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ON DEFICIENCIES OF SMALL FUNCTIONS FOR PAINLEVÉ TRANSCENDENTS OF THE FOURTH KIND

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Abstract. For transcendental meromorphic solutions of the fourth Painlevé equation, we estimate deficiencies of small functions. For every rational solution, we find all the transcendental meromorphic solutions whose solution curves do not intersect that of the rational solution.

1. Introduction

Consider the fourth Painlevé equation

$$\text{(IV)} \qquad ww'' = \frac{(w')^2}{2} + \frac{3}{2}w^4 + 4zw^3 + 2(z^2 - \alpha)w^2 + \beta$$

($' = d/dz$), where α and β are complex parameters. N. Steinmetz [15] examined value distribution properties of solutions meromorphic in $\mathbf{C}$. For an arbitrary transcendental meromorphic solution $\phi(z)$ of (IV), the deficiency of a complex number $a \in \mathbf{C}$ is estimated as follows:

Theorem A ([15; Sätze 1 und 2]). (1) *If* $(\beta, a) \neq (0,0)$, *then* $\delta(a, \phi) = 0$.
(2) *If* $\beta = 0$ *and if* $\phi(z)$ *does not satisfy the Riccati equation* $w' = \mp(w^2 + 2zw)$, *then* $\delta(0, \phi) \leq \frac{1}{2}$.

For the standard notation of the value distribution theory, see [6]. It is known that (IV) with $\beta = 0$ admits a family of solutions $V_0^{\pm} = \{v_c^{\pm}(z) \mid c \in \mathbf{C} \cup \{\infty\}\}$ with

$$v_c^{\pm}(z) = \exp(\mp z^2)\left(c \pm \int_0^z \exp(\mp t^2)\, dt\right)^{-1}$$

satisfying $w' = \mp(w^2 + 2zw)$ as well, if and only if $\alpha = \pm 1$ ([15]). Then, for every $v_c^{\pm}(z) \in V_0^{\pm}$, $c \in \mathbf{C}$, we have $N(r, 1/v_c^{\pm}) = 0$. Note that $v_{\infty}^{\pm}(z) \equiv 0$ is a rational solution of (IV) with $\beta = 0$. (For all the rational solutions of (IV), see [8], and for special transcendental solutions satisfying Riccati equations simultaneously, see [3] and [9].) In case $\beta = 0$, the result above says that the solution curves of

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$\phi(z) \notin V_0^{\pm}$ and the rational solution $v_{\infty}^{\pm}(z) \equiv 0$ intersect infinitely many times. For every $c \in \mathbf{C}$, the solution curves of $v_c^{\pm}(z) \in V_0^{\pm}$ and $v_{\infty}^{\pm}(z) \equiv 0$ have no intersecting points. From such a point of view, we pose the following problem.

Let $q(z)$ be an arbitrary rational solution (or rational function). For an arbitrary transcendental meromorphic solution $\phi(z)$, estimate the frequency of the intersecting points of the solution curves of $\phi(z)$ and $q(z)$. Furthermore, find all the transcendental meromorphic solutions whose solution curves do not intersect that of $q(z)$.

Let $f(z)$ and $g(z)$ be meromorphic functions in $\mathbf{C}$. We say that $g(z)$ is *small with respect to* $f(z)$ if $T(r,g) = S(r,f)$. The deficiency of the small function $g(z)$ is defined by

$$\delta(g,f) = \liminf_{r\to\infty} \frac{m\big(r, 1/(f-g)\big)}{T(r,f)}.$$

For Painlevé transcendents of the first and the second kind, deficiencies of small functions were estimated in [11]. In this paper, we treat the problem above for small functions in place of rational functions.

For each $\nu \in \mathbf{C}$, the equation

$$\frac{d^2u}{dt^2} + \left(-\frac{t^2}{4} + \nu + \frac{1}{2}\right)u = 0 \tag{1.1}$$

possesses linearly independent solutions $D_\nu(t)$, $D_{-\nu-1}(it)$, where $D_\nu(t)$ is the parabolic cylinder function with the asymptotic expression

$$D_\nu(t) = t^\nu \exp(-t^2/4)\big(1 + O(t^{-2})\big) \tag{1.2}$$

as $t \to \infty$ through the sector $|\arg t| < \frac{3}{4}\pi$. Especially, if $\nu \in \mathbf{N} \cup \{0\}$, then $D_\nu\big(\sqrt{2}\,z\big) = 2^{-\nu/2}\exp(-z^2/2)H_\nu(z)$, where $H_\nu(z)$ is the Hermite polynomial

$$H_\nu(z) = (-1)^\nu \exp(z^2)\,(d/dz)^\nu \exp(-z^2) \tag{1.3}$$

(cf. [1; Sections 8.2, 8.4 and 10.13], [12; Section 8]). Our main results are stated as follows:

Theorem 1.1. *Suppose that* $(\alpha,\beta) = \big(-(n+\frac{1}{2}\pm\frac{1}{2}), -2(n+\frac{1}{2}\mp\frac{1}{2})^2\big)$ *for some* $n \in \mathbf{Z}$. *Then* (IV) *admits a family of solutions* $\big\{\chi_{n,c}^{\mp}(z) \;\big|\; c \in \mathbf{C}\cup\{\infty\}\big\}$ *with the properties*:

(1) *for each* $c = c_1/c_2$, $\mathbf{c} = (c_1,c_2) \in \mathbf{C}^2 \setminus \{(0,0)\}$, *the solution* $\chi_{n,c}^{\mp}(z)$ *is expressible in the form*

$$\chi_{n,c}^{\mp}(z) = -z \mp \frac{\eta'_{n,\mathbf{c}}(z)}{\eta_{n,\mathbf{c}}(z)}, \qquad \eta_{n,\mathbf{c}}(z) = c_1 D_n\big(\sqrt{2}\,z\big) + c_2 D_{-n-1}\big(\sqrt{2}\,iz\big), \tag{1.4}$$

and it satisfies the Riccati equation

$$w' = \pm\big(w^2 + 2zw + (2n+1)\big) - 1;$$

(2) *especially,*

$$(1.5)\qquad g_n^{\mp}(z) = \begin{cases} \chi^{\mp}_{n,\infty}(z) = -z \mp \left(-z + \dfrac{H_n'(z)}{H_n(z)}\right) & \text{if } n \in \mathbf{N} \cup \{0\}, \\ \chi^{\mp}_{n,0}(z) = -z \mp \left(z + \dfrac{iH'_{-n-1}(iz)}{H_{-n-1}(iz)}\right) & \text{if } n \in \mathbf{N}_- \end{cases}$$

are rational solutions, where $\mathbf{N}_- = \{-l \mid l \in \mathbf{N}\}$*;*

(3) *if* $(n,c) \in \mathscr{P} = \big((\mathbf{N} \cup \{0\}) \times \mathbf{C}\big) \bigcup \big(\mathbf{N}_- \times (\mathbf{C} \cup \{\infty\} \setminus \{0\})\big)$, *then*

$$(1.6)\qquad C_1 r^2 \le T(r, \chi^{\mp}_{n,c}) \le C_2 r^2,$$

$$(1.7)\qquad N\big(r, 1/(\chi^{\mp}_{n,c} - g_n^{\mp})\big) = 0,$$

where $C_j = C_j(n,c)$, $j = 1,2$, *are some positive constants.*

Theorem 1.2. *Let* $\phi(z)$ *be an arbitrary transcendental meromorphic solution of* (IV). *Suppose that* $g(z)$ *is a meromorphic function small with respect to* $\phi(z)$, *and that the pair* $\big(\phi(z), g(z)\big)$ *coincides with none of* $\big(\chi^{\mp}_{n,c}(z), g_n^{\mp}(z)\big)$, $(n,c) \in \mathscr{P}$. *Then*

$$(1.8)\qquad \delta(g,\phi) \le \tfrac{1}{2} \qquad \text{if } \beta \ne 0,$$

$$(1.9)\qquad \delta(g,\phi) \le \tfrac{3}{4} \qquad \text{if } \beta = 0.$$

2. Preliminaries

From [6; Lemma 2.4.2] and [7; Theorem 6], we derive the following two lemmas.

Lemma 2.1. *Let* f *be a transcendental meromorphic function such that* $f^{p+1} = Q(z,f)$, $p \in \mathbf{N}$, *where* $Q(z,u)$ *is a polynomial in* u *and its derivatives with meromorphic coefficients* $\big\{a_\mu(z) \bigm| \mu \in M\big\}$. *Suppose that* $m(r,a_\mu) = S(r,f)$ *for all* $\mu \in M$, *and that the total degree of* $Q(z,u)$ *as a polynomial in* u *and its derivatives does not exceed* p. *Then,* $m(r,f) = S(r,f)$.

Lemma 2.2. *Let* $F(z,u)$ *be a polynomial in* u *and its derivatives with meromorphic coefficients* $\big\{b_\kappa(z) \bigm| \kappa \in K\big\}$. *Suppose that* $u = f$ *is a transcendental meromorphic function satisfying* $F(z,f) = 0$, *and that* $T(r,b_\kappa) = S(r,f)$ *for all* $\kappa \in K$. *If* $F(z,0) \not\equiv 0$, *then* $m(r,1/f) = S(r,f)$.

Consider a differential equation of the form

$$P(z,w,w',\dots,w^{(N)})=R(z,w)\qquad (N\in\mathbf{N}). \tag{2.1}$$

Here $P(z,w,w',\dots,w^{(N)})$ is a polynomial in w and its derivatives, $R(z,w)$ is rational in w, and the coefficients of P and R are meromorphic functions $\{c_\lambda(z) \mid \lambda\in\Lambda\}$. The following result is known by [13], [2; Satz 5] (see also [6; Theorem 13.1]).

Lemma 2.3. *Suppose that $w=f(z)$ is a meromorphic solution of (2.1), and that $T(r,c_\lambda)=S(r,f)$ for all $\lambda\in\Lambda$. Then $R(z,w)$ is a polynomial in w.*

Among parabolic cylinder functions the following relations hold (see [1; Section 8.2]).

Lemma 2.4. *We have*

$$D_\nu(t)=e^{-\nu\pi i}D_\nu(-t)+\frac{(2\pi)^{1/2}}{\Gamma(-\nu)}e^{-(\nu+1)\pi i/2}D_{-\nu-1}(it), \tag{2.2}$$

$$D_{-\nu-1}(it)=-e^{\nu\pi i}D_{-\nu-1}(-it)+\frac{(2\pi)^{1/2}}{\Gamma(\nu+1)}e^{\nu\pi i/2}D_\nu(-t). \tag{2.3}$$

For a movable pole of a solution of (IV) we have the following ([15]).

Lemma 2.5. *For an arbitrary meromorphic solution $\phi(z)$ of (IV), around a movable pole $z=z_\infty$, $\phi(z)=\mp(z-z_\infty)^{-1}-z_\infty+O(z-z_\infty)$.*

Multiplying (IV) by $2w'/w^2$, we have

$$\frac{d}{dz}\left(\frac{(w')^2}{w}-w^3-4zw^2-4(z^2-\alpha)w+\frac{2\beta}{w}\right)=-4w^2-8zw. \tag{2.4}$$

Using (2.4), we obtain the following lemma.

Lemma 2.6. *Suppose that both $\phi(z)$ and $g(z)$ are solutions of (IV). Then $\Phi(z)=\phi(z)-g(z)$ satisfies*

$$U'(z)=-4\Phi(z)^2-8\big(g(z)+z\big)\Phi(z) \tag{2.5}$$

with

$$U(z)=\frac{\Phi'(z)^2+2g'(z)\Phi'(z)-\big(g'(z)^2/g(z)\big)\Phi(z)}{\Phi(z)+g(z)} -\Phi(z)^3-G_2(z)\Phi(z)^2-G_1(z)\Phi(z)-\frac{2\beta\Phi(z)}{g(z)\big(\Phi(z)+g(z)\big)}, \tag{2.6}$$

$$G_2(z)=3g(z)+4z,\qquad G_1(z)=3g(z)^2+8zg(z)+4(z^2-\alpha). \tag{2.7}$$

3. Evaluation of $m(r, 1/\Phi)$

In this section we use the following notation: for a meromorphic function $f(z)$ and for a set $A \subset \mathbf{C}$,

$$N(r,f)|_A = \int_0^r \big(n(\rho,f)|_A - n(0,f)|_A\big)\,\frac{d\rho}{\rho} + n(0,f)|_A \log r,$$
$$n(\rho,f)|_A = \sum_{\substack{\sigma\in A,\,|\sigma|\le\rho \\ f(\sigma)=\infty}} \mu(\sigma,f),$$

where $\mu(\sigma, f)$ denotes the multiplicity of the pole $z = \sigma$ of $f(z)$.

Suppose that $\phi(z)$ is an arbitrary transcendental meromorphic solution of (IV), and that $g(z)$ is a meromorphic function small with respect to $\phi(z)$ not necessarily satisfying (IV). Then $W = \Phi(z) = \phi(z) - g(z)$ is a solution of the equation

$$2WW'' + 2g(z)W'' - (W')^2 - 2g'(z)W' = 3W^4 + g_3(z)W^3 + g_2(z)W^2 + g_1(z)W + g_0(z),$$

where

$$\begin{aligned}
g_3(z) &= 12g(z) + 8z,\\
g_2(z) &= 18g(z)^2 + 24zg(z) + 4(z^2-\alpha),\\
g_1(z) &= 12g(z)^3 + 24zg(z)^2 + 8(z^2-\alpha)g(z) - 2g''(z),\\
g_0(z) &= -2g(z)g''(z) + g'(z)^2 + 3g(z)^4 + 8zg(z)^3 + 4(z^2-\alpha)g(z)^2 + 2\beta.
\end{aligned}$$

By $T(r, g) = S(r, \phi)$ and Lemma 2.2, we have the following.

Proposition 3.1. *If $g_0(z) \not\equiv 0$, then $m(r, 1/\Phi) = S(r, \phi)$.*

In what follows we estimate $m(r, 1/\Phi)$ under the supposition

$$g_0(z) \equiv 0. \tag{3.1}$$

This relation means that $g(z)$ is also a solution of (IV).

3.1. Case $\beta \neq 0$. By (3.1) and Lemma 2.6,

$$U(z) = U(z_0) - \int_{z_0}^{z} \big(4\Phi(s)^2 + 8\big(g(s)+s\big)\Phi(s)\big)\,ds \tag{3.2}$$

for $z_0 \in \mathbf{C}$ satisfying $U(z_0) \neq \infty$, where $U(z)$ is given by (2.6). (Note that $g(z) \not\equiv 0$, since $\beta \neq 0$.) Consider the function

$$\Omega(z) = U(z)/\Phi(z). \tag{3.3}$$

Note that $\phi(z)^4 = Q\big(z, \phi(z)\big)$, where

$$3Q(z,u) = 2uu'' - (u')^2 - 8zu^3 - 4(z^2-\alpha)u^2 - 2\beta.$$

By Lemma 2.1 with $p=3$,

$$m(r,\Phi) = O\big(m(r,\phi) + m(r,g)\big) = S(r,\phi). \tag{3.4}$$

Furthermore, applying Lemma 2.2 to (IV) under the condition $\beta \neq 0$, we have $m\big(r, 1/(\Phi+g)\big) = m(r,1/\phi) = S(r,\phi)$. Hence, from (2.6), it follows that

$$\begin{aligned} m(r,\Omega) &= O\big(m(r,\Phi'/\Phi) + m(r,\Phi) + m\big(r,1/(\Phi+g)\big) + T(r,g) + \log r\big) \\ &= S(r,\phi). \end{aligned} \tag{3.5}$$

By Lemma 2.5 and (3.1), every pole of $\Phi(z)$ is simple and belongs to either of the sets

$$\Pi = \big\{\sigma \bigm| \phi(\sigma) = \infty,\ g(\sigma) \neq \infty\big\}, \qquad \Pi' = \big\{\sigma \bigm| g(\sigma) = \infty\big\}.$$

Note that, around $z = z_\infty \in \Pi$, $\Phi(z) = \mp(z - z_\infty)^{-1} + O(1)$. Substituting this into (3.3) and using the right-hand side of (3.2), we have $\Omega(z_\infty) = \mp 4$ for every $z_\infty \in \Pi$. Suppose that

$$\Omega(z) \not\equiv \mp 4. \tag{3.6}$$

Then, taking (3.5) into consideration, we have

$$N(r,\Phi)|_\Pi \leq N\big(r, 1/(\Omega^2 - 4^2)\big) \leq 2T(r,\Omega) + O(1) \leq 2N(r,\Omega) + S(r,\phi). \tag{3.7}$$

By (3.1) and Lemma 2.5, every pole of $g(z)$ is simple, which implies

$$N(r,\Phi)|_{\Pi'} \leq N(r,g) = S(r,\phi). \tag{3.8}$$

Let z'_∞ be an arbitrary pole of $\Omega(z)$. By (3.3), z'_∞ satisfies either (i) $\Phi(z'_\infty) = 0$ or (ii) $U(z'_\infty) = \infty$. Recall that every pole of $\Phi(z)$ belongs to $\Pi \cup \Pi'$, and that $\Omega(z)$ is analytic around each point of Π. By these facts and (3.2), the case (ii) implies that $z'_\infty \in \Pi'$. Hence every pole of $\Omega(z)$ belongs to $\{\sigma \mid \Phi(\sigma) = 0\} \cup \Pi'$, and

$$N(r,\Omega) \leq N(r,1/\Phi) + O\big(N(r,g)\big) = N(r,1/\Phi) + S(r,\phi). \tag{3.9}$$

From (3.7), (3.8) and (3.9), it follows that

$$\begin{aligned} N(r,\Phi) &\leq N(r,\Phi)|_\Pi + N(r,\Phi)|_{\Pi'} \leq 2N(r,\Omega) + S(r,\phi) \\ &\leq 2N(r,1/\Phi) + S(r,\phi) = 2\big(T(r,1/\Phi) - m(r,1/\Phi)\big) + S(r,\phi). \end{aligned}$$

Observing that $N(r,\Phi) = T(r,\Phi) + S(r,\phi) = T(r,\phi) + S(r,\phi)$ (cf. (3.4)), and that $T(r,1/\Phi) = T(r,\phi) + S(r,\phi)$, we have

Proposition 3.2. *Under the condition $\beta \neq 0$, if (3.6) holds, then $m(r,1/\Phi) \leq \frac{1}{2}T(r,\phi) + S(r,\phi)$.*

3.2. Case $\beta = 0$. Observing (3.1), we write (3.2) in the form

$$\begin{aligned} V(z) &= 2g''(z)\Phi(z) - 2g'(z)\Phi'(z) \\ &\quad + \big(\Phi(z)+g(z)\big)\left[U(z_0) - \int_{z_0}^{z} \big(4\Phi(s)^2 + 8(g(s)+s)\Phi(s)\big)\,ds\right], \end{aligned} \tag{3.10}$$

where

$$V(z) = \Phi'(z)^2 - \Phi(z)^4 - \big(G_2(z)+g(z)\big)\Phi(z)^3 - \big(G_1(z)+G_2(z)g(z)\big)\Phi(z)^2 \tag{3.11}$$

(for $G_1(z)$ and $G_2(z)$ cf. (2.7)). Put

$$\Theta(z) = \big(g'(z)+2\big)^{-1} V(z)/\Phi(z)^2. \tag{3.12}$$

(By (3.1), $g'(z)+2 \not\equiv 0$.) Observe that (3.4) is valid under the supposition $\beta = 0$ as well. Using (3.11), we have

$$m(r,\Theta) = O\big(m(r,\Phi'/\Phi) + m(r,\Phi) + T(r,g) + \log r\big) = S(r,\phi). \tag{3.13}$$

Let $z = z_\infty$ be a pole of $\Phi(z)$ belonging to

$$\Pi_0 = \big\{\sigma \;\big|\; \phi(\sigma) = \infty,\ g(\sigma) \neq \infty,\ g'(\sigma) \neq -2\big\}.$$

Then, by Lemma 2.5, around $z = z_\infty$, $\Phi(z) = \mp(z - z_\infty)^{-1} + O(1)$. Substituting this into (3.12) and using (3.10), we have $\Theta(z_\infty) = \mp 2$. Suppose that

$$\Theta(z) \not\equiv \mp 2. \tag{3.14}$$

Then, by (3.13),

$$N(r,\Phi)|_{\Pi_0} \le N\big(r, 1/(\Theta^2 - 2^2)\big) \le 2T(r,\Theta) + O(1) \le 2N(r,\Theta) + S(r,\phi).$$

Furthermore, set

$$\Pi_1 = \big\{\sigma \;\big|\; g(\sigma) = \infty\big\}, \qquad \Pi_2 = \big\{\sigma \;\big|\; g'(\sigma) = -2\big\}.$$

Since every pole of $\Phi(z)$ is simple,

$$\begin{aligned} N(r,\Phi) &\le N(r,\Phi)|_{\Pi_0} + N(r,\Phi)|_{\Pi_1\cup\Pi_2} \\ &\le 2N(r,\Theta) + S(r,\phi) + O\big(T(r,g)\big) \le 2N(r,\Theta) + S(r,\phi). \end{aligned} \tag{3.15}$$

Observing that every pole of $\Theta(z)$ belongs to $\{\sigma \mid \Phi(\sigma) = 0\} \cup \Pi_1 \cup \Pi_2$, we have

$$\begin{aligned} N(r,\Theta) &\le 2N(r,1/\Phi) + N\big(r,1/(g'+2)\big) + O\big(N(r,g)\big) \\ &\le 2\big(T(r,\Phi) - m(r,1/\Phi)\big) + S(r,\phi). \end{aligned} \tag{3.16}$$

Then, from (3.4), (3.15) and (3.16), we obtain the following.

Proposition 3.3. *Under the condition* $\beta = 0$, *if* (3.14) *holds, then* $m(r,1/\Phi) \le \frac{3}{4}T(r,\phi) + S(r,\phi)$.

4. Exceptional solutions

We wish to find a pair of solutions $\big(\phi(z), g(z)\big)$ such that (3.6) or (3.14) is not satisfied.

Lemma 4.1. *Suppose that* (IV) *admits a solution* $\psi(z)$ *which satisfies the equation*

$$w' = \pm(w^2 + 2zw - 2\alpha) - 2 \tag{4.1\,,\pm}$$

as well. Then $\beta = -2(\alpha \pm 1)^2$.

Proof. Observing that $\psi'(z) = \pm\big(\psi(z)^2 + 2z\psi(z) - 2\alpha\big) - 2$, and that $\psi''(z) = 2\big(\psi(z) + z\big)\big(\psi(z)^2 + 2z\psi(z) - (2\alpha \pm 2)\big) \pm 2\psi(z)$, we have $2\psi(z)\psi''(z) - \psi'(z)^2 = 3\psi(z)^4 + 8z\psi(z)^3 + 4(z^2 - \alpha)\psi(z)^2 - 4(\alpha \pm 1)^2$. Since $\psi(z)$ satisfies (IV), it follows that $\beta = -2(\alpha \pm 1)^2$. □

By the same computation as above, we obtain the lemma below.

Lemma 4.2. *If* $\beta = -2(\alpha \pm 1)^2$, *then every solution of* $(4.1\,,\pm)$ *satisfies* (IV).

Proposition 4.3. *Suppose that* $\Omega(z) \equiv \mp 4$ (*in case* $\beta \neq 0$) *or that* $\Theta(z) \equiv \mp 2$ (*in case* $\beta = 0$). *Then*

(1) $\beta = -2(\alpha \pm 1)^2$;

(2) *both* $g(z)$ *and* $\phi(z)$ *are solutions of* $(4.1\,,\pm)$.

Proof. Case $\beta \neq 0$: Suppose that $\Omega(z) \equiv \mp 4$, namely that $U(z) = \mp 4\Phi(z)$. By (3.2), we have

$$\Phi'(z) = \pm\big(\Phi(z)^2 + 2\big(g(z) + z\big)\Phi(z)\big). \tag{4.2\,,\pm}$$

On the other hand, from (2.6), it follows that

$$\begin{aligned} &\Phi'(z)^2 + 2g'(z)\Phi'(z) - g'(z)^2\Phi(z)/g(z) \\ &\quad - \big(\Phi(z)^3 + G_2(z)\Phi(z)^2 + G_1(z)\Phi(z)\big)\big(\Phi(z) + g(z)\big) - 2\beta\Phi(z)/g(z) \\ &\quad = \mp 4\Phi(z)\big(\Phi(z) + g(z)\big). \end{aligned} \tag{4.3\,,\mp}$$

Substitution of $(4.2\,,\pm)$ into $(4.3\,,\mp)$ yields $G_2^*(z)\Phi(z)^2 + G_1^*(z)\Phi(z) = 0$, where $G_2^*(z) = g'(z) \mp \big(g(z)^2 + 2zg(z) - 2\alpha\big) + 2$, and $G_1^*(z)$ is a rational function of z, $g(z)$ and $g'(z)$. By the supposition $T(r, g) = S(r, \phi) = S(r, \Phi)$, we obtain $G_2^*(z) \equiv 0$, which implies that $g(z)$ satisfies $(4.1\,,\pm)$. Furthermore, substituting $\Phi(z) = \phi(z) - g(z)$ into $(4.2\,,\pm)$ we see that $\phi(z)$ is also a solution of $(4.1\,,\pm)$. From Lemma 4.1, the assertion (1) follows immediately.

Case $\beta = 0$: Suppose that $\Theta(z) \equiv \mp 2$. From (3.11) and (3.12), it follows that

$$\Phi'(z)^2 = \Phi(z)^2 F\big(z, \Phi(z)\big), \qquad F(z,W) = W^2 + h_1(z)W + h_0(z), \tag{4.4}$$

where $h_1(z) = 4\big(g(z) + z\big)$, $h_0(z) = 6g(z)^2 + 12zg(z) + 4(z^2 - \alpha) \mp 2\big(g'(z) + 2\big)$. Differentiating (4.4), we observe that $W = \Phi(z)$ is a solution of the equation

$$\big(2W'' - 2WF(z,W) - W^2 F_W(z,W)\big)^2 = W^2 F_z(z,W)^2 / F(z,W)$$

$(F_W = \partial F/\partial W,\ F_z = \partial F/\partial z)$, whose coefficients are small with respect to $\Phi(z)$. By Lemma 2.3, $W^2\big(h_1'(z)W + h_0'(z)\big)^2/F(z,W)$ is a polynomial in W. Hence, we have $F(z,W) = \big(W + h_1(z)/2\big)^2$ or $h_0(z) = h_1(z) \equiv 0$ (cf. [14], [5], see also [4], [10]). Since the latter case is impossible, we have $4h_0(z) - h_1(z)^2 \equiv 0$, from which it follows that $g(z)$ satisfies $(4.1, \pm)$. Then, by Lemma 4.1, $\alpha = \mp 1$. It remains to show that $\phi(z)$ is also a solution of $(4.1, \pm)$. By (4.4), $\Phi(z)$ satisfies either of the equations

$$\Phi'(z) = -\Phi(z)\big(\Phi(z) + h_1(z)/2\big), \tag{4.5, $-$}$$

$$\Phi'(z) = \Phi(z)\big(\Phi(z) + h_1(z)/2\big). \tag{4.5, $+$}$$

To determine the sign, suppose that $\alpha = -1$ and that $g(z)$ satisfies $(4.1, +)$. If $(4.5, -)$ occurs, then

$$\phi'(z) = -\phi(z)^2 - 2z\phi(z) + G_0(z)$$

with $G_0(z) = 2g(z)^2 + 4zg(z)$. By this relation and (IV) with $(\alpha, \beta) = (-1, 0)$, we have $(-2G_0(z) - 8)\phi(z)^2 + 2G_0'(z)\phi(z) - G_0(z)^2 = 0$, which contradicts the fact $T(r, G_0) = S(r, \phi)$. Hence, if $\alpha = -1$, then $(4.5, +)$ holds, and in consequence $\phi(z)$ satisfies $(4.1, +)$. Similarly, we can verify that both $g(z)$ and $\phi(z)$ satisfy $(4.1, -)$ if $\alpha = 1$. □

We examine properties of solutions of $(4.1, \pm)$ (see also [3], [9], [15; Section 4], [16]).

Lemma 4.4. *Every solution of* $(4.1, \pm)$ *is expressible in the form*

$$w = -z \mp u'(z)/u(z), \tag{4.6}$$

where $u(z)$ *is a nontrivial solution of*

$$u'' - \big(z^2 + (2\alpha \pm 1)\big)u = 0. \tag{4.7, $\pm$}$$

Conversely, for an arbitrary nontrivial solution $u(z)$ *of* $(4.7, \pm)$, *the function given by* (4.6) *satisfies* $(4.1, \pm)$.

Proof. Putting $w = -z \mp u'/u$ into $(4.1, \pm)$, we obtain the lemma. □

Lemma 4.5. *For arbitrary solutions* $\chi_1(z)$, $\chi_2(z)$ *of* $(4.1, \pm)$ *satisfying* $\chi_1(z) \not\equiv \chi_2(z)$, *we have* $N\big(r, 1/(\chi_1 - \chi_2)\big) = 0$.

Proof. Note that $\chi_j(z)$, $j = 1, 2$, are expressed as $\chi_j(z) = -z \mp u_j'(z)/u_j(z)$, where $u_j(z)$, $j = 1, 2$, are linearly independent solutions of $(4.7, \pm)$. From the fact $u_1(z)u_2'(z) - u_1'(z)u_2(z) \equiv C_0 \in \mathbf{C} \setminus \{0\}$, the conclusion immediately follows. □

Lemma 4.6. *If* $\alpha \notin \mathbf{Z}$, *then every solution* $\chi(z)$ *of* $(4.1, \pm)$ *satisfies* $C_1 r^2 \le T(r, \chi) \le C_2 r^2$, *where* C_1 *and* C_2 *are some positive constants.*

Proof. Note that, by the transformation $z = t/\sqrt{2}$, $(4.7, \pm)$ is changed into

$$\frac{d^2u}{dt^2} + \Big(-\frac{t^2}{4} - \frac{1}{2}(2\alpha \pm 1)\Big)u = 0 \tag{4.8, $\pm$}$$

(cf. (1.1), (1.2). By Lemma 4.4, $\chi(z)$ is expressible in the form

$$\begin{aligned} &\chi(z) = -z \mp \eta_{\mathbf{c}}'(z)/\eta_{\mathbf{c}}(z), \\ &\eta_{\mathbf{c}}(z) = c_1 D_\nu\big(\sqrt{2}\,z\big) + c_2 D_{-\nu-1}\big(\sqrt{2}\,iz\big), \quad \nu = -\alpha \mp \tfrac{1}{2} - \tfrac{1}{2} \end{aligned} \tag{4.9}$$

for some $\mathbf{c} = (c_1, c_2) \in \mathbf{C}^2 - \{(0,0)\}$. If $c_1c_2 \ne 0$, using (1.2), we have

$$\begin{aligned} \eta_{\mathbf{c}}(z) = {}& 2^{\nu/2} c_1 z^\nu \exp(-z^2/2)\big(1 + o(1)\big) \\ & \times \big[1 + o(1) + 2^{-\nu-1/2} e^{(-\nu-1)\pi i/2}(c_2/c_1) z^{-2\nu-1} \exp(z^2)\big] \end{aligned}$$

as $z \to \infty$ through the sector $|\arg z + \frac{1}{4}\pi| < \frac{1}{2}\pi$. Hence $\eta_{\mathbf{c}}(z)$ possesses a sequence of simple zeros $z_k = \sqrt{2\pi}\, e^{-\pi i/4} k^{1/2}\big(1 + o(1)\big)$, $k \in \mathbf{N}$, which implies $T(r, \chi) \ge C_1 r^2$ for some $C_1 > 0$. In case $c_1c_2 = 0$, we also obtain the same estimate. For example, if $c_1 \ne 0$, $c_2 = 0$, then, using (2.2) with $\nu = -\alpha \mp \frac{1}{2} - \frac{1}{2} \notin \mathbf{Z}$, we obtain an analogous sequence of simple zeros in the sector $|\arg z + \frac{3}{4}\pi| < \frac{1}{2}\pi$. Furthermore, for each $\mathbf{c} \in \mathbf{C}^2 \setminus \{(0,0)\}$, $T(r, \eta_{\mathbf{c}}) = O(r^2)$ (cf. [12; Sections 8 and 22]). Hence $T(r, \chi) = O(r^2)$, which completes the proof. □

Proposition 4.7. *Suppose that* $-\alpha \mp \frac{1}{2} - \frac{1}{2} = n \in \mathbf{Z}$. *Then all the solutions of* $(4.1, \pm)$ *constitute the family* $\big\{\chi^{\mp}_{n,c}(z) \bigm| c \in \mathbf{C} \cup \{\infty\}\big\}$ *with the properties* (1), (2), (3) *of Theorem* 1.1.

Proof. Under the supposition $n \in \mathbf{Z}$, every solution of $(4.1, \pm)$ is also written in the form (4.9) with $\nu = n$, so that it has the property (1). If $n \in \mathbf{N} \cup \{0\}$, $\mathbf{c} = (1, 0)$, then $\eta_{\mathbf{c}}(z) = D_n\big(\sqrt{2}\,z\big) = 2^{-n/2}\exp(-z^2/2)H_n(z)$. If $-n \in \mathbf{N}$, $\mathbf{c} = (0, 1)$, then $\eta_{\mathbf{c}}(z) = D_{-n-1}\big(\sqrt{2}\,iz\big) = 2^{(n+1)/2}\exp(z^2/2)H_{-n-1}(iz)$. In these cases, we obtain rational solutions given by (1.5), which implies the property (2). If $n \in \mathbf{N} \cup \{0\}$, $c_2 \ne 0$, or if $-n \in \mathbf{N}$, $c_1 \ne 0$, then we can verify (1.6) by the same argument as in the proof of Lemma 4.6. By Lemma 4.5, we have (1.7). Thus the property (3) is verified. □

5. Proofs of the main results

5.1. Proof of Theorem 1.1. From Lemmas 4.1, 4.2 and Proposition 4.7, we immediately obtain Theorem 1.1.

5.2. Proof of Theorem 1.2. By Propositions 3.1, 3.2 and 3.3, for an arbitrary solution $\phi(z)$ of (IV) and a small function $g(z)$ with respect to $\phi(z)$, the conclusion of Theorem 1.2 is valid unless $g(z)$ is a solution of (IV) satisfying $\Omega(z) \equiv \mp 4$ (in case $\beta \neq 0$) or $\Theta(z) \equiv \mp 2$ (in case $\beta = 0$). By Proposition 4.3, these exceptional cases may occur only when $\beta = -2(\alpha \pm 1)^2$, and then both $g(z)$ and $\phi(z)$ satisfy $(4.1, \pm)$. By Proposition 4.7, in case $n = -\alpha \mp \frac{1}{2} - \frac{1}{2} \in \mathbf{Z}$, a pair $\bigl(\phi(z), g(z)\bigr)$ of solutions of $(4.1, \pm)$ satisfies $T(r, g) = S(r, \phi)$ if and only if $\bigl(\phi(z), g(z)\bigr) = \bigl(\chi^{\mp}_{n,c}(z), g^{\mp}_n(z)\bigr)$ for some $(n, c) \in \mathscr{P}$ (cf. Theorem 1.1). By Lemma 4.6, if $\alpha \notin \mathbf{Z}$, then such a pair of solutions of $(4.1, \pm)$ does not exist. This completes the proof.

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