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LINDELÖF THEOREMS FOR MONOTONE SOBOLEV FUNCTIONS

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Abstract. This paper deals with Lindelöf type theorems for monotone functions in weighted Sobolev spaces.

1. Introduction

Let $\mathbf{R}^n$, $n \geq 2$, denote the n-dimensional Euclidean space. We use the notation $\mathbf{D}$ to denote the upper half space of $\mathbf{R}^n$, that is,

$$\mathbf{D} = \{x = (x_1, \ldots, x_{n-1}, x_n) : x_n > 0\}.$$

Denote by $B(x,r)$ the open ball centered at x with radius r, and set $\sigma B(x,r) = B(x, \sigma r)$ for $\sigma > 0$ and $S(x,r) = \partial B(x,r)$.

A continuous function u on $\mathbf{D}$ is called monotone in the sense of Lebesgue (see [5]) if for every relatively compact open set $G \subset \mathbf{D}$,

$$\max_{\overline{G}} u = \max_{\partial G} u \qquad \text{and} \qquad \min_{\overline{G}} u = \min_{\partial G} u.$$

If u is monotone in $\mathbf{D}$ and $p > n-1$, then

$$(1) \qquad |u(x) - u(x')| \leq Mr \left(\frac{1}{r^n} \int_{B(y,2r)} |\nabla u(z)|^p \, dz \right)^{1/p}$$

for every $x, x' \in B(y,r)$, whenever $B(y,2r) \subset \mathbf{D}$ (see [6, Theorem 1] and [4, Theorem 2.8]). For further results of monotone functions, we refer to [3], [13] and [15].

Our aim in the present note is to extend the second author's result [12, Theorem 2] and the most recent results by Manfredi–Villamor [8].

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Theorem 1. *Let u be a monotone function on $\mathbf{D}$ satisfying*

$$\int_{\mathbf{D}} |\nabla u(z)|^p z_n^\alpha \, dz < \infty, \tag{2}$$

where $p > n-1$ and $0 \le n+\alpha-p < 1$. Define

$$E_{n+\alpha-p} = \left\{ \xi \in \partial\mathbf{D} : \limsup_{r\to 0} r^{p-\alpha-n} \int_{B(\xi,r)\cap\mathbf{D}} |\nabla u(z)|^p z_n^\alpha \, dz > 0 \right\}.$$

If $\xi \in \partial\mathbf{D} - E_{n+\alpha-p}$ and there exists a curve γ in $\mathbf{D}$ tending to ξ along which u has a finite limit, then u has a nontangential limit at ξ.

Remark 1. We know that $E_{n+\alpha-p}$ has $(n+\alpha-p)$-dimensional Hausdorff measure zero, and hence it is of $C_{1-\alpha/p,p}$-capacity zero; for these results, see Meyers [9], [10] and the second author's book [13].

We shall give a generalization of Theorem 1 (see Theorem 2 below). We proceed to the proof of Theorem 1 for the sake of clarity.

Throughout this paper, let M denote various positive constants independent of the variables in question, and $M(\varepsilon)$ a positive constant which depends on ε.

2. Proof of Theorem 1

A sequence $\{X_j\}$ is called regular at $\xi \in \partial\mathbf{D}$ if $X_j \to \xi$ and

$$|X_j - \xi| < c|X_{j+1} - \xi|$$

for some constant $c > 0$.

First we give the following result, which can be proved by (1).

Lemma 1. *Let u be a monotone function on $\mathbf{D}$ satisfying* (2) *with $n-1 < p \le \alpha + n$. If $\xi \in \partial\mathbf{D} - E_{n+\alpha-p}$ and there exists a regular sequence $\{X_j\} \subset \mathbf{D}$ with $X_j = \xi + (0,\ldots,0,r_j)$ such that $u(X_j)$ has a finite limit, then u has a nontangential limit at ξ.*

Proof of Theorem 1. For $r > 0$ sufficiently small, take $C(r) \in \gamma \cap S(\xi,r)$. Letting $C_1(r) = \xi + (0,\ldots,0,r)$, take an end point $C_2(r) \in \partial\mathbf{D}$ of a quarter of circle containing $C_1(r)$ and $C(r)$.

Let $\varrho_{\mathbf{D}}(x)$ denote the distance of $x \in \mathbf{D}$ from the boundary $\partial\mathbf{D}$, that is, $\varrho_{\mathbf{D}}(x) = x_n$. We take a finite covering $\{B(X_j, 4^{-1}\varrho_{\mathbf{D}}(X_j))\}$ of circular arc $\widehat{C(r)C_1(r)}$ such that

(i) $X_1 = C(r)$ and $X_{N+1} = C_1(r)$;

(ii) $|z-\xi| < 2r$ and $|z - C_2(r)| \sim \varrho_{\mathbf{D}}(z)$ for $z \in A(\xi,r) = \bigcup_j 2B_j$, where $B_j = B(X_j, 4^{-1}\varrho_{\mathbf{D}}(X_j))$;

(iii) $B_j \cap B_{j+1} \neq \emptyset$ for each j;
(iv) $\sum_j \chi_{2B_j}$ is bounded, where χ_A denotes the characteristic function of A;
see Heinonen [2] and Hajłasz–Koskela [1]. By the monotonicity of u we see that

$$|u(x) - u(X_j)| \le M \varrho_{\mathbf{D}}(X_j) \left(\frac{1}{\varrho_{\mathbf{D}}(X_j)^n} \int_{2B_j} |\nabla u(z)|^p \, dz \right)^{1/p}$$

for $x \in B_j$. We have by Hölder's inequality

$$\begin{aligned}
\big|u\big(C_1(r)\big) - u\big(C(r)\big)\big| &\le |u(X_1) - u(X_2)| + |u(X_2) - u(X_3)| \\
&\quad + \cdots + |u(X_N) - u(X_{N+1})| \\
&\le M \sum_j \varrho_{\mathbf{D}}(X_j)^{1-(n-\delta)/p} \left(\int_{2B_j} |\nabla u(z)|^p \varrho_{\mathbf{D}}(X_j)^{-\delta} \, dz \right)^{1/p} \\
&\le M \left(\sum_j \varrho_{\mathbf{D}}(X_j)^{p'(p-n+\delta)/p} \right)^{1/p'} \\
&\quad \times \left(\int_{A(\xi,r)} |\nabla u(z)|^p \varrho_{\mathbf{D}}(z)^{-\delta} \, dz \right)^{1/p} \\
&\le M \left(\sum_j \varrho_{\mathbf{D}}(X_j)^{p'(p-n+\delta)/p} \right)^{1/p'} \\
&\quad \times \left(\int_{B(\xi,2r)\cap\mathbf{D}} |\nabla u(z)|^p \varrho_{\mathbf{D}}(z)^{\alpha} |C_2(r) - z|^{-\delta-\alpha} \, dz \right)^{1/p}
\end{aligned}$$

for $\delta > 0$, where $1/p + 1/p' = 1$. Here note that

$$\begin{aligned}
\sum_j \varrho_{\mathbf{D}}(X_j)^{p'(p-n+\delta)/p} &\le M \int_{A(\xi,r)} \varrho_{\mathbf{D}}(z)^{p'(p-n+\delta)/p-n} \, dz \\
&\le M \int_{A(\xi,r)} |C_2(r) - z|^{p'(p-n+\delta)/p-n} \, dz \\
&\le M r^{p'(p-n+\delta)/p}
\end{aligned}$$

when $\delta > n - p$. Moreover,

$$\int_{2^{-j}}^{2^{-j+1}} |C_2(r) - z|^{-\delta-\alpha} \, dr \le \int_{2^{-j}}^{2^{-j+1}} \big|r - |z|\big|^{-\delta-\alpha} \, dr \le M 2^{-j(1-\delta-\alpha)} \tag{3}$$

when $-\alpha < \delta < 1 - \alpha$. Hence it follows that

$$\int_{2^{-j}}^{2^{-j+1}} \big|u\big(C_1(r)\big) - u\big(C(r)\big)\big|^p \, dr/r \le M 2^{-j(p-n-\alpha)} \int_{B(\xi,2^{-j+2})\cap\mathbf{D}} |\nabla u(z)|^p \varrho_{\mathbf{D}}(z)^{\alpha} \, dz.$$

Since $\xi \in \partial \mathbf{D} - E_{n+\alpha-p}$, we can find a sequence $\{r_j\}$ such that $2^{-j} < r_j < 2^{-j+1}$ and

$$\lim_{j\to\infty} \left| u\big(C_1(r_j)\big) - u\big(C(r_j)\big) \right| = 0.$$

By our assumption we see that $u\big(C_1(r_j)\big)$ has a finite limit as $j \to \infty$. If we note that $\{C_1(r_j)\}$ is regular at ξ, then Lemma 1 proves the required conclusion of the theorem.

3. Monotone functions on a measure space $(\mathbf{D}; \mu)$

Let μ be a Borel measure on $\mathbf{R}^n$ satisfying the doubling condition:

$$\mu(2B) \le M\mu(B)$$

for every ball $B \subset \mathbf{R}^n$. We further assume that

$$\frac{\mu(B')}{\mu(B)} \ge M\left(\frac{\operatorname{diam}(B')}{\operatorname{diam}(B)}\right)^s \tag{4}$$

for all $B' = B(\xi', r')$ and $B = B(\xi, r)$ with $\xi', \xi \in \partial\mathbf{D}$ and $B' \subset B$, where $s > 1$ and $\operatorname{diam}(B)$ denotes the diameter of B.

A pair $(u, g) \in L^1_{\mathrm{loc}}(\mathbf{D}; \mu) \times L^p_{\mathrm{loc}}(\mathbf{D}; \mu)$ is said to satisfy p-Poincaré inequality if $g \ge 0$ on $\mathbf{D}$ and

$$\frac{1}{\mu(B)} \int_B |u(x) - u_B|\, d\mu(x) \le M \operatorname{diam}(B) \left(\frac{1}{\mu(\sigma B)} \int_{\sigma B} g(z)^p\, d\mu(z) \right)^{1/p}$$

for every ball B with $\sigma B \subset \mathbf{D}$, where $\sigma > 1$ and

$$u_B = ⨍_B u(y)\, d\mu(y) = \frac{1}{\mu(B)} \int_B u(y)\, d\mu(y).$$

We need a stronger property than Poincaré inequalities; a continuous function u is called monotone in $\mathbf{D}$ if there exists a nonnegative function $g \in L^p_{\mathrm{loc}}(\mathbf{D}; \mu)$ such that

$$|u(x) - u_B| \le Mr \left(\frac{1}{\mu(\sigma B)} \int_{\sigma B} g(z)^p\, d\mu(z) \right)^{1/p} \tag{5}$$

for every $x \in B$ with $\sigma B \subset \mathbf{D}$, where $\sigma > 1$ and $B = B(y, r)$.

Now we show the following result, which gives of course a generalization of Theorem 1.

Theorem 2. *Let u be a monotone function on $\mathbf{D}$ with g satisfying* (5) *and*

$$\int_{\mathbf{D}} g(z)^p \, d\mu(z) < \infty. \tag{6}$$

Suppose $p > s-1$, and set

$$E = \left\{ \xi \in \partial\mathbf{D} : \limsup_{r \to 0} \bigl(r^{-p}\mu\bigl(B(\xi,r)\bigr)\bigr)^{-1} \int_{B(\xi,r)\cap\mathbf{D}} g(z)^p \, d\mu(z) > 0 \right\}.$$

If $\xi \in \partial\mathbf{D} - E$ and there exists a curve γ in $\mathbf{D}$ tending to ξ along which u has a finite limit, then u has a nontangential limit at ξ.

Remark 2. Let $1 \le q < p/(n-1)$. Let w be a Muckenhoupt (A_q) weight, and define

$$d\mu(y) = w(y)\, dy.$$

If u is monotone in the sense of Lebesgue, then $(u, |\nabla u|)$ satisfies the monotonicity property (5) by applying Hölder's inequality to (1) with p replaced by p/q (see also Manfredi–Villamor [8]). If in addition u satisfies (6) with $g = |\nabla u|$, then we apply Theorem 1 with p replaced by p/q to obtain the same conclusion as Theorem 2.

Remark 3. In Theorem 2, since $\mu(E) = 0$, we see that E is of $C_{1,p,\mu}$-capacity zero; here the weighted p-capacity $C_{1,p,\mu}(E)$ is defined by

$$C_{1,p,\mu}(E) = \inf\left\{ \int |f(y)|^p \, d\mu : \int_{B(x,1)} |x-y|^{1-n} f(y)\, dy \ge 1 \text{ for all } x \in E \right\},$$

which has the property

$$C_{1,p,\mu}\bigl(B(x,r)\bigr) \le M r^{-p} \mu\bigl(B(x,r)\bigr). \tag{7}$$

For proofs of these facts, see Meyers [9] and [10].

Proof of Theorem 2. By the monotonicity of u we see that

$$\bigl|u(x) - u\bigl(C(r)\bigr)\bigr| \le M \operatorname{diam}(B) \left(\frac{1}{\mu(\sigma B)} \int_{\sigma B} g(z)^p \, d\mu(z) \right)^{1/p}$$

for $x \in B = B\bigl(C(r), 2^{-1}\sigma^{-1}\varrho_{\mathbf{D}}\bigl(C(r)\bigr)\bigr)$. We take a finite covering $\{B_j\}$ of circular arc $\widehat{C(r)C_1(r)}$ as in the proof of Theorem 1; in this case

$$B_j = B\bigl(X_j, 2^{-1}\sigma^{-1}\varrho_{\mathbf{D}}(X_j)\bigr).$$

We find by Hölder's inequality

$$\begin{aligned}
\big|u\big(C_1(r)\big) - u\big(C(r)\big)\big| &\le |u(X_1) - u(X_2)| + |u(X_2) - u(X_3)| \\
&\quad + \cdots + |u(X_N) - u(X_{N+1})| \\
&\le M \sum_j \varrho_{\mathbf{D}}(X_j)^{1+\delta/p} \mu(\sigma B_j)^{-1/p} \\
&\quad \times \left(\int_{\sigma B_j} g(z)^p \varrho_{\mathbf{D}}(z)^{-\delta} \, d\mu(z) \right)^{1/p} \\
&\le M \left(\sum_j \varrho_{\mathbf{D}}(X_j)^{p'(1+\delta/p)} \mu(\sigma B_j)^{-p'/p} \right)^{1/p'} \\
&\quad \times \left(\int_{A(\xi,r)} g(z)^p \varrho_{\mathbf{D}}(z)^{-\delta} \, d\mu(z) \right)^{1/p} \\
&\le M \left(\sum_j \varrho_{\mathbf{D}}(X_j)^{p'(1+\delta/p)} \mu(\sigma B_j)^{-p'/p} \right)^{1/p'} \\
&\quad \times \left(\int_{B(\xi,2r)\cap \mathbf{D}} g(z)^p |C_2(r) - z|^{-\delta} \, d\mu(z) \right)^{1/p}
\end{aligned}$$

for $\delta > 0$, where $1/p + 1/p' = 1$. If we take $\delta > s - p$, then we see from (4) that

$$\begin{aligned}
\sum_j \varrho_{\mathbf{D}}(X_j)^{p'(p+\delta)/p} \mu(\sigma B_j)^{-p'/p} &\le M \int_0^{2r} t^{p'(p+\delta)/p} \mu\big(B\big(C_2(r), t\big)\big)^{-p'/p} \, dt/t \\
&\le M r^{p's/p} \mu\big(B(\xi, 4r)\big)^{-p'/p} \int_0^{2r} t^{p'(p+\delta-s)/p} \, dt/t \\
&\le M r^{p'\delta/p} \big(r^{-p} \mu\big(B(\xi, r)\big)\big)^{-p'/p}.
\end{aligned}$$

Hence it follows from (3) with $0 < \delta < 1$ and $\alpha = 0$ that

$$\int_{2^{-j}}^{2^{-j+1}} \big|u\big(C_1(r)\big) - u\big(C(r)\big)\big|^p \, dr/r \le M \big(2^{jp} \mu\big(B(\xi, 2^{-j})\big)\big)^{-1} \int_{B(\xi, 2^{-j+2})} g(z)^p \, d\mu(z).$$

Thus we can show that u has a nontangential limit at ξ, in the same manner as Theorem 1.

Remark 4. Let u be a monotone Sobolev function on $\mathbf{D}$ satisfying

$$\int_{\mathbf{D}} |\nabla u(x)|^p \, d\mu(x) < \infty.$$

Define

$$E_1 = \left\{ \xi \in \partial \mathbf{D} : \int_{B(\xi,1)\cap \mathbf{D}} |\xi - y|^{1-n} |\nabla u(y)| \, dy = \infty \right\}$$

and

$$E_2 = \left\{ \xi \in \partial \mathbf{D} : \limsup_{r\to 0} \bigl(r^{-p} \mu\bigl(B(\xi,r)\bigr)\bigr)^{-1} \int_{B(\xi,r)\cap \mathbf{D}} |\nabla u(y)|^p \, d\mu(y) > 0 \right\}.$$

Then we can show as in [11], [12] that u has a nontangential limit at every $\xi \in \partial \mathbf{D} - (E_1 \cup E_2)$. Note here that $E_1 \cup E_2$ is of $C_{1,p,\mu}$-capacity zero.

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