delta_r$. In view of (13), (40) and (41) then imply, by Jensen's formula,

$$\int_0^{r_0} \frac{m(\Delta_r) + \varrho(\Delta_r)}{r}\, dr - c_0 \leq \frac{(1+\kappa)M}{2\pi r_0} \int_{\partial \Delta_{r_0}} G_{\mathscr{E}}(\zeta, \infty)\, |d\zeta|.$$

As in the proof of Lemma 2, the right side of this relation equals

$$(1+\kappa)M \int_0^{r_0} \frac{\omega_{\mathscr{E}}(\Delta_r, \infty)}{r}\, dr.$$

Therefore, since $\varrho \geq 0$,

$$\int_0^{r_0} \frac{m(\Delta_r)}{r}\,dr \leq (1+\kappa)M \int_0^{r_0} \frac{\omega_{\mathscr{E}}(\Delta_r,\infty)}{r}\,dr + c_0 \tag{42}$$

(cf. (21)).

We have supposed that $H(z)$ vanishes at each of the points $\zeta_n$ on $\sigma$, therefore $m(\Delta_r)$ is at least equal to the *number of such points* in the disk $\Delta_r$. Assume, then, that the circle of radius $r < r_0$ about $\zeta_0$ cuts $\sigma$ on one side of $\zeta_0$ between the points $\zeta_{-p}$ and $\zeta_{-p-1}$, $p \geq 1$, and on the other side between $\zeta_q$ and $\zeta_{q+1}$, $q \geq 1$. Then $m(\Delta_r) \geq p+q$, but by (37),

$$\omega_{\mathscr{E}}(\Delta_r,\infty) \leq \frac{p+q+1}{N}.$$

In this situation, we thus have

$$m(\Delta_r) \geq N\omega_{\mathscr{E}}(\Delta_r,\infty) - 1, \tag{43}$$

and it is seen in the same way that this relation also holds when $\partial\Delta_r$ only surrounds points $\zeta_n$ lying to one side of $\zeta_0$, or when it encloses none of the $\zeta_n$.

Choosing now an $r_1$, $0 < r_1 < r_0$, to be specified in a moment, we have, by (43),

$$\begin{aligned}\int_0^{r_0} \frac{m(\Delta_r)}{r}\,dr &\geq \int_{r_1}^{r_0} \frac{m(\Delta_r)}{r}\,dr \geq N\int_{r_1}^{r_0} \frac{\omega_{\mathscr{E}}(\Delta_r,\infty)}{r}\,dr - \log\frac{r_0}{r_1}\\ &= N\int_0^{r_0} \frac{\omega_{\mathscr{E}}(\Delta_r,\infty)}{r}\,dr - N\int_0^{r_1} \frac{\omega_{\mathscr{E}}(\Delta_r,\infty)}{r}\,dr - \log\frac{r_0}{r_1},\end{aligned}$$

and substitution of this into (42) yields, after transposition,

$$\begin{aligned}N\int_0^{r_0} \frac{\omega_{\mathscr{E}}(\Delta_r,\infty)}{r}\,dr &\leq (1+\kappa)M\int_0^{r_0} \frac{\omega_{\mathscr{E}}(\Delta_r,\infty)}{r}\,dr\\ &\qquad + N\int_0^{r_1} \frac{\omega_{\mathscr{E}}(\Delta_r,\infty)}{r}\,dr + \log\frac{r_0}{r_1} + c_0.\end{aligned} \tag{44}$$

If $r_0$ is small enough (independently of the position of $\zeta_0$ on $\Gamma$), we have, for $0 < r < r_0$,

$$(1-\varepsilon)\frac{\omega_{\mathscr{E}}(\Delta_{r_0},\infty)}{r_0} \leq \frac{\omega_{\mathscr{E}}(\Delta_r,\infty)}{r} \leq (1+\varepsilon)\frac{\omega_{\mathscr{E}}(\Delta_{r_0},\infty)}{r_0} \tag{45}$$

where $\varepsilon > 0$ is arbitrary, thanks to the continuity of $d\omega_{\mathscr{E}}(\zeta,\infty)/|d\zeta|$ (and to its non-vanishing) due, in turn, to the Kellogg smoothness of $\Gamma$. Since the latter property ensures that $d\omega_{\mathscr{E}}(\zeta,\infty)/|d\zeta|$ is bounded away from 0 on $\Gamma$, we can, for any *fixed* $k$, make $r_0$—determined by (39)—and hence the $\varepsilon$ in (45) *as small as we like* by taking $N$ sufficiently large (independently of $\zeta_0$).

From (44), (45) and (39) we get

$$(1-\varepsilon)(2k+1) \le (1+\varepsilon)(1+\kappa)\frac{M}{N}(2k+1) + (1+\varepsilon)(2k+1)\frac{r_1}{r_0} + \log\frac{r_0}{r_1} + c_0,$$

that is, by (38),

$$2k+1 \le \frac{1+\varepsilon}{1-\varepsilon}\left(\lambda + \frac{r_1}{r_0}\right)(2k+1) + \frac{1}{1-\varepsilon}\left(\log\frac{r_0}{r_1} + c_0\right). \tag{46}$$

Here $\lambda$ was given as $<1$, so we can *first* assign a value $\tau > 0$ to the ratio $r_1/r_0$, so small as to make $\lambda + \tau < \sqrt{\lambda}$ (say), and *then* decide on a value for $\varepsilon$, $0 < \varepsilon < \frac{1}{2}$, small enough for us to still have

$$\frac{1+\varepsilon}{1-\varepsilon}(\lambda+\tau) < \sqrt{\lambda}; \tag{47}$$

with *these* choices of $\tau$ and $\varepsilon$ (46) will read

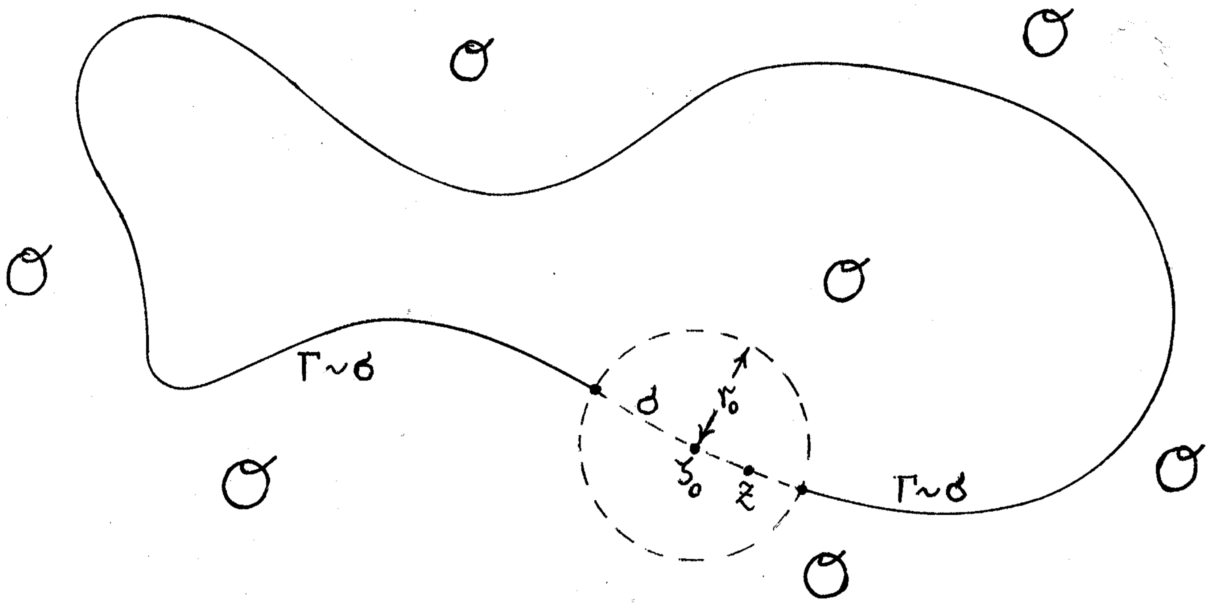
$$2k+1 \le \sqrt{\lambda}\,(2k+1) + 2\left(\log\frac{1}{\tau} + c_0\right). \tag{48}$$

But this relation *cannot hold* for arbitrarily large $k$, $\sqrt{\lambda}$ being $<1$. *Fix*, then, *a value of $k$ for which* (48) *fails*, and *then* consider the $N$ for which (45) holds with the $\varepsilon$ just agreed upon, small enough to guarantee (47). As just observed, *all sufficiently large $N$ will have that property*. For *such $N$, $H(z)$ cannot vanish* at all the points $\zeta_n$ on the arc $\sigma$ determined in the way described earlier, by using our chosen value of $k$ in (39). *Otherwise*, (48) *would hold*, but it *does not*, due to the choice of $k$. The lemma is proved.

**Remark 1.** The result just obtained is analogous to Lemma 1 of [7], and the method followed here can also be applied to yield the latter. Such an argument would be simpler and shorter than the original one in [7].

**Remark 2.** In an earlier and more elaborate proof of Lemma 5 an intermediate result was used which is no longer needed here but seems interesting in its own right.

Fixing any $\zeta_0 \in \Gamma$ we consider, as above, the arc $\sigma$ cut off on $\Gamma$ by a circle of *small* radius $r_0$ about $\zeta_0$, and imagine that $\sigma$ has then been *removed* from $\Gamma$, yielding a new domain $\mathscr{O} \supset \mathscr{D} \cup \mathscr{E}$ bounded by $\Gamma \sim \sigma$.




Taking any $z$ on the excised arc $\sigma$, let us consider *the ratio*

$$\frac{G_{\mathscr{O}}(z, \infty)}{\omega_{\mathscr{E}}(\sigma, \infty)}.$$

*When* $r_0$ *is small this ratio is practically equal to*

$$\frac{\pi}{2}\sqrt{1 - |z - \zeta_0|^2/r_0^2}\,,$$

*and the more nearly so as* $r_0$ *gets smaller. In particular, the limiting value of* $G_{\mathscr{O}}(\zeta_0, \infty)/\omega_{\mathscr{E}}(\sigma, \infty)$ *for* $r_0 \to 0$ *is* $\frac{1}{2}\pi$.

This result must be known, but I did not know it! To prove it, one first writes

$$G_{\mathscr{O}}(z, \infty) = \int_\sigma G_{\mathscr{O}}(z, \zeta)\, d\omega_{\mathscr{E}}(\zeta, \infty).$$

When $r_0$ is small, $d\omega_{\mathscr{E}}(\zeta, \infty)$ can be safely replaced by $\big(\omega_{\mathscr{E}}(\sigma, \infty)/2r_0\big)\,|d\zeta|$ in the integral.

In order to get an idea of $G_{\mathscr{O}}(z, \zeta)$ when $z$ and $\zeta$ *both* lie on $\sigma$, we do a blow-up, using an affine transformation from the $z$-plane to the $w$-plane which sends $\zeta_0$ to $0$, the circle of radius $r_0$ about $\zeta_0$ to $|w| = 1$, and the tangent of $\Gamma$ at $\zeta_0$ to the real axis. This mapping takes $\mathscr{O}$ to a domain $\Omega$ bounded by the affine image of $\Gamma \sim \sigma$. When $r_0$ is small, $\Omega$ practically coincides, at moderate distances from the origin, with the *complement* $\Omega_0$ of $(-\infty, -1] \cup [1, \infty)$ in $\mathbf{C}$, and the more so as $r_0$ gets smaller.

Denoting by $w$ the affine image of $z \in \sigma$ and by $w'$ that of $\zeta \in \sigma$, we of course have

$$G_{\mathscr{O}}(z, \zeta) = G_{\Omega}(w, w');$$

*here*, however, $G_{\Omega}(w, w')$ is nearly equal to $G_{\Omega_0}(w, w')$ (in the appropriate sense) when $r_0$ is so small as to make $\Omega$ about the same as $\Omega_0$. The function $G_{\Omega_0}(w, w')$ may thus be used to obtain an approximation to $G_{\mathscr{O}}(z, \zeta)$, appearing in the above integral representation of $G_{\mathscr{O}}(z, \infty)$. An explicit formula for $G_{\Omega_0}(w, w')$ is, however, available. Using it (after suitable transformation) in the integral, one obtains the above approximation to the ratio in question.

This procedure of course involves a dominated convergence argument, but the results to be given in the next section furnish what is needed to carry that through.

I think that the same kind of relation can be proved, and in more or less the same way, for sufficiently smooth closed surfaces in $\mathbf{R}^n$.

The limiting value $\frac{1}{2}\pi$ of the ratio $G_{\mathscr{O}}(\zeta_0, \infty)/\omega_{\mathscr{E}}(\sigma, \infty)$ is directly related to the upper limit 1 on the quantity $\lambda$ figuring in (38). This value $\frac{1}{2}\pi$ is *not*, however, universal, but depends on the smoothness of $\Gamma$. If, for instance, $\Gamma$ has a *corner* (of opening $\neq \pi$) at $\zeta_0$ but is otherwise smooth enough, a conformal mapping described on pp. 189–190 of [9] can be used to show that the limiting ratio still exists but is *not* $\frac{1}{2}\pi$; instead it is a complicated function of the opening at $\zeta_0$. I think this example could be used to show that the final results of this paper, established for polynomials whose degree $M$ satisfies (38) with $\lambda < 1$, are no longer valid for curves $\Gamma$ with less smoothness unless the ratio $M/N$ is subjected to a more severe restriction. That would be worthwhile.

**8.** We wish to continue along the lines of Section 1 in [7]. Taking a point $\zeta_0$ on $\Gamma$ we consider, as in the last section, the arc $\sigma$ of $\Gamma$ cut off by a circle of (small) radius $r_0$ about $\zeta_0$ and, as in Remark 2 at the end of that section, the complement $\mathscr{O}$ (on the Riemann sphere) of the closed arc $\Gamma \sim \sigma$. An estimate of $G_{\mathscr{O}}(z, z')$ will be needed, and its derivation is simplified if one first makes the affine transformation described in Remark 2. We therefore put

$$w = \frac{\omega}{r_0}(z - \zeta_0), \tag{49}$$

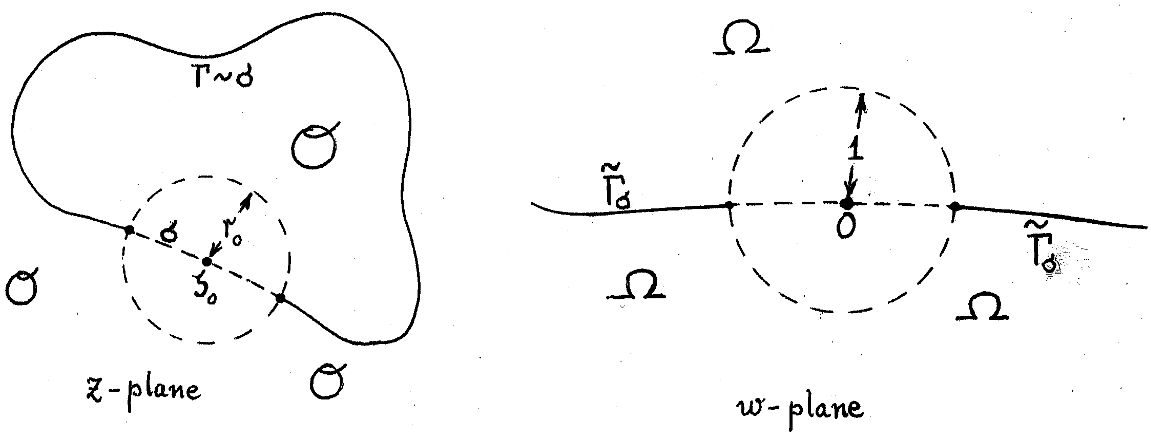
with a constant $\omega$, $|\omega| = 1$, so chosen as to make the tangent to $\Gamma$ at $\zeta_0$ correspond to the real axis in the $w$-plane.

Under the mapping

$$z \longrightarrow w,$$

$\Gamma \sim \sigma$ goes over to an arc $\widetilde{\Gamma}_\sigma$ in the $w$-plane, and $\mathscr{O}$ to that arc's complement $\Omega$. Note that $\infty \in \Omega$, since $\infty \in \mathscr{O}$. If, in that mapping, $z$ and $z'$ correspond respectively to $w$ and $w'$, we (of course) have

$$G_{\mathscr{O}}(z, z') = G_{\Omega}(w, w'), \tag{50}$$




and we proceed to estimate the Green's function standing on the right for certain pairs $w$, $w'$. For this we are helped by the circumstance that $\Omega$ includes the unit disk while $\partial\Omega = \widetilde{\Gamma}_\sigma$ has two points on the boundary of that disk.

**Lemma 6.** *When $r_0 > 0$ is small enough (independently of the position of $\zeta_0$ on $\Gamma$), we have*

$$G_\Omega(w, 0) \le \log \frac{20}{|w|} \qquad \text{for } |w| \le 1. \tag{51}$$

*Proof.* Let $w \to Z = f(w)$ be a conformal mapping of the simply connected domain $\Omega$ onto the unit disk in the $Z$-plane, with $f(0) = 0$. Then

$$G_\Omega(w, 0) = \log \frac{1}{|f(w)|} \qquad \text{for } w \in \Omega. \tag{52}$$

The function $f$ maps $\{|w| < 1\} \subset \Omega$ conformally onto a certain domain $\mathscr{U}$, $0 \in \mathscr{U}$, lying inside the disk $\{|Z| < 1\}$. Letting

$$\delta = \inf\{|f(w)|\,;\ |w| = 1\}$$

be the distance from $\partial\mathscr{U}$ to $0$, we see that the function

$$F(w) = \frac{f(w)}{f'(0)},$$

univalent for $|w| < 1$, has $F(0) = 0$, $F'(0) = 1$, and takes $\{|w| < 1\}$ conformally onto a domain $\big(1/f'(0)\big)\cdot\mathscr{U}$, whose boundary is distant by $\delta/|f'(0)|$ units from $0$. It follows by the Koebe $\frac{1}{4}$-theorem ([13, p. 140] or [14, p. 288]) that $\delta/|f'(0)| \ge \frac{1}{4}$, i.e., that

$$\delta \ge \tfrac{1}{4}|f'(0)|. \tag{53}$$

In order to get a lower bound on $|f'(0)|$, we consider the mapping

$$Z \to w = \phi(Z)$$

*inverse* to $w \to Z = f(w)$, which takes $\{|Z| < 1\}$ conformally onto $\Omega$. For the function

$$\Phi(Z) = \frac{\phi(Z)}{\phi'(0)},$$

univalent in the unit disk, we do have $\Phi(0) = 0$ and $\Phi'(0) = 1$, but Koebe's theorem cannot be directly applied to $\Phi(Z)$ since $\infty \in \Omega$, causing $\Phi(Z)$ to have *a pole* at some $Z$, $|Z| < 1$.

To deal with this difficulty, we must take account of the fact that for any $\zeta_0$ on $\Gamma$, one has points on $\Gamma$ *distant by at least a certain* $h > 0$ *from* $\zeta_0$, where $h$ depends on $\Gamma$ but is *independent* of the choice of $\zeta_0$ thereon. One may, for instance, take $h = \frac{1}{2}\,\mathrm{diam.}\,\Gamma$ (observation of H.L. Pedersen). When $r_0 < h$ we will thus have points $w_0$ on $\widetilde{\Gamma}_\sigma$ with

$$|w_0| \geq \frac{h}{r_0} > 1,$$

and by taking $r_0$ small enough we can ensure that $|w_0| \geq 5$, say.

Considering henceforth any small $r_0 > 0$ for which that is possible, we *fix* a $w_0$, $|w_0| \geq 5$, on $\widetilde{\Gamma}_\sigma$, and put

$$\Psi(Z) = \frac{w_0 \Phi(Z)}{w_0 - \phi'(0)\Phi(Z)}$$

for $|Z| < 1$. Since $\phi'(0)\Phi(Z) = \phi(Z)$ never takes the value $w_0$ there, $\Psi(Z)$ is *analytic* in the unit disk, and (like $\Phi(Z)$) *univalent* there; moreover, $\Psi(0) = 0$, $\Psi'(0) = 1$. Koebe's theorem is therefore applicable to $\Psi(Z)$.

The boundary $\widetilde{\Gamma}_\sigma$ of $\Omega$ has on it a point $w_1$ with $|w_1| = 1$, so $\Phi(Z)$ never takes the value $w_1/\phi'(0)$ for $|Z| < 1$. The function $\Psi(Z)$ thus never takes the value

$$\frac{w_0 w_1}{w_0 - w_1} \cdot \frac{1}{\phi'(0)}$$

in the unit disk, and by the $\frac{1}{4}$-theorem we have

$$\left| \frac{w_0 w_1}{w_0 - w_1} \right| \cdot \frac{1}{|\phi'(0)|} \geq \frac{1}{4}.$$

Using the values $|w_0| \geq 5$, $|w_1| = 1$, we find from this that

$$\frac{1}{|\phi'(0)|} \geq \frac{1}{5},$$

i.e., that $|f'(0)| \geq \frac{1}{5}$, since $\phi$ is the inverse function to $f$.

Substituting the last relation into (53), we obtain

$$\delta \geq \frac{1}{20},$$

i.e., $|f(w)| \geq 1/20$ for $|w| = 1$, and thence

$$\left|\frac{w}{f(w)}\right| \leq 20 \qquad \text{for } |w| \leq 1 \tag{54}$$

by the principle of maximum, $w/f(w)$ being analytic in the unit disk, and continuous up to its boundary. Referring now to (52), we have the lemma by (54).

**Corollary 1.** *Let* $|w| \leq 1$, $|w_0| < 1$. *Then*

$$G_\Omega(w, w_0) \leq \log \frac{40}{|w - w_0|} \tag{55}$$

*as long as* $r_0 > 0$ *is sufficiently small* (*independently of the position of* $\zeta_0$ *on* $\Gamma$).

*Proof.* We can write

$$G_\Omega(w, 0) = \log \frac{1}{|w|} + h(w, 0) \tag{56}$$

for $w \in \Omega$, with $h(w, 0)$ *harmonic* at the *finite points* of that domain and, like $G_\Omega(w, 0)$, continuous down to $\widetilde{\Gamma}_\sigma = \partial\Omega$. The behaviour of $h(w, 0)$ at $\infty$ is governed by the requirement that $G_\Omega(w, 0)$ *remain finite* there; we thus have

$$h(w, 0) = \log |w| + O(1) \qquad \text{for } w \to \infty.$$

Fixing a $w_0$, $|w_0| < 1$, we have similarly

$$G_\Omega(w, w_0) = \log \frac{1}{|w - w_0|} + h(w, w_0) \tag{57}$$

in $\Omega$, with $h(w, w_0)$ harmonic at the finite points of $\Omega$, continuous down to $\widetilde{\Gamma}_\sigma$, and *behaving in the same way as* $h(w, 0)$ *for* $w \to \infty$.

The difference

$$h(w, w_0) - h(w, 0)$$

is thus *harmonic and bounded* in $\Omega$ (including at $\infty$). It is also continuous down to $\widetilde{\Gamma}_\sigma$, where $G_\Omega(w, w_0)$ and $G_\Omega(w, 0)$ both vanish, so we have

$$h(w, w_0) - h(w, 0) = \log\left|\frac{w - w_0}{w}\right| \qquad \text{for } w \in \widetilde{\Gamma}_\sigma$$

by (56) and (57). Since $|w| \geq 1$ when $w \in \widetilde{\Gamma}_\sigma$, the left side of this last relation is $\leq \log(1 + |w_0|) < \log 2$ on $\widetilde{\Gamma}_\sigma$, and the principle of maximum thence yields

$$h(w, w_0) - h(w, 0) \leq \log 2 \tag{58}$$

for all $w \in \Omega$.

When $|w| = 1$, we have, by (51) (from Lemma 6) and (56),

$$h(w, 0) \leq \log 20;$$

this must then hold for $|w| \leq 1$ by the principle of maximum, making

$$h(w, w_0) \leq \log 40 \qquad \text{for } |w| \leq 1$$

by (58). The desired result now follows from (57).

Let us return, now, to our original domain $\mathscr{O}$, the complement of $\Gamma \sim \sigma$. From (49), (50) and (55) we immediately obtain

**Corollary 2.** *For $z$ and $z'$ inside the circle of radius $r_0$ about $\zeta_0$ (and, in particular, on the arc $\sigma$), we have*

$$G_{\mathscr{O}}(z, z') \leq \log \frac{40 r_0}{|z - z'|}, \tag{59}$$

*as long as $r_0$ is small enough (independently of the position of $\zeta_0$).*

**9.** As previously, $\mathscr{E}$ denotes the exterior of the curve $\Gamma$, while the arc $\sigma$ of $\Gamma$ and the complement $\mathscr{O}$ of $\Gamma \sim \sigma$ are as in the last section.

**Lemma 7.** *If $r_0$ (the radius of the circle about $\zeta_0 \in \Gamma$ cutting off the arc $\sigma$) is small enough, we have*

$$\int_\sigma G_{\mathscr{O}}(z, \zeta)\, d\omega_{\mathscr{E}}(\zeta, \infty) \leq 2(1 + \log 40)\omega_{\mathscr{E}}(\sigma, \infty) \tag{60}$$

*for any point $z$ on $\sigma$.*

*Proof.* Under the conditions of the lemma, the integral in (60) is

$$\leq \int_\sigma \log \frac{40 r_0}{|z - \zeta|}\, d\omega_{\mathscr{E}}(\zeta, \infty)$$

according to (59).

Here one may proceed as in the proof of Lemma 3 (Section 4), taking (wlog) the centre $\zeta_0$ of the circle cutting off $\sigma$ to be at *the origin*, and the *tangent* to $\Gamma$ at $\zeta_0$ as the *axis of abscissae*. (The reader may refer to the figure accompanying the proof of Lemma 3.) We can then write $\zeta = \xi + i\eta(\xi)$ for the points $\zeta$ of $\sigma$, and the last integral takes the form

$$\int_{-r_0'}^{r_0''} \left(\log \frac{40r_0}{|z-\zeta|}\right) \frac{d\omega_{\mathscr{E}}(\zeta,\infty)}{|d\zeta|} \sqrt{1+\left(\eta'(\xi)\right)^2}\, d\xi,$$

where $0 < r_0' \le r_0$, $0 < r_0'' \le r_0$, both $r_0'$ and $r_0''$ being very close to $r_0$.

In the last expression the logarithm is made *larger* when we replace its argument by $40r_0/|x-\xi|$ with $x = \Re z$; also, $d\omega_{\mathscr{E}}(\zeta,\infty)/|d\zeta|$ is very near to $\omega_{\mathscr{E}}(\sigma,\infty)/2r_0$ along $\sigma$, and $|\eta'(\xi)|$ very small there, when $r_0$ is small. In that circumstance, what we have is thus

$$\le 2\omega_{\mathscr{E}}(\sigma,\infty) \int_{-r_0'}^{r_0''} \left(\log \frac{40r_0}{|x-\xi|}\right) \frac{d\xi}{2r_0}.$$

This integral is increased when $r_0'$ and $r_0''$ are both replaced by $r_0$, and then it is as large as possible when $x = 0$. Evaluation of the resulting expression now leads directly to (60).

Here is an analogue of Theorem 1 from [7].

**Theorem 2.** *Let $\zeta_0 \in \Gamma$, and suppose that for some $c_0 \ge 0$,*

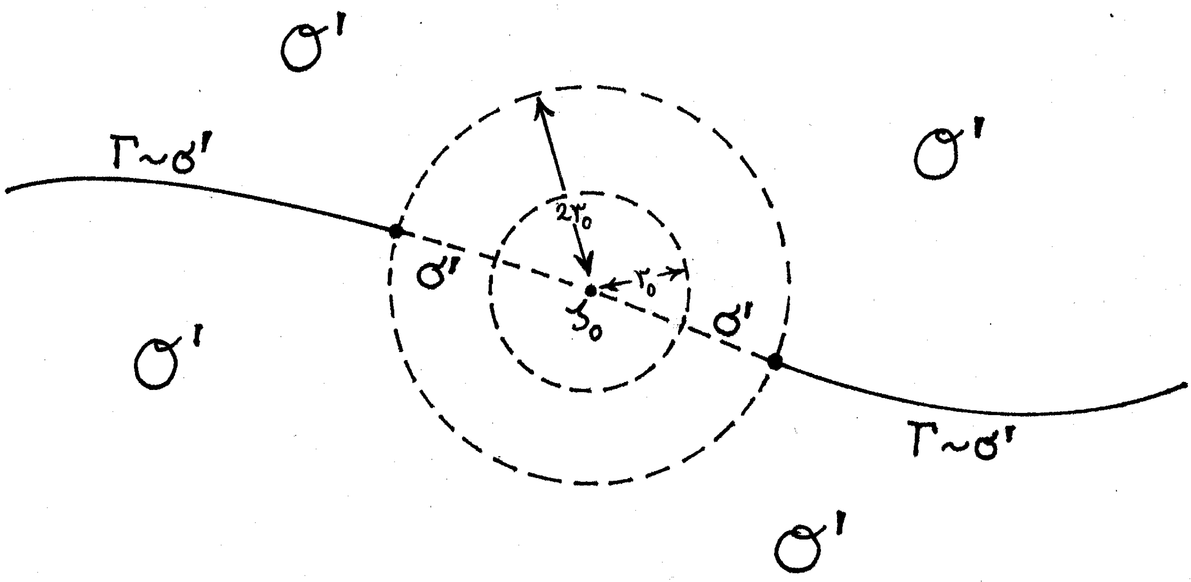
$$\log|P(\zeta_0)| - (\mathfrak{M}F)(\zeta_0) \ge -c_0 > -\infty, \tag{61}$$

*where $P(z)$ is a polynomial whose degree $M$ satisfies* (38) *with some $\kappa > 0$ and $\lambda < 1$, $N$ being large* (*depending on $\lambda$ and the curve $\Gamma$*). *Then, if $\sigma$ is an arc of $\Gamma$, cut off by a circle about $\zeta_0$ and for which* (39) *holds with a suitable integer $k > 0$* (*depending on $c_0$ and $\lambda$*), *we must have*

$$\log|P(\zeta_n)| - (\mathfrak{M}F)(\zeta_n) \ge -c_1 > -\infty \tag{62}$$

*for at least one of the points $\zeta_n$* (*specified by* (37)) *lying on $\sigma$. Here, $c_1$ depends on $c_0$ and $\lambda$.*

*Proof.* By reduction to Lemma 5 (Section 7), using Lagrange interpolation. For this it will be helpful to consider, along with the arc $\sigma$, *another* arc, $\sigma'$, of $\Gamma$, cut off by a circle of radius $2r_0$ about $\zeta_0$, where $r_0$ is the radius of the circle about that point cutting off $\sigma$. We then denote by $\mathscr{O}'$ the complement of $\Gamma \sim \sigma'$ on the Riemann sphere, and we shall work with $G_{\mathscr{O}'}(z,z')$ whose properties are the same as those of the function $G_{\mathscr{O}}(z,z')$ considered in the last section. During the following discussion the quantity $k$ figuring in (39) will be taken as *fixed*, and we shall see at the end how it should be chosen.




Taking $G_{\mathscr{O}'}(z,z')$, we form the function

$$V(z) = \int_{\sigma'} G_{\mathscr{O}'}(z,\zeta)\,d\varrho(\zeta) - (\mathfrak{M}F)(z) + (1-\{\kappa M\})G_{\mathscr{O}'}(z,\infty),$$

where $\{\kappa M\} = \kappa M - [\kappa M]$ is the *fractional part* of $\kappa M$. According to (13), $V(z)$ is *harmonic* in $\mathscr{O}'$, *except at* $\infty$, for the logarithmic singularities of $G_{\mathscr{O}'}(z,\zeta)$ at the points $\zeta$ of $\sigma'$ serve to exactly cancel the contributions to the potential $(\mathfrak{M}F)(z)$ from its Riesz mass on that arc. At $\infty$, $V(z)$ is equal to $([\kappa M]+1)\log|z|$ plus a bounded function, where $[\kappa M]+1$ is an *integer*. We therefore have a function $\psi(z)$, analytic in the *simply connected* region $\mathscr{O}'$ save at $\infty$ where, however, it behaves like const. $z^{1+[\kappa M]}$, with $\log|\psi(z)| = V(z)$, i.e.,

$$\log|\psi(z)| = \int_{\sigma'} G_{\mathscr{O}'}(z,\zeta)\,d\varrho(\zeta) + (1-\{\kappa M\})G_{\mathscr{O}'}(z,\infty) - (\mathfrak{M}F)(z); \tag{63}$$

$\psi(z)$ is, in particular, *never zero* in $\mathscr{O}'$.

Let us estimate the first two right-hand terms of (63) for $z$ on $\sigma'$. For this we use Lemma 7 and, to deal with the integral, Lemma 2 (from Section 3). Referring to (18) and then replacing $\mathscr{O}$ by $\mathscr{O}'$ and $\sigma$ by $\sigma'$ in (60), we find in the first place that

$$\int_{\sigma'} G_{\mathscr{O}'}(z,\zeta)\,d\varrho(\zeta) \le 2(1+\log 40)(1+\kappa)M\omega_{\mathscr{E}}(\sigma',\infty)$$

for $z$ on $\sigma'$. Here $\omega_{\mathscr{E}}(\sigma',\infty)$ has about *twice* the value of $\omega_{\mathscr{E}}(\sigma,\infty)$ given by (39) for large values of $N$ (i.e., for small values of $r_0$), and the last relation then becomes

$$\int_{\sigma'} G_{\mathscr{O}'}(z,\zeta)\,d\varrho(\zeta) \le \frac{5(1+\log 40)(1+\kappa)M(2k+1)}{N} \le 5(1+\log 40)(2k+1)$$

in view of (38), since $\lambda < 1$. Again, $(1 - \{\kappa M\})G_{\mathscr{O}'}(z,\infty) \leq G_{\mathscr{O}'}(z,\infty)$ with, for $z \in \sigma'$,

$$G_{\mathscr{O}'}(z,\infty) = \int_{\sigma'} G_{\mathscr{O}'}(z,\zeta)\, d\omega_{\mathscr{E}}(\zeta,\infty)$$

(cf. Remark 2 at the end of Section 7). The integral on the right is (as above)

$$\leq 5(1 + \log 40)\frac{2k+1}{N}$$

for large $N$ so we have, with the preceding estimate,

$$\int_{\sigma'} G_{\mathscr{O}'}(z,\zeta)\, d\varrho(\zeta) + (1 - \{\kappa M\})G_{\mathscr{O}'}(z,\infty) \leq 5(1 + \log 40)(2k+2) \tag{64}$$

for $z$ on $\sigma'$ and $N$ large.

Write

$$D = 5(1 + \log 40)(2k+2);$$

$D$, like $k$, stays fixed. *Supposing that* (62) *is violated for every* $\zeta_n$ *on* $\sigma$ (sic!), we will then have, by (63) and (64),

$$|P(\zeta_n)\psi(\zeta_n)| < e^{D-c_1} \tag{65}$$

for *each* of the $\zeta_n$ on $\sigma$. This means that the values $P(\zeta_n)\psi(\zeta_n)$ will all be nearly zero if $c_1$ is very large, and that in turn gives us the idea of constructing a *new* function, analytic and *bounded* in $\mathscr{O}'$, taking *the same* values at those points.

For that purpose we take, for each $\zeta_n \in \sigma$, a function $g(z,\zeta_n)$ mapping $\mathscr{O}'$ (sic!) conformally onto the unit disk and sending $\zeta_n$ to $0$. Then, if we put

$$F(z) = \prod_{\zeta_n \in \sigma} g(z,\zeta_n) \qquad \text{for } z \in \mathscr{O}', \tag{66}$$

the function

$$Q(z) = \sum_{\zeta_n \in \sigma} \frac{P(\zeta_n)\psi(\zeta_n)}{(z-\zeta_n)F'(\zeta_n)} F(z), \tag{67}$$

*analytic and bounded* in $\mathscr{O}'$ (where $|F(z)| < 1$), will satisfy

$$P(\zeta_n)\psi(\zeta_n) - Q(\zeta_n) = 0 \qquad \text{for } \zeta_n \in \sigma. \tag{68}$$

Here $Q(z)$ can be estimated with the help of Corollary 2 (end of last section), because

$$G_{\mathscr{O}'}(z,\zeta_n) = \log \frac{1}{|g(z,\zeta_n)|}.$$

Using this and the relation

$$F'(\zeta_n) = g'(\zeta_n, \zeta_n) \prod_{\zeta_l \in \sigma,\, l \neq n} g(\zeta_l, \zeta_n)$$

we get, after replacing $\mathscr{O}$ by $\mathscr{O}'$ and $r_0$ by $2r_0$ in (59),

$$(69) \qquad |F'(\zeta_n)| \geq \frac{1}{80r_0} \prod_{\zeta_l \in \sigma,\, l \neq n} \frac{|\zeta_l - \zeta_n|}{80r_0}.$$

Now for small enough $r_0$, $|\zeta_l - \zeta_n|/2r_0$ is practically equal, by (37), (39) and the continuity of $d\omega_{\mathscr{E}}(\zeta, \infty)/|d\zeta|$, to

$$\frac{|l-n|/N}{(2k+1)/N} = \frac{|l-n|}{2k+1},$$

and the *number of points* $\zeta_n$ on $\sigma$ is *at most* $2k+2$, again by (37) and (39). For small $r_0 > 0$, the right-hand product in (69) has thus at most $2k+1$ factors, each $> 1/(80k+80)$, say, and is hence (and by far!) $> (80k+80)^{-2k-1}$. In other words,

$$(70) \qquad |F'(\zeta_n)| \geq \frac{L}{r_0} \qquad \text{for } \zeta_n \in \sigma$$

with a certain fixed constant $L > 0$.

Substituted into (67), (65) and (70) yield, with (66),

$$|Q(z)| \leq \frac{r_0}{L} e^{D-c_1} \sum_{\zeta_n \in \sigma} \frac{1}{|z - \zeta_n|}$$

for $z \in \mathscr{O}'$. On $\partial\mathscr{O}' = \Gamma \sim \sigma'$, the expression on the right is $\leq (2k+2)e^{D-c_1}/L$, since $|z - \zeta_n| \geq r_0$ there for each of the $\zeta_n$ on $\sigma$ (and that is *why* we brought $\sigma'$ into this discussion!). Therefore, since $Q(z)$ is bounded in $\mathscr{O}'$ (besides being continuous up to $\partial\mathscr{O}'$ like the functions $g(z, \zeta_n)$ used to form $F(z)$), the (extended) principle of maximum implies that

$$(71) \qquad |Q(z)| \leq \frac{2k+2}{L} e^{D-c_1} \qquad \text{for } z \in \mathscr{O}'.$$

*Given* the quantity $c_0$ figuring in (61), we now take $c_1$ *so large* (depending on $D$, $L$ and $k$, i.e., on $k$ *alone*) as to make the right side of (71) $< \frac{1}{2}e^{-c_0}$, say, and then form

$$(72) \qquad H_1(z) = P(z) - \frac{Q(z)}{\psi(z)} \qquad \text{for } z \in \mathscr{O}'.$$

This function is analytic in $\mathscr{O}' \cap \mathbf{C}$, and by (68),

$$H_1(\zeta_n) = 0 \qquad \text{for } \zeta_n \in \sigma. \tag{73}$$

Again,

$$e^{-(\mathfrak{M}F)(z)} H_1(z) = e^{-(\mathfrak{M}F)(z)} P(z) - \frac{e^{-(\mathfrak{M}F)(z)}}{\psi(z)} Q(z) \tag{74}$$

by (72), with the second term on the right of modulus

$$\le e^{-\int_{\sigma'} G_{\mathscr{O}'}(z,\zeta)\, d\varrho(\zeta)}\, e^{-(1-\{\kappa M\})G_{\mathscr{O}'}(z,\infty)} |Q(z)| \le |Q(z)| \le \tfrac{1}{2} e^{-c_0}$$

according to (71), (63) and the choice of $c_1$ just made. From (61) and (74) we therefore have

$$e^{-(\mathfrak{M}F)(\zeta_0)} |H_1(\zeta_0)| \ge \tfrac{1}{2} e^{-c_0}. \tag{75}$$

On the other hand,

$$e^{-(\mathfrak{M}F)(z)} |P(z)| \le e^{(1+\kappa)MG_{\mathscr{E}}(z,\infty)}$$

(see near the beginning of Section 7), so, since $c_0 \ge 0$, (74) and the estimate just given also yield

$$e^{-(\mathfrak{M}F)(z)} |H_1(z)| \le e^{(1+\kappa)MG_{\mathscr{E}}(z,\infty)} + \tfrac{1}{2} e^{-c_0} \le 2 e^{(1+\kappa)MG_{\mathscr{E}}(z,\infty)} \tag{76}$$

for $z \in \mathscr{O}'$.

Put, finally,

$$H(z) = \tfrac{1}{2} H_1(z).$$

By (76), the function $H(z)$, *analytic* in $\mathscr{O} \cap \mathbf{C} \subseteq \mathscr{O}' \cap \mathbf{C}$, *satisfies the hypothesis* of Lemma 5, save for (41). In place of the latter, we have, however,

$$\log |H(\zeta_0)| - (\mathfrak{M}F)(\zeta_0) \ge -c_0 - \log 4,$$

from (75). Lemma 5 can thus still be applied, and it tells us *that* (73) *cannot hold* provided that $k$ is chosen large enough to begin with. Fixing *such* a value of $k$ and *then* taking $c_1$ in consequence so as to make the right side of (71) $< \frac{1}{2} e^{-c_0}$, we see *that* (65) *cannot hold either* for every $\zeta_n$ on $\sigma$. That relation will therefore *fail* for some $\zeta_n \in \sigma$, and then (62) *will be satisfied* at that $\zeta_n$.

We are done.

**10.** Let us now return to the closed set $E$ on $\Gamma$, described by (14). Concerning its points, we have

**Theorem 3.** *Suppose that $N$ is large, and that $P(z)$ is a polynomial of degree $M$ where* (38) *holds with some $\kappa > 0$ and $\lambda < 1$. Then there are an integer $k > 0$ and a constant $\gamma < \infty$, both depending only on $\lambda$ and the curve $\Gamma$, such that, for any $\zeta_0 \in E$, we have*

$$\log^+ |P(\zeta_n)| \geq \tfrac{1}{3}(\mathfrak{M}F)(\zeta_0) - \gamma \tag{77}$$

*for at least one of the points $\zeta_n$ specified by* (37) *and lying on an arc $\sigma$ of $\Gamma$, cut off by a circle about $\zeta_0$ and for which* (39) *holds.*

*Proof.* When $\log^+ |P(\zeta_0)| = 0$ there is nothing that needs to be done, for then $(\mathfrak{M}F)(\zeta_0) = 0$ by (14), and (77) holds for any $\zeta_n \in \sigma$ as long as $\gamma \geq 0$.

Otherwise, $\log^+ |P(\zeta_0)| = \log |P(\zeta_0)|$, and then we have (61) with $c_0 = 0$. Assuming $N$ sufficiently large we can thus, by Theorem 2, choose $k$ corresponding to the given value of $\lambda$ so that (62) *will hold* for an appropriate $\zeta_n \in \sigma$ (the arc determined by $k$ according to (39)), and with a $c_1 < \infty$ corresponding to the value $c_0 = 0$. For such a $\zeta_n$, that makes

$$\log |P(\zeta_n)| \geq (\mathfrak{M}F)(\zeta_n) - c_1.$$

A lower estimate for the *right side* of this relation is furnished by Theorem 1 (Section 5). According to (29), it is

$$\begin{aligned}
&\geq \tfrac{1}{3}(\mathfrak{M}F)(\zeta_0) - \tfrac{1}{3}(1+\kappa)MB\omega_{\mathscr{E}}(\widehat{\zeta_0,\zeta_n},\infty) - c_1 \\
&\geq \tfrac{1}{3}(\mathfrak{M}F)(\zeta_0) - \tfrac{1}{3}(1+\kappa)MB\omega_{\mathscr{E}}(\sigma,\infty) - c_1 \\
&\geq \frac{1}{3}(\mathfrak{M}F)(\zeta_0) - \frac{1+\kappa}{3}MB\frac{2k+1}{N} - c_1
\end{aligned}$$

with a constant $B$ depending on $\Gamma$; we have used (39). Referring finally to (38), we see that

$$\log |P(\zeta_n)| \geq \tfrac{1}{3}(\mathfrak{M}F)(\zeta_0) - \gamma$$

with

$$\gamma = \tfrac{1}{3}(2k+1)B\lambda + c_1,$$

and (77) is proved.

We come finally to the main result of this paper.

**Theorem 4.** *Under the conditions of the preceding theorem, we have*

$$\int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,\infty) \leq \frac{\alpha}{N}\sum_{n=1}^{N} \log^+ |P(\zeta_n)| + \beta, \tag{78}$$

*where $\alpha$ and $\beta$ are constants depending on $\kappa$, $\lambda$ and the curve $\Gamma$.*

*Proof.* With $N$ large we take the arcs $\widehat{\zeta_N, \zeta_1}$, $\widehat{\zeta_n, \zeta_{n+1}}$, $n = 1, 2, \ldots N-1$, and look at the intersections

$$E_N = E \cap (\widehat{\zeta_N, \zeta_1}), \qquad E_n = E \cap (\widehat{\zeta_n, \zeta_{n+1}}), \qquad n = 1, 2, \ldots N-1, \tag{79}$$

where $E$ is given by (14).

From Lemma 4 (Section 6) we have, by (35),

$$\kappa M \int_\Gamma \log^+ |P(\zeta)| \, d\omega_{\mathscr{E}}(\zeta, \infty) \leq \int_\Gamma (\mathfrak{M}F)(\zeta) \, d\varrho(\zeta), \tag{80}$$

and since the Riesz mass $\varrho$ corresponding to $\mathfrak{M}F$ is supported on $E$ (and is *absolutely continuous*—see the corollary to Lemma 2, end of Section 3), the right side is equal to

$$\sum_{n=1}^{N} \int_{E_n} (\mathfrak{M}F)(\zeta) \, d\varrho(\zeta). \tag{81}$$

Theorem 3 and Lemma 2 (Section 3) are used to obtain an upper bound on each of the integrals in (81) for which $\varrho(E_n) > 0$; the remaining ones contribute nothing to that sum.

Let us, *for the moment*, concentrate on a *particular* $E_n$ with $\varrho(E_n) > 0$. We take a point $\zeta_0 \in \overline{E}_n$ where $(\mathfrak{M}F)(\zeta)$ assumes its *maximum* on that closed set, making

$$\int_{E_n} (\mathfrak{M}F)(\zeta) \, d\varrho(\zeta) \leq (\mathfrak{M}F)(\zeta_0) \varrho(E_n).$$

By (18), (79) and (37) the right side is

$$\leq (\mathfrak{M}F)(\zeta_0)(1+\kappa) M \omega_{\mathscr{E}}(E_n, \infty) \leq \frac{(1+\kappa)M}{N} (\mathfrak{M}F)(\zeta_0),$$

so we get

$$\int_{E_n} (\mathfrak{M}F)(\zeta) \, d\varrho(\zeta) \leq \frac{(1+\kappa)M}{N} (\mathfrak{M}F)(\zeta_0). \tag{82}$$

Here the right side can be estimated by Theorem 3. According to that result, if we take an arc $\sigma$ of $\Gamma$ about $\zeta_0$ ($\in E$ !), of such size as to make

$$\omega_{\mathscr{E}}(\sigma, \infty) = \frac{2k+1}{N},$$

we *will have*

$$(\mathfrak{M}F)(\zeta_0) \leq 3(\log^+ |P(\zeta_l)| + \gamma) \tag{83}$$

at one (at least) of the points $\zeta_l$ on $\sigma$; this $\zeta_l$ of course depends on the particular set $\overline{E}_n$ where we have taken our point $\zeta_0$, *so we henceforth denote the index* $l$ (for which (83) holds) by $l(n)$. Combining (82) and (83), we get

$$\int_{E_n} (\mathfrak{M}F)(\zeta)\, d\varrho(\zeta) \le \frac{3(1+\kappa)M}{N}(\log^+|P(\zeta_{l(n)})| + \gamma) \tag{84}$$

for each $n$ with $\varrho(E_n) > 0$.

Our idea is now to use (84) to estimate the sum (81), but the trouble is that *more* than *one* value of $n$ may correspond to *the same* index $l(n)$. In order to see *how many* such values there can be, we again fix our attention on one $\overline{E}_n$ with $\varrho(E_n) > 0$, and on the point $\zeta_0$ therein. Here $\zeta_0$ and $\zeta_{l(n)}$ *both* lie on the arc $\sigma$ formed in the way described above, so *an* arc, say $\sigma_n$, *joining* $\zeta_0$ *to* $\zeta_{l(n)}$, *is contained in* $\sigma$. Therefore,

$$\omega_{\mathscr{E}}(\sigma_n, \infty) \le \omega_{\mathscr{E}}(\sigma, \infty) = \frac{2k+1}{N},$$

and it follows by (37) that *there cannot be more than* $2k+2$ *of the different points* $\zeta_j$ *on the closed arc* $\sigma_n$.

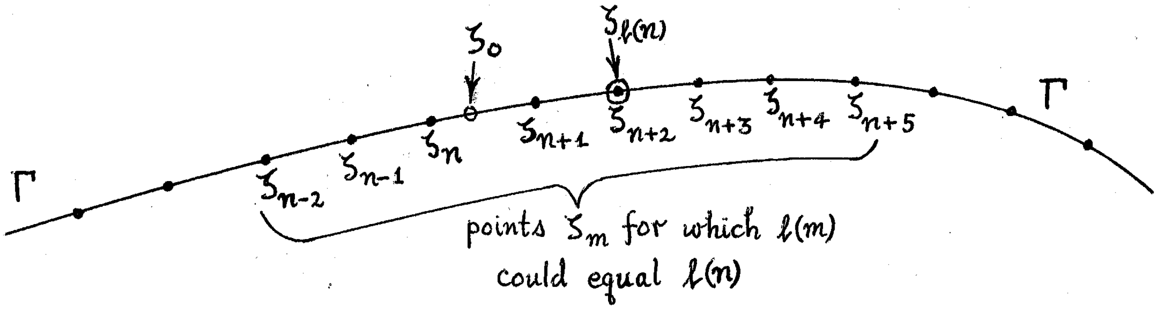



figure drawn for $k = 1$

That being the case for *any* $n$ with $\varrho(E_n) > 0$, let us, for such a *fixed* $n$, count off successively $2k+1$ of the points $\zeta_m$ lying immediately *to one side* of $\zeta_{l(n)}$, and $2k+2$ such points lying immediately to the *other side*. Together with $\zeta_{l(n)}$, this gives us altogether $4k+4$ points $\zeta_m$ for which the corresponding index $l(m)$ could equal $l(n)$ if $\varrho(E_m) > 0$; for *any other* $\zeta_m$ the set $\overline{E}_m$, if non-empty, *will consist entirely of* points $\zeta'$ lying between $\zeta_m$ and $\zeta_{m+1}$ and separated from $\zeta_{l(n)}$ by *more than* $2k$ of the $\zeta_j$ along the curve $\Gamma$. *Each* of the *indices* $l(n)$ is, in other words, *shared by at most* $4k+4$ *of the values of* $n$. This means that

$$\sum_{n=1}^{N} \log^+|P(\zeta_{l(n)}| \le (4k+4) \sum_{n=1}^{N} \log^+|P(\zeta_n)|.$$

Using this together with (84) to evaluate the sum (81), we find, by (80), that

$$\kappa M \int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,\infty) \le \frac{3(1+\kappa)(4k+4)}{N} M \sum_{n=1}^{N} \bigl(\log^+ |P(\zeta_n)| + \gamma\bigr),$$

i.e., that

$$\int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{E}}(\zeta,\infty) \le \frac{3(1+\kappa)(4k+4)}{\kappa N} \sum_{n=1}^{N} \log^+ |P(\zeta_n)| + \frac{3(1+\kappa)}{\kappa}(4k+4)\gamma.$$

The theorem is proved.

**Corollary.** *Under the conditions of the preceding two theorems we have, for any polynomial $P(z)$ of degree $M$ and any $z_0 \in \mathscr{D}$, the inside of $\Gamma$,*

$$\log^+ |P(z_0)| \le \frac{A(z_0)}{N} \sum_{n=1}^{N} \log^+ |P(\zeta_n)| + B(z_0) \tag{85}$$

*with two functions $A(z_0)$, $B(z_0)$, each bounded in the interior of $\mathscr{D}$ and depending only on $z_0$, $\lambda$ and the curve $\Gamma$.*

*Proof.* For $z_0 \in \mathscr{D}$, we have

$$\log^+ |P(z_0)| \le \int_\Gamma \log^+ |P(\zeta)|\, d\omega_{\mathscr{D}}(\zeta, z_0).$$

When $z_0$ ranges through $\mathscr{D}$, the variation of the integral on the right is governed by Harnack for each $P$. Refer to the observations at the beginning of Section 1 and to (78).

**11.** It is perhaps worthwhile to end this paper with a qualitative variant of the last results.

**Theorem 5.** *Given a sequence of polynomials $P_j(z)$ having the respective degrees $M_j$ let us, for each $j$, take $N_j$ points $\zeta_{n,j}$, $n = 1, 2, \ldots N_j$, around the curve $\Gamma$, in such fashion as to have*

$$\omega_{\mathscr{E}}(\widehat{\zeta_{n,j}, \zeta_{n+1,j}}, \infty) = \omega_{\mathscr{E}}(\widehat{\zeta_{N_j,j}, \zeta_{1,j}}, \infty) = \frac{1}{N_j}$$

*for $n = 1, 2, \ldots N_j - 1$. Suppose that*

$$\frac{M_j}{N_j} \le \lambda' < 1$$

*for all $j$, with some fixed $\lambda'$. Then, if the averages*

$$\frac{1}{N_j} \sum_{n=1}^{N_j} \log^+ |P_j(\zeta_{n,j})|$$

*are bounded, the $P_j(z)$ form a normal family in $\mathscr{D}$, the inside of $\Gamma$.*

*Proof.* Take a $\kappa > 0$ and $\lambda < 1$ for which

$$(1+\kappa)\lambda' < \lambda;$$

corresponding to this $\lambda$ and to the curve $\Gamma$, Theorem 4 and its corollary will hold for sufficiently large values of $N$. According to an observation made at the beginning of Section 7 there is no loss of generality in our assuming that $N_j \to \infty$ as $j \to \infty$, and we may thus restrict our attention to the $j$ for which the results just mentioned are valid when $N = N_j$. The present theorem then follows on taking $N = N_j$ and $\zeta_n = \zeta_{n,j}$ in (85), for here (38) is satisfied with $M = M_j$ and $N = N_j$.

**Remark 1.** In this result, the requirement that the ratios $M_j/N_j$ be bounded away from 1 (and not merely $< 1$) is essential. To see this, take any large composite integer $N$ having a divisor $K$ such that

$$\frac{K}{N} = \varepsilon$$

is *small*, and put

$$\omega = e^{2\pi i/N}, \qquad \chi = e^{2\pi i/K},$$

making $\omega^L = \chi$ for $L = N/K$.

Then

$$R(z) = \frac{z^N - 1}{z^K - 1} = \prod_{\substack{1 \le n \le N \\ L \nmid n}} (z - \omega^n)$$

is a polynomial of degree $M = N - K$ with $R(0) = 1$. $R(z)$ vanishes at each of the points $\omega^n$ on the unit circle, *except* those of the form $\chi^m = \omega^{mL}$, $m = 1, 2, \ldots K$. At any one of *these*,

$$|R(\chi^m)| = \left|\frac{N\chi^{m(N-1)}}{K\chi^{m(K-1)}}\right| = \frac{N}{K},$$

so

$$\frac{1}{N}\sum_{n=1}^{N} \log^+ |R(\omega^n)| = \frac{K}{N}\log\frac{N}{K} = \varepsilon \log\frac{1}{\varepsilon}.$$

Put now

$$P(z) = \frac{1}{\varepsilon} R(z).$$

Then $P(0) = 1/\varepsilon$, but

$$\frac{1}{N}\sum_{n=1}^{N} \log^+ |P(\omega^n)| = \varepsilon \log\frac{1}{\varepsilon^2}.$$

By arranging matters so as to make $M/N = 1 - (K/N) = 1 - \varepsilon$ *close to* 1 we render the logarithmic average *small* but $P(0)$ *large*. Therefore *no bound* can, for $|z| < 1$, be deduced on $P(z)$ from a bound on the logarithmic average when $M/N$ gets arbitrarily close to 1.

**Remark 2.** The procedure of this paper can, most likely, be extended to more general closed curves $\Gamma$—$\mathfrak{M}F$ sits upon $F$ like a rider on the back of a camel. As noted at the end of Section 3, Lemma 2 holds without any special assumptions about $\Gamma$; that is so by the result of Grishin. Without resorting to the geometric considerations of Section 4, one can also obtain a substitute for Theorem 1:

$$(\mathfrak{M}F)(z_2) \leq 3(\mathfrak{M}F)(z_1) + (1+9\kappa)M \int_0^{2|z_1-z_2|} \frac{\omega_{\mathscr{E}}(\Delta_r, \infty)}{r}\, dr,$$

where $\Delta_r$ denotes the disk of radius $r$ about $z_2$ (sic!); this is a simple consequence of the Poisson–Jensen formula, Lemma 2, and the positivity of $(\mathfrak{M}F)(z) + \kappa M G_{\mathscr{E}}(z,\infty)$. It seems quite probable that suitable versions of the other intermediate results used in this paper could also be established. Any extension of Theorem 5 obtainable in such fashion would probably involve a restriction more stringent than

$$\frac{M_j}{N_j} \leq \lambda' < 1$$

on the degrees $M_j$ of the polynomials $P_j(z)$; see the observation at the end of Remark 2 in Section 7.

A more useful generalization would involve replacement of the simple closed curve $\Gamma$ by one going out to $\infty$ (in both directions). For this one would have to take a *fixed* infinite set of nodes $\zeta_n$ on $\Gamma$ instead of the collections $\{\zeta_{n,j}\}$ accumulating ever more densely on the bounded curve $\Gamma$ figuring in Theorem 5. In place of polynomials $P(z)$, entire functions subject to a growth restriction depending on $\Gamma$ and the nodes $\zeta_n$ could be allowed. I think that the simple result involving a parabola, formulated in the introduction, could, for instance, be deduced quite easily using a modification of the method followed in the preceding sections.

It seems to me, however, that a *real* extension of the above work would involve *getting rid of the curve* $\Gamma$ *altogether* and looking directly at certain distributions of given nodes $\zeta_n$. At present, I have only the foggiest glimmerings of how that might be carried out.

In conclusion, I must thank the referee who pointed out a bad mistake in an earlier version of Section 1 and also insisted on my giving appropriate credit to Grishin in the remark at the end of Section 3.

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