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GROWTH ESTIMATES FOR SOLUTIONS OF LINEAR COMPLEX DIFFERENTIAL EQUATIONS

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Abstract. Two methods are used to find growth estimates (in terms of the p-characteristic) for the analytic solutions of

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0$$

in the disc $\{z \in \mathbf{C} : |z| < R\}$, $0 < R \leq \infty$. By restricting to special cases, these estimates yield known results in the complex plane without appealing to the commonly used Wiman–Valiron theory.

1. Introduction

Our main problem is classical: We study the growth of the solutions of the complex differential equations of the form

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0, \tag{1.1}$$

where the coefficients $A_0(z), \dots, A_{k-1}(z)$ are analytic in the disc $D_R = \{z \in \mathbf{C} : |z| < R\}$, $0 < R \leq \infty$. We use the special notation $\mathbf{C}$ $(= D_\infty)$ for the complex plane and D $(= D_1)$ for the unit disc.

A typical way of classifying the growth is by means of Nevanlinna theory, see [7] and [12]. In this direction, H. Wittich [15] considers the case where the coefficients, and hence the solutions, are entire functions. His classical result, originally published in 1966, is stated as follows.

Theorem A ([12, Theorem 4.1]). *All solutions of* (1.1) *are entire and of finite order of growth if and only if the coefficients* $A_0(z), \dots, A_{k-1}(z)$ *in* (1.1) *are polynomials.*

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Assuming that all solutions of (1.1) are entire and of finite order of growth, it follows by the standard order reduction procedure and logarithmic derivative estimates that the coefficients of (1.1) are polynomials. We note that the original proof in [15] of the reverse implication does not make use of Wiman–Valiron theory, whereas the proof in [12] does. As a consequence of our main results we offer two new proofs of the reverse implication, see Section 6.

Today, much more is known about the relation between the degrees of the coefficient polynomials and the possible orders of growth of the solutions. For example, we have:

Theorem B ([12, Proposition 7.1]). *Suppose that the coefficients* $A_0(z),\dots,A_{k-1}(z)$ *in* (1.1) *are polynomials and let* f *be a solution of* (1.1). *Then the order of growth* $\varrho(f)$ *of* f *satisfies*

$$\varrho(f)\le 1+\max_{j=0,\dots,k-1}\frac{\deg(A_j)}{k-j}.$$

For more delicate results in this direction, we refer to the work of G. G. Gundersen, E. Steinbart and S. Wang, see, e.g., [6].

Besides Nevanlinna theory, a commonly used tool to obtain growth estimates is Wiman–Valiron theory, which is very powerful in the complex plane, but is known to be insufficient in any finite disc. The main purpose of the present paper is to find two growth estimates for the solutions of (1.1) in any disc D_R, where $0<R\le\infty$. This is done by using two methods, both independent of Wiman–Valiron theory. By restricting to special cases, our main results also imply some classical results such as Theorem B above.

The basic ideas that we use have been applied earlier to special cases

$$f''+A(z)f=0 \tag{1.2}$$

and

$$f^{(k)}+A(z)f=0,\quad k\in\mathbf{N}, \tag{1.3}$$

of (1.1) in D or in $\mathbf{C}$ only.

In [3], an alternative way, independent of Wiman–Valiron theory, is used to prove the following well-known result.

Theorem C ([16, Kapitel 5]). *Let* $k\in\mathbf{N}$ *and let* $A(z)$ *be a polynomial. Then every non-trivial solution* f *of* (1.3) *is an entire function of order of growth* $1+\deg(A)/k$.

Indeed, the upper bound $\varrho(f)\le 1+\deg(A)/k$ in [3] is proved by generalizing the method of successive approximations, while the lower bound $\varrho(f)\ge 1+\deg(A)/k$ follows by the sharp logarithmic derivative estimates developed earlier in [5].

In Section 6 we give an alternative proof for the following well-known result.

Theorem D ([16, Kapitel 5], [6]). *Suppose that the coefficients* $A_0(z), \ldots, A_{k-1}(z)$ *in* (1.1) *are polynomials satisfying*

$$\frac{\deg(A_j)}{k-j} \le \frac{\deg(A_0)}{k} \tag{1.4}$$

for all $j = 1, \ldots, k-1$. *Let* f *be a transcendental solution of* (1.1). *Then*

$$\varrho(f) = 1 + \frac{\deg(A_0)}{k}.$$

We prove growth estimates for the solutions of (1.1) relying on two methods, allowing the coefficient functions to have arbitrarily rapid growth. The first method is based on a representation theorem for solutions of (1.1), while the second relies on a comparison theorem by H. Herold [10]. The representation theorem is introduced in Section 3 and it generalizes analogous earlier results for the solutions of (1.2) and (1.3). The estimates are stated in terms of the p-characteristic, see [14] and Section 2 of the present paper. The main results and their proofs are given in Sections 3–5. A further discussion in Section 6 completes the paper.

2. The p-characteristic

Our estimates in Sections 4 and 5 are stated in terms of the p-characteristic of the solutions of (1.1). To this end, let f be analytic in the disc D_R, where $0 < R \le \infty$. For $0 < p < \infty$, we define the *p-characteristic* as

$$m_p(r,f) := \left(\frac{1}{2\pi} \int_0^{2\pi} \left(\overset{+}{\log} |f(re^{i\theta})|\right)^p d\theta \right)^{1/p}, \quad 0 \le r < R,$$

see, e.g., [14]. Obviously, $m_1(r,f) = m(r,f)$ is the Nevanlinna proximity function.

In estimating the growth of the p-characteristic, the following elementary result appears to be useful, see [4, p. 57].

Lemma E. *Let* $a_n \ge 0$ *for* $n = 1, \ldots, N$. *Then*

$$\left(\sum_{n=1}^{N} a_n \right)^p \le \left(\sum_{n=1}^{N} a_n^p \right), \quad 0 < p \le 1,$$

and

$$\left(\sum_{n=1}^{N} a_n \right)^p \le N^{p-1} \left(\sum_{n=1}^{N} a_n^p \right), \quad 1 \le p < \infty.$$

3. A representation theorem

Let us first recall that any solution of (1.3), where $A(z)$ is analytic either in D or in $\mathbf{C}$, possesses an integral representation, see, e.g., [8, Theorem 4.1] and the proof of [2, Lemma 4], respectively. These integral representations for solutions of (1.3) are rather simple generalizations of the corresponding representation result for solutions of (1.2), see, e.g., the proof of [1, Lemma 1].

The proof of the representation theorem is essentially independent of the size of the domain, hence it can be carried out in any disc D_R, $0 < R \le \infty$. We first state the result for the solutions of (1.3).

Theorem F. *Let f be a solution of* (1.3) *in D_R, $0 < R \le \infty$. Then, for any $z, z_0 \in D_R$,*

$$f(z) = \sum_{n=0}^{k-1} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n - \frac{1}{(k-1)!}\int_{z_0}^{z}(z-\xi)^{k-1}A(\xi)f(\xi)\,d\xi,$$

where the path of integration is a piecewise smooth curve in D_R joining z_0 and z.

Next, we give a representation theorem for the solutions of the general equation (1.1).

Theorem 3.1. *Let f be a solution of* (1.1) *in D_R, $0 < R \le \infty$. Then, for any $z, z_0 \in D_R$,*

$$f(z) = \sum_{n=0}^{k-1} c_n(z-z_0)^n + \sum_{j=0}^{k-1}\sum_{n=0}^{j} d_{j,n}\int_{z_0}^{z}(z-\xi)^{k-j+n-1}A_j^{(n)}(\xi)f(\xi)\,d\xi,$$

where the constants $c_n \in \mathbf{C}$ depend on the initial values of f at z_0, $d_{j,n} \in \mathbf{Q}$, and the path of integration is a piecewise smooth curve in D_R joining z_0 and z.

Proof. Write (1.1) in the form

$$f^{(k)} = -A_{k-1}(z)f^{(k-1)} - \cdots - A_1(z)f' - A_0(z)f. \tag{3.1}$$

We next integrate (3.1) k times from z_0 to z along any fixed piecewise smooth curve in D_R joining z_0 and z. This will be done in several steps.

After integrating $f^{(k)}$, we have

$$\int_{z_0}^{z}\cdots\int_{z_0}^{z} f^{(k)}(\xi)\,d\xi^k = f(z) - \sum_{n=0}^{k-1}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n. \tag{3.2}$$

For any continuous function φ,

$$\int_{z_0}^{z}\cdots\int_{z_0}^{z}\varphi(\xi)\,d\xi^k = \frac{1}{(k-1)!}\int_{z_0}^{z}(z-\xi)^{k-1}\varphi(\xi)\,d\xi, \tag{3.3}$$

as is easily verified by differentiation. Hence

$$\int_{z_0}^{z} \cdots \int_{z_0}^{z} A_0(\xi) f(\xi)\, d\xi^k = \frac{1}{(k-1)!} \int_{z_0}^{z} (z-\xi)^{k-1} A_0(\xi) f(\xi)\, d\xi. \tag{3.4}$$

Integrating $A_j(z) f^{(j)}$, where $j \in \{1, \dots, k-1\}$, is not so easy. Let us look at the simplest case $j = 1$ first. By (3.3),

$$\begin{aligned}
&\int_{z_0}^{z} \cdots \int_{z_0}^{z} A_1(\xi) f'(\xi)\, d\xi^k \\
&\qquad = \int_{z_0}^{z} \cdots \int_{z_0}^{z} \bigl(A_1(\xi) f(\xi)\bigr)'\, d\xi^k - \int_{z_0}^{z} \cdots \int_{z_0}^{z} A_1'(\xi) f(\xi)\, d\xi^k \\
&\qquad = \int_{z_0}^{z} \cdots \int_{z_0}^{z} A_1(\xi) f(\xi)\, d\xi^{k-1} - \frac{A_1(z_0) f(z_0)}{(k-1)!} (z-z_0)^{k-1} \\
&\qquad\quad - \int_{z_0}^{z} \cdots \int_{z_0}^{z} A_1'(\xi) f(\xi)\, d\xi^k \qquad\qquad (3.5) \\
&\qquad = \frac{1}{(k-2)!} \int_{z_0}^{z} (z-\xi)^{k-2} A_1(\xi) f(\xi)\, d\xi \\
&\qquad\quad - \frac{1}{(k-1)!} \int_{z_0}^{z} (z-\xi)^{k-1} A_1'(\xi) f(\xi)\, d\xi \frac{A_1(z_0) f(z_0)}{(k-1)!} (z-z_0)^{k-1}.
\end{aligned}$$

If $k > 2$, further terms $A_j(z) f^{(j)}$ must be integrated as well. In general, by the binomial formula,

$$\begin{aligned}
A_j f^{(j)} &= (A_j f)^{(j)} - \sum_{n=1}^{j} \binom{j}{n} A_j^{(n)} f^{(j-n)} \\
&= (A_j f)^{(j)} \\
&\quad - \sum_{n=1}^{j} \binom{j}{n} \left((A_j^{(n)} f)^{(j-n)} - \sum_{m=1}^{j-n} \binom{j-n}{m} A_j^{(n+m)} f^{(j-n-m)} \right) \\
&= (A_j f)^{(j)} - \sum_{n=1}^{j} \binom{j}{n} (A_j^{(n)} f)^{(j-n)} \qquad\qquad (3.6) \\
&\quad + \sum_{n=1}^{j} \binom{j}{n} \sum_{m=1}^{j-n} \binom{j-n}{m} A_j^{(n+m)} f^{(j-n-m)} \\
&\;\;\vdots \\
&= \sum_{n=0}^{j} a_{j,n} (A_j^{(n)} f)^{(j-n)}
\end{aligned}$$

for some constants $a_{j,n} \in \mathbf{Q}$. Hence, applying (3.3) and (3.6),

$$\begin{aligned}
\int_{z_0}^{z} \cdots \int_{z_0}^{z} A_j(\xi) f^{(j)}(\xi)\, d\xi^k &= \sum_{n=0}^{j} a_{j,n} \int_{z_0}^{z} \cdots \int_{z_0}^{z} (A_j^{(n)} f)^{(j-n)}(\xi)\, d\xi^k \\
&= \sum_{n=0}^{j} a_{j,n} \int_{z_0}^{z} \cdots \int_{z_0}^{z} A_j^{(n)}(\xi) f(\xi)\, d\xi^{k-j+n} \\
&\quad + \sum_{n=k-j}^{k-1} b_{j,n} (z-z_0)^n \\
&= \sum_{n=0}^{j} \frac{a_{j,n}}{(k-j+n-1)!} \int_{z_0}^{z} (z-\xi)^{k-j+n-1} A_j^{(n)}(\xi) f(\xi)\, d\xi \\
&\quad + \sum_{n=k-j}^{k-1} b_{j,n} (z-z_0)^n,
\end{aligned} \tag{3.7}$$

where $b_{j,n} \in \mathbf{C}$ are constants depending on the initial values of f at z_0. Combining (3.1), (3.2), (3.4), (3.5) and (3.7), we get the assertion. □

4. Estimates based on the representation theorem

The growth estimates in [8, Theorem 4.2] and in [2, Lemma 4] are proved for solutions of (1.3) in D and in $\mathbf{C}$, respectively, by applying the classical Gronwall lemma to Theorem F. Again the proof is essentially independent of the size of the domain, and so it can be carried out in any disc D_R, $0 < R \le \infty$. We first state the growth estimates for the solutions of (1.3).

Theorem G. *Let f be a solution of* (1.3) *in D_R.*

(a) *If $0 < R \le 1$, then*

$$|f(re^{i\theta})| \le \left(\sum_{n=0}^{k-1} \frac{|f^{(n)}(0)|}{n!} r^n \right) \exp\left(\frac{1}{(k-1)!} \int_0^r |A(se^{i\theta})| (R-s)^{k-1}\, ds \right)$$

for all $\theta \in [0, 2\pi)$ and $r \in [0, R)$.

(b) *If $1 < R \le \infty$, then*

$$|f(re^{i\theta})| \le \left(\sum_{n=0}^{k-1} \frac{|f^{(n)}(e^{i\theta})|}{n!} r^n \right) \exp\left(\frac{1}{(k-1)!} \int_0^r |A(se^{i\theta})| s^{k-1}\, ds \right)$$

for all $\theta \in [0, 2\pi)$ and $r \in (1, R)$.

At the present stage, it is rather obvious that Theorem 3.1 yields analogous growth estimates for the solutions of the general equation (1.1) in D_R.

Theorem 4.1. *Let f be a solution of* (1.1) *in D_R.*

(a) *If $0<R\leq 1$, then there exist a constant $C_1=C_1(k)>0$, depending on the initial values of f at $z_0=0$, and a constant $C_2=C_2(k)>0$, such that*

$$|f(re^{i\theta})|\leq C_1\exp\bigg(C_2\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_0^r|A_j^{(n)}(se^{i\theta})|(R-s)^{k-j+n-1}\,ds\bigg)$$

for all $\theta\in[0,2\pi)$ and $r\in[0,R)$.

(b) *If $1<R\leq\infty$, then there exist a constant $C_1=C_1(k)>0$, depending on the initial values of f at $z_0=e^{i\theta}$, and a constant $C_2=C_2(k)>0$, such that*

$$|f(re^{i\theta})|\leq C_1r^{k-1}\exp\bigg(C_2\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_0^r|A_j^{(n)}(se^{i\theta})|s^{k-j+n-1}\,ds\bigg)$$

for all $\theta\in[0,2\pi)$ and $r\in(1,R)$.

Proof. (a) Theorem 3.1, in the case when $z_0=0$ and the path of integration is the line segment $[0,z]$, yields

$$|f(re^{i\theta})|\leq C_1+\int_0^r\bigg(C_2\sum_{j=0}^{k-1}\sum_{n=0}^{j}|A_j^{(n)}(se^{i\theta})|(R-s)^{k-j+n-1}\bigg)|f(se^{i\theta})|\,ds, \tag{4.1}$$

where $C_1>0$ is a constant depending on the initial values of f at $z_0=0$, and $C_2=\max\{|d_{j,n}|\}>0$. The assertion in Part (a) follows by applying the Gronwall lemma [12, p. 86] to (4.1).

(b) Similarly as above, with $z_0=e^{i\theta}$ and the path of integration being the line segment $[e^{i\theta},z]$, we obtain

$$|f(re^{i\theta})|\leq C_1r^{k-1}+\int_1^r\bigg(C_2\sum_{j=0}^{k-1}\sum_{n=0}^{j}|A_j^{(n)}(se^{i\theta})|(r-s)^{k-j+n-1}\bigg)|f(se^{i\theta})|\,ds, \tag{4.2}$$

where $C_1>0$ is a constant depending on the initial values of f at $z_0=e^{i\theta}$ and $C_2=\max\{|d_{j,n}|\}>0$. We note that, for all $1\leq s\leq r$, $j\in\{0,\dots,k-1\}$ and $n\in\{0,\dots,j\}$,

$$\frac{(r-s)^{k-j+n-1}}{r^{k-1}}\leq\frac{1}{r^{j-n}}\leq\frac{s^{k-j+n-1}}{s^{k-1}}. \tag{4.3}$$

Dividing (4.2) by r^{k-1} and using (4.3), we obtain

$$\frac{|f(re^{i\theta})|}{r^{k-1}}\leq C_1+\int_1^r\bigg(C_2\sum_{j=0}^{k-1}\sum_{n=0}^{j}|A_j^{(n)}(se^{i\theta})|s^{k-j+n-1}\bigg)\frac{|f(se^{i\theta})|}{s^{k-1}}\,ds. \tag{4.4}$$

Applying the Gronwall lemma to (4.4), we get

$$|f(re^{i\theta})|\leq C_1r^{k-1}\exp\bigg(C_2\sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_1^r|A_j^{(n)}(se^{i\theta})|s^{k-j+n-1}\,ds\bigg),$$

from which we conclude the assertion in Part (b). □

We next give growth estimates for the solutions f of (1.1) in terms of the p-characteristic $m_p(r,f)$, where $1\le p<\infty$.

Corollary 4.2. *Let f be a solution of (1.1) in D_R and let $1\le p<\infty$.*

(a) *If $0<R\le 1$, then there exist a constant $C_1=C_1(k)>0$, depending on the initial values of f at $z_0=0$, and on p, and a constant $C_2=C_2(k)>0$, depending on p, such that*

$$m_p(r,f)^p \le C_1 + C_2 \sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_0^{2\pi}\int_0^r |A_j^{(n)}(se^{i\theta})|^p (R-s)^{p(k-j+n-1)}\,ds\,d\theta$$

for all $r\in[0,R)$.

(b) *If $1<R\le\infty$, then there exist a constant $C_1=C_1(k)>0$, depending on f and on p, and a constant $C_2=C_2(k)>0$, depending on p, such that*

$$m_p(r,f)^p \le C_1\big(\log r^{k-1}\big)^p + C_2 \sum_{j=0}^{k-1}\sum_{n=0}^{j}\int_0^{2\pi}\int_0^r |A_j^{(n)}(se^{i\theta})|^p s^{p(k-j+n-1)}\,ds\,d\theta$$

for all $r\in(1,R)$.

Proof. (a) The case $p=1$ is an immediate consequence of Theorem 4.1(a). If $1<p<\infty$, Theorem 4.1(a) together with Lemma E and the Hölder inequality yield the assertion.

(b) We apply Theorem 4.1(b) for a fixed $\theta\in[0,2\pi)$. For each θ we might get a different constant C_1, but, as f is a fixed function analytic in D_R, where $R>1$, the functions $f,f',\dots,f^{(k-1)}$ are bounded on ∂D. Hence a finite constant C_1, depending on f and on p, can be found. □

5. Estimates based on Herold's comparison theorem

In this section we prove another growth estimate for the solutions of (1.1). For this we need the following version of Herold's comparison theorem [10, Satz 1], which can be easily verified by a careful examination of the original proof.

Theorem H. *Let $p_j(x)$, $j=1,\dots,n$, be complex valued functions defined on $[a,b)$, let $E\subset[a,b)$ be a finite set of points, and let $P_j(x)$, $j=1,\dots,n$, be real valued non-negative functions such that $|p_j(x)|\le P_j(x)$ for all $x\in[a,b)\setminus E$. Moreover, let $P_j(x)$ be continuous for all $x\in[a,b)\setminus E$. If $v(x)$ is a solution of the differential equation*

$$v^{(n)}-\sum_{j=1}^{n}p_j(x)v^{(n-j)}=0,$$

and $V(x)$ satisfies

$$V^{(n)}-\sum_{j=1}^{n}P_j(x)V^{(n-j)}=0$$

on $[a,b)\backslash E$, *where*

$$|v^{(k)}(a)| \le V^{(k)}(a)$$

for all $k = 0, \dots, n-1$, *then*

$$|v^{(k)}(x)| \le V^{(k)}(x)$$

for all $x \in [a,b)\backslash E$ *and* $k = 0, \dots, n-1$.

We apply Theorem H to obtain the next result, which is a generalization of [13, Lemma 2] (unit disc estimate) and of [1, Lemma 2] (complex plane estimate). The estimates in [13] and [1] are for the solutions of equation (1.2). Note that, differing from [13], the proof here does not appeal to the maximal theorem due to Hardy and Littlewood [4, Theorem 1.8].

Theorem 5.1. *Let* f *be a solution of* (1.1) *in* D_R, *where* $0 < R \le \infty$, *let* $n_c \in \{1, \dots, k\}$ *be the number of nonzero coefficients* $A_j(z)$, $j = 0, \dots, k-1$, *and let* $\theta \in [0, 2\pi)$ *and* $\varepsilon > 0$. *If* $z_\theta = \nu e^{i\theta} \in D_R$ *is such that* $A_j(z_\theta) \ne 0$ *for some* $j = 0, \dots, k-1$, *then, for all* $\nu < r < R$,

$$|f(re^{i\theta})| \le C \exp\left(n_c \int_\nu^r \max_{j=0,\dots,k-1} |A_j(te^{i\theta})|^{1/(k-j)}\, dt \right), \tag{5.1}$$

where $C > 0$ *is a constant satisfying*

$$C \le (1+\varepsilon) \max_{j=0,\dots,k-1} \left(\frac{|f^{(j)}(z_\theta)|}{(n_c)^j \max\limits_{n=0,\dots,k-1} |A_n(z_\theta)|^{j/(k-n)}} \right). \tag{5.2}$$

Proof. Let $\nu < r < R$, and denote

$$h_\theta(x) := \max_{j=0,\dots,k-1} |A_j(xe^{i\theta})|^{1/(k-j)} \tag{5.3}$$

for convenience. Take ϱ such that $r < \varrho < R$, and let $\varepsilon_0 > 0$. Then h_θ is Riemann integrable on $[\nu, \varrho]$, and so there exists a partition $P := \{\nu = x_0, x_1, \dots, x_{n-1}, x_n = \varrho\}$ of $[\nu, \varrho]$ such that $x_j \ne r$ for all $j = 0, \dots, n$, and

$$U(P, h_\theta) - \int_\nu^\varrho h_\theta(s)\, ds < \varepsilon_0, \tag{5.4}$$

where $U(P, h_\theta)$ is the upper Riemann sum of h_θ, corresponding to the partition P. Define the auxiliary function g_θ: $[\nu, \varrho] \longrightarrow \mathbf{R}$,

$$g_\theta(t) := n_c \cdot \sup_{x_j \le x \le x_{j+1}} h_\theta(x), \quad x_j \le t \le x_{j+1}, \quad j = 0, \dots, n-1.$$

Then $g_\theta(t)$ is a step function, which satisfies $g_\theta(t) \ge n_c \cdot h_\theta(t)$ for all $t \in [\nu, \varrho]$. Moreover,

$$U(P, h_\theta) = \frac{1}{n_c} \int_\nu^\varrho g_\theta(s)\, ds,$$

and so, by (5,4),

$$\frac{1}{n_c} \int_\nu^r g_\theta(s)\, ds < \int_\nu^r h_\theta(s)\, ds + \varepsilon_0. \tag{5.5}$$

Next, we define the auxiliary function

$$V(t) := \exp\biggl(\int_\nu^t g_\theta(s)\, ds\biggr)$$

on $[\nu, \varrho)$ and the constants

$$\delta_j := \begin{cases} 0, & \text{if } A_j(z) \equiv 0, \\ 1, & \text{otherwise,} \end{cases}$$

where $j = 0, \dots, k-1$. Then, since $g_\theta^{(l)}(t) \equiv 0$ for all $t \in [\nu, \varrho) \setminus P$ when $l \ge 1$, $V(t)$ satisfies the equation

$$V^{(k)} - \frac{1}{n_c} g_\theta(t) \delta_{k-1} V^{(k-1)} - \cdots - \frac{1}{n_c} g_\theta(t)^{k-1} \delta_1 V' - \frac{1}{n_c} g_\theta(t)^k \delta_0 V = 0$$

on $[\nu, \varrho) \setminus P$. Since $g_\theta(\nu) \ne 0$, the constant

$$C_0 := \max_{j=0,\dots,k-1} \biggl(\frac{|f^{(j)}(z_\theta)|}{g_\theta(\nu)^j} \biggr)$$

is well-defined. Furthermore,

$$C_0 \le \max_{j=0,\dots,k-1} \Biggl(\frac{|f^{(j)}(z_\theta)|}{(n_c)^j \max\limits_{n=0,\dots,k-1} |A_n(z_\theta)|^{j/(k-n)}} \Biggr), \tag{5.6}$$

and

$$\begin{aligned} |f(\nu e^{i\theta})| &\le C_0 V(\nu) = C_0, \\ |f'(\nu e^{i\theta})| &\le C_0 V'(\nu) = C_0 g_\theta(\nu), \\ &\vdots \\ |f^{(k-1)}(\nu e^{i\theta})| &\le C_0 V^{(k-1)}(\nu) = C_0 g_\theta(\nu)^{k-1}. \end{aligned}$$

Clearly $v(t) := f(te^{i\theta})$ solves the equation

$$v^{(k)} + p_{k-1}(t)v^{(k-1)} + \ldots + p_0(t)v = 0,$$

where $p_j(t) := e^{i(k-j)\theta}A_j(te^{i\theta})$, $j = 0, \ldots, k-1$. Moreover, since

$$|p_j(t)| = |A_j(te^{i\theta})| \leq \frac{1}{n_c} g_\theta(t)^{k-j}\delta_j$$

for all $j = 0, \ldots, k-1$, and

$$\begin{aligned} |v(\nu)| &\leq C_0 V(\nu), \\ |v'(\nu)| &\leq C_0 V'(\nu), \\ &\vdots \\ |v^{(k-1)}(\nu)| &\leq C_0 V^{(k-1)}(\nu), \end{aligned}$$

we have, by Theorem H and (5.5),

$$\begin{aligned} |f(re^{i\theta})| = |v(r)| &\leq C_0 V(r) = C_0 \exp\biggl(\int_\nu^r g_\theta(s)\,ds\biggr) \\ &\leq C_0 \exp\biggl(\int_\nu^r n_c \cdot h_\theta(s)\,ds + n_c\varepsilon_0\biggr) = C \exp\biggl(n_c \int_\nu^r h_\theta(s)\,ds\biggr), \end{aligned}$$

where, by (5.6) and choosing ε_0 to be sufficiently small, C satisfies (5.2). Since $\nu < r < R$ is arbitrary, the assertion follows. □

Example 5.2. The function $f(z) = e^{kz}$ is the solution of the initial value problem

$$f^{(k)} - k^k f = 0, \quad f^{(j)}(0) = k^j, \quad j = 0, \ldots, k-1.$$

On the other hand, Theorem 5.1 gives $|f(z)| \leq (1+\varepsilon)e^{k|z|}$, where $\varepsilon > 0$ is arbitrary.

The following corollary is analogous to Corollary 4.2.

Corollary 5.3. *Let f be a solution of* (1.1) *in D_R, where $0 < R \leq \infty$, and let $1 \leq p < \infty$. Then, for all $0 \leq r < R$,*

$$m_p(r,f)^p \leq C\biggl(\sum_{j=0}^{k-1} \int_0^{2\pi}\int_0^r |A_j(se^{i\theta})|^{p/(k-j)}\,ds\,d\theta + 1\biggr),$$

where $C = C(k) > 0$ is a constant depending on p, and on the initial values of f at the point z_θ, where $A_j(z_\theta) \neq 0$ for some $j = 0, \ldots, k-1$.

Proof. The case $p = 1$ follows directly by Theorem 5.1. Suppose then that $1 < p < \infty$. Using Theorem 5.1, Lemma E, and the Hölder inequality, we obtain

$$m_p(r,f)^p \leq C\biggl(\int_0^{2\pi}\int_0^r h_\theta(s)^p\,ds\,d\theta + 1\biggr),$$

where $C > 0$ is a constant of the type described in the statement of the corollary and h_θ is defined in (5.3). The assertion follows by using Lemma E again. □

6. Alternative proofs, and further discussion

We give alternative proofs of Theorems B–D and of the reverse implication in Theorem A, as described in Section 1. Moreover, we discuss some other applications of our growth estimates.

Theorems A–D. As indicated in Section 1, the proofs of Theorems A–D are typically based on Wiman–Valiron theory, although there are alternative ways to prove Theorems A–C. We offer new proofs of Theorems A–D by applying the growth estimates in Sections 4 and 5.

Theorem B can be proved by applying Corollary 5.3 in the case when $p=1$ and $R=\infty$. On the other hand, Corollary 4.2(b) gives the upper bound

$$\max_{0\le j\le k-1}\{\deg(A_j)+k-j\}=\max_{0\le j\le k-1}\left\{\left(\frac{\deg(A_j)}{k-j}+1\right)(k-j)\right\}$$

for the order of growth of the entire solutions of (1.1). This agrees with the estimate in Theorem B, provided that the maximum is attained at $j=k-1$.

In this setting, Corollary 5.3 seems to be stronger than Corollary 4.2. However, this is not the case in general, see the discussion below. We also note the trivial fact that both of these corollaries can be used to obtain the reverse implication in Theorem A.

Finally, we give an alternative proof for Theorem D, from which Theorem C immediately follows. To see that the inequality

$$\varrho(f)\le 1+\frac{\deg(A_0)}{k}$$

holds, we use Corollary 5.3 and (1.4). The proof of the inequality

$$\varrho(f)\ge 1+\frac{\deg(A_0)}{k}$$

is an easy modification of the reasoning in [3, Section 7]. The original idea was pointed out to the first author by G. G. Gundersen. Namely, we write (1.1) in the form

$$A_0(z)=-\frac{f^{(k)}(z)}{f(z)}-A_{k-1}(z)\frac{f^{(k-1)}(z)}{f(z)}-\cdots-A_1(z)\frac{f'(z)}{f(z)}$$

and use the logarithmic derivative estimates [5, Corollary 2] to obtain

$$\deg(A_0)\le\max_{j=1,\dots,k}\bigl\{\deg(A_j)+j\bigl(\varrho(f)-1\bigr)+\varepsilon\bigr\},\tag{6.1}$$

where $A_k(z):=1$ and $\varepsilon>0$ is arbitrary. Comparing (6.1) and (1.4) and letting $\varepsilon\longrightarrow 0$, we get the assertion.

Remark. Under the assumptions of Theorem D, all transcendental solutions of (1.1) are of finite type.

Iterated order of growth. Our growth estimates can be used to obtain information on the iterated order of growth of the solutions of (1.1) in the spirit of [12, Theorem 7.3] (see also [11]), where Wiman–Valiron theory plays a fundamental role. For example, if the expressions $|A_j(z)|$ are of growth $O(e^{|z|^s})$ at most, that is, if the functions $A_j(z)$ are entire and of finite order of growth, then all solutions of (1.1) are entire and are of finite iterated 2-order (also known as the hyper order) by [12, Theorem 7.3]. To offer another viewpoint, we note that, in the case $p = 1$, the expressions $|A_j^{(n)}(se^{i\theta})|s^{k-j+n-1}$ in Corollary 4.2(b) and the expressions $|A_j(se^{i\theta})|^{1/(k-j)}$ in Corollary 5.3 are all of the growth $O(e^{|z|^s})$ at most. Therefore, Corollary 4.2(b) and Corollary 5.3 can both be applied to get an alternative proof, independent of Wiman–Valiron theory, for the finiteness of the iterated 2-order of the solutions of (1.1).

Remark. The discussion above can be further extended to the case when the coefficients $A_j(z)$ are of finite order (or of finite iterated order) of growth in any disc D_R, $0 < R \leq \infty$. To the best of our knowledge, iterated orders have not been treated earlier in this sense in a finite disc.

Real differential equations. Theorems 4.1 and 5.1 restricted to the real line yield immediate growth estimates for the solutions of linear real differential equations of the form (1.1) on the interval $(-R, R)$, where $0 < R \leq \infty$.

Oscillation. Theorem F together with a growth estimate based on the Herold comparison theorem, see [2, Lemma 3], are used in [2] to obtain a certain oscillation result for entire solutions of (1.3). We have been able to extend these auxiliary results for the general equation (1.1), see Theorems 3.1 and 5.1 above. The question as to whether or not these extended results could be applied to obtain an analogous oscillation result to that in [2] for the solutions of (1.1) is left open.

Unit disc. In a preprint [9] by the authors, the estimates in Corollaries 4.2 and 5.3 are used together with some function spaces arguments to study the analytic solutions of (1.1) in D. It is shown that neither of these estimates is better than the other in the case $1 \leq p < 2$.

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