Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 27, 2002, 373–380

ON THE L^1 APPROXIMATION OF $|f(z)|$ BY $\operatorname{Re} f(z)$ FOR ANALYTIC FUNCTIONS

Edgar Reich

University of Minnesota, School of Mathematics, 127 Vincent Hall
206 Church Street S.E., Minneapolis, MN 55455, U.S.A.; reich@math.umn.edu

Abstract. For angular regions of the plane, the integral of $|f(z)| - \operatorname{Re} f(z)$, where $f(z)$ is analytic and L^1, can be estimated from below in terms of the L^1 norm of $|f(z)|$. We obtain necessary and sufficient conditions on the shape of a type of region of infinite area for which a natural generalization of the foregoing exists, and examine the consequences.

1. Introduction

Let Ω be a region of the complex plane. We denote by $L_a^1(\Omega)$ the class of functions $f(z)$ analytic in Ω and belonging to $L^1(\Omega)$ and set

$$\delta_\Omega[f] = \iint_\Omega [|f(z)| - \operatorname{Re} f(z)]\, dx\, dy, \qquad f \in L_a^1(\Omega). \tag{1.1}$$

Of course, $\delta_\Omega[f] \ge 0$. Also, obviously, if Ω has finite area, and $f(z)$ is identically 1 (or any non-negative constant) in Ω, then $\delta_\Omega[f] = 0$. When Ω has infinite area, non-zero constants no longer belong to $L_a^1(\Omega)$. In that case we consider the sequence,

$$\delta_\Omega[f_n] = \iint_\Omega [|f_n(z)| - \operatorname{Re} f_n(z)]\, dx\, dy, \qquad f_n \in L_a^1(\Omega),\ n = 1, 2, \dots, \tag{1.2}$$

where

$$\lim_{n\to\infty} f_n(z) = 1 \text{ uniformly on every compact subset of } \Omega. \tag{1.3}$$

We will see that, depending on the size of the "opening" of Ω at infinity, as measured by comparison to a class of standard openings when a comparison in the sense to be described below is possible, there are only two alternatives: Either there exists a sequence $f_n \in L_a^1(\Omega)$ satisfying (1.3) for which $\lim \delta_\Omega[f_n] = 0$, or $\lim \delta_\Omega[f_n] = +\infty$ for *every* sequence $f_n \in L_a^1(\Omega)$ satisfying (1.3).

The standard openings are defined in terms of the family of parabolic-shaped regions, Ω_β, $0 \le \beta \le 1$, starting with the limiting case of a half-strip Ω_0, and ending with the limiting case of an angular region, Ω_1,

$$\begin{aligned} &\Omega_0 = \{z = x + iy : |x| < 1,\ y > 0\}, \\ &\Omega_\beta = \{z = x + iy : |x| < y^\beta,\ y > 0\}, \qquad (0 < \beta \le 1). \end{aligned} \tag{1.4}$$

The result is as follows:

2000 Mathematics Subject Classification: Primary 30A10, 30C62, 30C75.

Theorem 1.1. *With the conditions and notations* (1.2), (1.3), (1.4),
(i) *if* $\Omega \supset \Omega_\beta$, $(\frac{1}{3} < \beta \le 1)$, *then* $\lim \delta_\Omega[f_n] = +\infty$ *for every choice of* $\{f_n\}$;
(ii) *if* $\Omega \subset \Omega_\beta$, $(0 \le \beta \le \frac{1}{3})$, *there exist* $\{f_n\}$, *such that* $\lim \delta_\Omega[f_n] = 0$.

Since Ω_{β_2} contains a translated copy of Ω_{β_1} when $0 \le \beta_1 \le \beta_2 \le 1$, it is sufficient to restrict consideration to values of β to any interval of the form, $\frac{1}{3} < \beta < \beta_o$, in proving (i). However, since the situation when $\beta < 1$ is much subtler than when $\beta = 1$, we obtain a better appreciation of it if we first consider the case separately when Ω is an angular region. This will be done in Section 2. The case, $\frac{1}{3} < \beta < 1$, follows in Section 3. The method used in Section 3 differs radically from that used in Section 2. It would be difficult to anticipate the key result, (3.1)–(3.2), of Section 3 from the key result (2.2) of Section 2, but once they are compared, they are seen to fit together well.

Except for the special value, $\beta = \frac{1}{3}$, the proof of (ii) is very easy: A simple computation shows that it is enough to set $f_n(z) = e^{iz/n}$. Namely,

$$\begin{aligned} \delta_{\Omega_\beta}[e^{iz/n}] &= \iint_{\Omega_\beta} e^{-y/n}\left(1 - \cos\frac{x}{n}\right) dx\,dy \le \frac{2}{n^2}\iint_{\Omega_\beta} x^2 e^{-y/n}\,dx\,dy \\ &= \frac{4}{3n^2}\int_0^\infty y^{3\beta} e^{-y/n}\,dy = \frac{4}{3} n^{3\beta-1}\Gamma(3\beta+1). \end{aligned} \tag{1.5}$$

So, if $0 \le \beta < \frac{1}{3}$, then $\lim_{n\to\infty} \delta_{\Omega_\beta}[e^{iz/n}] = 0$. For $\beta = \frac{1}{3}$, $\{\delta_{\Omega_\beta}[e^{iz/n}]\}$ is bounded, but does not go to zero as $n \to \infty$. The problem is handled in a roundabout manner in Section 4.

As may be gathered from the list of references, there is a strong relationship of our problem with certain results of the theory of extremal planar quasiconformal mappings. The material in Section 2 and a part of the construction of Section 3 actually duplicate work in some of these references but, as far as I am aware, Part (i) of Theorem 1.1 is stronger than anything that has up to now been established in quasiconformal mapping theory. On the other hand, Part (ii) of Theorem 1.1 *is* equivalent to known facts in [5], [4], and [1]. Except for Section 4, however, it has been possible to keep the reasoning here completely elementary and the exposition self-contained.

It should be noted that for the case of arbitrary regions Ω, even if we considered only simply-connected regions with infinite area, a comparison of Ω with a region of type Ω_β in the sense of Theorem 1.1 is of course, in general, not possible. Whether other alternatives for $\lim \delta_\Omega[f_n]$ can occur in the general case is an open problem.

2. Angular regions and the case $\beta = 1$

Let $\mathscr{R}_\alpha$ be the angular region

$$\mathscr{R}_\alpha = \{z : 0 < \arg z < \alpha\}, \qquad (0 < \alpha < 2\pi).$$

Theorem 2.1. *For every $f \in L^1_a(\mathscr{R}_\alpha)$ we have*

$$\delta_{\mathscr{R}_\alpha}[f] \geq \left(1 - \frac{|\sin\alpha|}{\alpha}\right) \iint_{\mathscr{R}_\alpha} |f(z)|\, dx\, dy. \tag{2.1}$$

Proof. ([2, p. 124]) Let $\Sigma = \{w = u + iv : 0 < v < \alpha\}$. Then

$$A = \iint_{\mathscr{R}_\alpha} f(z)\, dx\, dy = \iint_\Sigma e^{-2iv} g(w)\, du\, dv, \qquad g(w) = e^{2w} f(e^w),$$

and

$$B = \iint_{\mathscr{R}_\alpha} |f(z)|\, dx\, dy = \iint_\Sigma |g(w)|\, du\, dv = \int_0^\alpha dv \int_{-\infty}^{\infty} |g(u+iv)|\, du.$$

Since $B < \infty$, $\int_{-\infty}^{\infty} g(u+iv)\, du$ exists for almost all v, and since $g(w)$ is analytic in Σ,

$$\int_{-\infty}^{\infty} g(u+iv)\, du = \kappa = \text{const} \qquad \text{for a.a. } v,\ (0 < v < \alpha).$$

Hence,

$$A = \kappa \int_0^\alpha e^{-2iv}\, dv = \kappa e^{-i\alpha} \sin\alpha, \qquad B \geq |\kappa|\alpha.$$

It follows that

$$\frac{|A|}{B} \leq \frac{|\sin\alpha|}{\alpha}.$$

This implies (2.1). Therefore, since $\mathscr{R}_{\pi/2}$ is congruent to Ω_1,

$$\delta_{\Omega_1}[f] \geq \left(1 - \frac{2}{\pi}\right) \iint_{\Omega_1} |f(z)|\, dx\, dy, \qquad f \in L^1_a(\Omega_1). \tag{2.2}$$

This proves Part (i) of Theorem 1.1 for the case $\beta = 1$. When $0 \leq \beta < 1$, no inequality like (2.2) holds; that is, it is not possible to replace the coefficient $(1 - 2/\pi)$ on the right side by any positive constant, even if that constant were allowed to depend on β. This can be seen by trying, $f_n(z) = e^{iz/n}$, $n \to \infty$. But we will see in Section 3, following, that the matter can be handled by means of a different type of relationship. [1]

[1] It is not difficult to show that the constant $(1 - 2/\pi)$ in (2.2) is best possible.

3. The case $\frac{1}{3} < \beta < 1$

We will see that (2.2) can be generalized as follows:

Theorem 3.1. *When $\frac{1}{3} < \beta < 1$, there exists a non-negative function $P_\beta(z)$, measurable as a function of z, $z \in \Omega_\beta$, such that*

$$\delta_{\Omega_\beta}[f] \geq \iint_{\Omega_\beta} P_\beta(z)|f(z)|\,dx\,dy, \qquad f \in L^1_a(\Omega_\beta). \tag{3.1}$$

The function $P_\beta(z)$ has infinite L^1 norm,

$$\iint_{\Omega_\beta} P_\beta(z)\,dx\,dy = +\infty. \tag{3.2}$$

When $0 \leq \beta \leq \frac{1}{3}$, no such function P_β exists.

The proof will consist in an explicit construction of $P_\beta(z)$, and will take up the remainder of this section. It is clear that Part (i) of Theorem 1.1 will then follow immediately.

We start with some preliminaries.

Pavlović's inequality.[2] *If ζ_1, ζ_2, w are complex numbers with*

$$|\zeta_1|^2 + |\zeta_2|^2 \leq 2,$$

then

$$|\zeta_1 - \zeta_2|^2|w| \leq 4\big[|w| - \mathrm{Re}(\zeta_1 w)\big] + 4\big[|w| - \mathrm{Re}(\zeta_2 w)\big]. \tag{3.3}$$

Suppose $\tau \in L^\infty(\Omega)$, $\|\tau\|_\infty \leq 1$, $f \in L^1(\Omega)$, $\iint_\Omega f(z) = \iint_\Omega \tau(z)f(z)$. Then

$$\delta_\Omega[f] \geq \frac{1}{8}\iint_\Omega |1 - \tau(z)|^2|f(z)|\,dx\,dy. \tag{3.4}$$

Proof. The right-hand side of (3.3) minus the left-hand side of (3.3) equals $2Q_1 + 4Q_2 + Q_3$, where

$$\begin{aligned} Q_1 &= (2 - |\zeta_1|^2 - |\zeta|^2)|w| \geq 0,\\ Q_2 &= |(\zeta_1 + \zeta_2)w| - \mathrm{Re}\big[(\zeta_1 + \zeta_2)w\big] \geq 0,\\ Q_3 &= (|\zeta_1 + \zeta_2| - 2)^2|w| \geq 0. \end{aligned}$$

Relation (3.4) follows from (3.3) on setting $\zeta_1 = 1$, $\zeta_2 = \tau$, $w = f$, and integrating over Ω.

---

2 Miroslov Pavlović, personal communication.

In outline, the procedure to be used to take advantage of (3.4) is the following. We look for a complex-valued function H on $\Omega \cup \partial\Omega$, vanishing on $\partial\Omega$ and sufficiently regular so that $H_{\bar z}(z)$ is bounded in Ω and so that Green's formula,

$$0 = \frac{1}{2i} \int_{\partial\Omega} H(z) f(z)\, dz = \iint_{\Omega} H_{\bar z} f(z)\, dx\, dy,$$

holds whenever, say, f belongs to $L^1_a(\Omega)$ and is continuous on $\Omega \cup \partial\Omega$. It is crucial to be able to determine H in such a manner that, in addition,

$$\operatorname{ess\,sup}\{\operatorname{Re} H_{\bar z} : z \in \Omega\} < 0. \tag{3.5}$$

By (3.5), and the boundedness of $H_{\bar z}$, there will then exist a constant $c > 0$, such that $|1 + cH_{\bar z}| \le 1$. Set $\tau(z) = 1 + cH_{\bar z}$. Then, by (3.4),

$$\delta_\Omega[f] \ge \frac{c^2}{8} \iint_\Omega |H_{\bar z}|^2 |f(z)|\, dx\, dy \tag{3.6}$$

for all $f \in L^1_a(\Omega)$ that are continuous on $\Omega \cup \partial\Omega$. By means of an approximation argument one establishes that (3.6) holds for all $f \in L^1_a(\Omega)$.

It so happens that a function H that allows us to carry out the above in slightly modified form for $\Omega = \Omega_\beta$, $\frac{1}{3} < \beta < 1$, has already been determined [3] in connection with a question of Hahn–Banach extensions. One starts with the disjoint decomposition,

$$\Omega_\beta = E \cup \Omega_{11} \cup \Omega_{12} \cup \Omega_{22} \cup \Omega_{21},$$

where

$$\Omega_{mn} = \{z \in \Omega_\beta : (-1)^m \operatorname{Re} z < 0,\ (-1)^n \operatorname{Im} z > (-1)^n\},$$
$$E = \{z \in \Omega_\beta : \operatorname{Re} z = 0 \text{ or } \operatorname{Im} z = 1\},$$

and defines H by

$$H(x+iy) = \begin{cases} (y^\beta - x)\bigl(-\frac{1}{2}(1-\beta)xy + iy^{1-\beta}\bigr), & z \in \Omega_{11}, \\ (y^\beta - x)\bigl(-\frac{1}{2}(1-\beta)xy^{\beta-2} + iy^{\beta-1}\bigr), & z \in \Omega_{12}, \end{cases} \tag{3.7}$$

$$H(z) = -\overline{H(-\bar z)}, \qquad z \in \Omega_{21} \cup \Omega_{22}. \tag{3.8}$$

This H is continuous in Ω_β, vanishes on $\partial\Omega_\beta$, and is in C^∞ in the separate regions Ω_{mn}. The resulting function $H_{\bar z}$ is well defined and bounded in all Ω_{mn}, that is, in Ω_β outside a set of two-dimensional measure zero. One applies Green's formula to the portion of Ω_β between parallel horizontals. In the limit, as Ω_β is exhausted, the conclusion is that

$$\iint_{\Omega_\beta} H_{\bar z} f(z)\,dx\,dy = 0, \qquad f \in L^1_a(\Omega_\beta).$$

By (3.7), (3.8),

$$\operatorname{Re} H_{\bar z} = \begin{cases} -\frac{1}{2} + \frac{1}{4}(1-\beta)(2xy + 2xy^{-\beta} - y^{\beta+1}), & z \in \Omega_{11}, \\ \frac{1}{4}(1-3\beta)y^{2\beta-2}, & z \in \Omega_{12}, \end{cases} \tag{3.9}$$

$$\operatorname{Im} H_{\bar z} = \begin{cases} -\frac{1}{2}y^{1-\beta} + \frac{1}{4}(1-\beta)\bigl[x - (1+\beta)y^\beta\bigr]x, & z \in \Omega_{11}, \\ -\frac{1}{2}y^{\beta-1} + \frac{1}{4}(1-\beta)\bigl[2(1-\beta)y^\beta - (2-\beta)x\bigr]xy^{\beta-3}, & z \in \Omega_{12}, \end{cases} \tag{3.10}$$

$$H_{\bar z}(-x+iy) = \overline{H_{\bar z}(x+iy)}, \qquad z \in \Omega_{21} \cup \Omega_{22}. \tag{3.11}$$

Evidently, $\operatorname{Re} H_{\bar z} < 0$ in $\Omega_{12} \cup \Omega_{22}$. For $z \in \Omega_{11}$, we note that $\operatorname{Re} H_{\bar z}$ is a linear function of x for every fixed y, $(0 < x < y^\beta)$, whose maximum occurs at $x = y^\beta$. Thus, $\operatorname{Re} H_{\bar z} \le \frac{1}{4}(1-3\beta) < 0$ in $\Omega_{11} \cup \Omega_{21}$. We therefore conclude that (3.1) holds with $P_\beta(z) = \frac{1}{8}c^2|H_{\bar z}|^2$. As $P_\beta(x+iy)$ is bounded and an even function of x, it suffices to consider its behavior in Ω_{12}, as $y \to +\infty$, in order to prove (3.2). By (3.9), (3.10),

$$\int_0^{y^\beta} \bigl[\operatorname{Re} H_{\bar z}(x+iy)\bigr]^2\,dx = \left(\frac{1-3\beta}{4}\right)^2 y^{5\beta-4},$$

$$\int_0^{y^\beta} \bigl[\operatorname{Im} H_{\bar z}(x+iy)\bigr]^2\,dx = \frac{1}{4}y^{3\beta-2} + \frac{(2\beta-1)(1-\beta)}{12}y^{5\beta-4} + \frac{(1-\beta)^2(2-7\beta+8\beta^2)}{240}y^{7\beta-6},$$

for $y > 1$. Thus, for $y > 1$,

$$\int_{-y^\beta}^{y^\beta} P_\beta(x+iy)\,dx = Cy^{3\beta-2} + O(y^{5\beta-4}), \qquad \text{as } y \to +\infty,\ (\tfrac{1}{3} < \beta < 1), \tag{3.12}$$

where C is a positive constant depending on β. Assertion (3.2) follows.

Remark. *In the case $\frac{1}{3} < \beta < 1$, the exponent of the term $y^{3\beta-2}$ in (3.12) is best possible as $y \to \infty$.*

Proof. By (3.1), (3.12), $\delta_{\Omega_\beta}[e^{iz/n}]$ has as lower bound $\frac{1}{2}C\int_1^\infty y^{3\beta-2}e^{-y/n}\,dy$, when n is sufficiently large. But this term has the same order of magnitude, as $n \to \infty$, as the *upper* bound (1.5), namely $n^{3\beta-1}$.

4. The case $\beta = \frac{1}{3}$

Let $\mathscr{G} = \{z = x + iy : y > |x|^3\} = \Omega_{1/3}$. Our object is to prove Part (i) of Theorem 1.1, that is, to show that there exists a sequence $\varphi_n \in L_a^1(\mathscr{G})$, such that $\lim_{n\to\infty} \delta_{\mathscr{G}}[\varphi_n] = 0$. Although $\{\varphi_n\}$ will not be explicitly determined, we will point out that the conclusion follows from results of [5] and [1] about quasiconformal mappings of $\mathscr{G}$. Let F denote the horizontal stretch of $\mathscr{G}$ onto $\mathscr{G}' = F(\mathscr{G})$ by the factor $K > 1$; i.e.

$$F(x + iy) = Kx + iy, \qquad z = x + iy \in \mathscr{G}.$$

In line with standard terminology, F is called *extremal* if the maximal dilatation M of any arbitrary quasiconformal mapping ζ of $\mathscr{G}$ onto $\mathscr{G}'$ which agrees with F on $\partial\mathscr{G}$ satisfies $M \geq K$. The mapping $\zeta\colon \mathscr{G} \to \mathscr{G}'$ is called *uniquely extremal* if $M = K$ implies $\zeta(z) \equiv F(z)$. We refer to the following result, first proved in [5] [3]:

Theorem 4.1 ([5, Sections 1–3]). *The horizontal stretch mapping $F\colon \mathscr{G} \to \mathscr{G}'$ is uniquely extremal.*

Since the mapping F has complex dilatation

$$\frac{F_{\bar z}}{F_z} = \frac{K-1}{K+1} = k > 0,$$

the desired conclusion now follows from the following special case of a fundamental theorem of Božin, Lakic, Marković, and Mateljević:

Theorem 4.2 ([1, p. 312). *Let φ be a non-zero function analytic in the simply connected region $\mathscr{R}$ and let F be a quasiconformal mapping of $\mathscr{R}$ with complex dilatation $\mu(z) = k\,\overline{\varphi(z)}/|\varphi(z)|$, $0 < k < 1$. Then F is uniquely extremal if and only if there exists a sequence $\varphi_n \in L_a^1(\mathscr{R})$ such that*

(i) $\lim \varphi_n(z) = \varphi(z)$, *uniformly on every compact subset of $\mathscr{R}$, and*
(ii) $k \iint_{\mathscr{R}} |\varphi_n(z)|\,dx\,dy - \operatorname{Re} \iint_{\mathscr{R}} \mu(z)\varphi_n(z)\,dx\,dy \to 0$.

In our case, $\varphi(z) \equiv 1$.

References

[1] Božin, V., N. Lakic, V. Marković, and M. Mateljević: Unique extremality. - J. Anal. Math. 75, 1998, 299–338.

[2] Reich, E.: Quasiconformal Maps of the Disk with Given Boundary Values. - Springer Lecture Notes in Math. 505, 1976, 101–137.

[3] Reich, E.: Uniqueness of Hahn–Banach extensions from certain spaces of analytic functions. - Math. Z. 167, 1979, 81–89.

[3] An alternative proof can be found in [4].

[4] Reich, E.: On criteria for unique extremality of Teichmüller mappings. - Ann. Acad. Sci. Fenn. Ser. A I Math. 6, 1981, 289–301.

[5] Reich, E., and K. Strebel: On the extremality of certain Teichmüller mappings. - Comment. Math. Helv. 45, 1970, 353–362.

Received 20 September 2001