Annales Academiæ Scientiarum Fennicæ
Mathematica
Volumen 29, 2004, 283–293

QUOTIENT DECOMPOSITION OF $Q_p^{\#}$ FUNCTIONS

Hasi Wulan

Shantou University, Department of Mathematics
Shantou, Guangdong 515063, P. R. China; wulan@stu.edu.cn

Abstract. For $0 < p < \infty$, $Q_p^{\#}$ is the class of meromorphic functions f defined in the unit disk Δ satisfying that $\sup_{a\in\Delta} \iint_\Delta (f^{\#}(z))^2 g^p(z,a)\, dA(z) < \infty$, where $g(z,a)$ is the Green function of Δ. A sufficient and necessary condition for the quotient $f = f_1/f_2$ of two bounded analytic functions f_1 and f_2 to belong to $Q_p^{\#}$ is given. Also, we prove that there exists a class X of meromorphic functions on Δ such that $Q_1^{\#} \subsetneqq X \subsetneqq N$, where N is the class of normal functions. This observation gives an affirmative answer to a question in the literature.

1. Throughout this paper, Δ, $\partial\Delta$ and $dA(z)$ are the unit disk on the complex plane, the boundary of the unit disk and the Euclidean area element on Δ, respectively. For a meromorphic function f on Δ, the Ahlfors–Shimizu characteristic function is defined as

$$T(r,f) = \frac{1}{\pi}\int_0^r t^{-1} \iint_{|z|<t} \bigl(f^{\#}(z)\bigr)^2\, dA(z)\, dt, \qquad 0 < r < 1.$$

A function f meromorphic on Δ is said to belong to the Nevanlinna class $\mathscr{N}$ if

$$T(1,f) = \lim_{r\to 1} T(r,f) < \infty.$$

The well-known R. Nevanlinna quotient theorem (cf. [Ne, p. 188]) says that every function in $\mathscr{N}$ is the quotient of two functions in

$$H^\infty = \Bigl\{ f : f \text{ analytic in } \Delta \text{ and } \|f\|_\infty = \sup_{z\in\Delta} |f(z)| < \infty \Bigr\}.$$

If a meromorphic function f belongs to $\mathscr{N}$, then $f = IO/J$, where I, J are inner functions whose greatest common divisor is 1 and O is an outer function in $\mathscr{N}$. Conversely, such a function $f = IO/J$ belongs to $\mathscr{N}$. Up to some unimodular constants, the functions I, J, O are uniquely determined in this case. An inner function is a function I analytic on Δ, having the properties $|I(z)| \le 1$ for all $z \in \Delta$ and $|I(e^{i\theta})| = 1$ a.e. on $\partial\Delta$. An outer function is a function of the form

$$O_\psi(z) = \exp\biggl(\int_{\partial\Delta} \frac{\zeta + z}{\zeta - z} \log\psi(\zeta)\, \frac{|d\zeta|}{2\pi} \biggr),$$

2000 Mathematics Subject Classification: Primary 30D35, 30D45.

where $\psi \geq 0$ a.e. on $\partial\Delta$ and $\log\psi \in L^1(\partial\Delta)$.

In his thesis, Carleson [Ca] considered the classes T_α, $0 \leq \alpha < 1$, of meromorphic functions f on Δ satisfying

$$\|f\|_\alpha = \int_0^1 (1-r)^{-\alpha} \iint_{|z|<r} \left(f^\#(z)\right)^2 dA(z)dr < \infty, \tag{1.1}$$

and the class T_1 of meromorphic functions f on Δ with

$$\|f\|_1 = \sup_{0<r<1} \iint_{|z|<r} \left(f^\#(z)\right)^2 dA(z) < \infty. \tag{1.2}$$

Obviously, we have $T_1 \subset T_\alpha \subset T_\beta \subset T_0$ for all $\alpha, \beta \in (0,1)$ with $\alpha > \beta$. The class T_0 coincides with the Nevanlinna class $\mathscr{N}$.

Carleson [Ca] found a partial quotient theorem for T_α and further Aleman [Al] gave a complete result to the quotient decomposition for T_α, showing that each function in T_α, $0 < \alpha \leq 1$, is the quotient of two functions in $H^\infty \cap T_\alpha$. Using a result of Cima and Colwell [CC] for the class of normal functions, Yamashita [Ya1] gave criteria for a Blaschke quotient B_1/B_2 to be of UBC, the class of all meromorphic functions of uniformly bounded characteristic on Δ (cf. [Ya2]). Xiao [Xi] considered the subclass BIT_α of T_α and proved that each function in $BIT_\alpha, 0 < \alpha < 1$, is the quotient of two functions in $H^\infty \cap BIT_\alpha$. Recently, we studied [AW] the same problem for $Q_p^\#$ classes which have attracted considerable attention and proved that each function in $Q_p^\#$, $0 < p < \infty$, is the quotient of two functions in $H^\infty \cap Q_p$. We first show in Section 2 that the converse is not true, that is, there exist functions $f_1, f_2 \in H^\infty \cap Q_p$ but $f = f_1/f_2 \notin Q_p^\#$ for any p, $0 < p < \infty$. The aim of Section 2 is to give a sufficient and necessary condition for $f = f_1/f_2 \in T_0$ to belong to $Q_p^\#$ $(Q_{p,0}^\#)$ for all $p \in (0,\infty)$. In Section 3 we prove that there exists a class X of meromorphic functions on Δ such that $Q_1^\# \subsetneqq X \subsetneqq N$, where N is the class of normal functions. Notice that this observation gives an affirmative answer to the question in [Wu]. In Sections 4 and 5 we generalize a result in [DG] for Q_p spaces and give its counterpart for the subharmonic case.

2. The Green function on Δ with pole at $a \in \Delta$ is given by $g(z,a) = \log\left(1/|\varphi_a(z)|\right)$, where $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ is a Möbius transformation of Δ. For $0 < r < 1$, let $\Delta(a,r) = \{z \in \Delta : |\varphi_a(z)| < r\}$ be the pseudohyperbolic disk with center a and radius r. For $0 < p < \infty$, we define the classes $Q_p^\#$ and $Q_{p,0}^\#$ of meromorphic functions f on Δ, respectively, for which

$$\sup_{a\in\Delta} \iint_\Delta \left(f^\#(z)\right)^2 \left(g(z,a)\right)^p dA(z) < \infty \tag{2.1}$$

and

$$\lim_{|a|\to 1} \iint_\Delta \big(f^\#(z)\big)^2 \big(g(z,a)\big)^p \, dA(z) = 0, \tag{2.2}$$

where $f^\#(z) = |f'(z)|/\big(1+|f(z)|^2\big)$ is the spherical derivative of f. We have that $Q_1^\# = \mathrm{UBC}$ and for each $p \in (1,\infty)$ the class $Q_p^\#$ is the class of normal meromorphic functions N (cf. [AL]) for which

$$\|f\|_N = \sup_{z\in\Delta}(1-|z|^2)f^\#(z) < \infty.$$

Replacing $f^\#(z)$ by $|f'(z)|$ in the above expressions (2.1) and (2.2) for analytic functions f on Δ, we obtain the spaces Q_p and $Q_{p,0}$ (cf. [AL] and [AXZ]).

Let us define for an outer function O two cut-off outer functions (cf. [Al] and [Xi]):

$$O_+(z) = \exp\left[\int_{\partial\Delta} \frac{\zeta+z}{\zeta-z} \log\big(\max\{|O(\zeta)|,1\}\big)\frac{|d\zeta|}{2\pi}\right]$$

and

$$O_-(z) = \exp\left[\int_{\partial\Delta} \frac{\zeta+z}{\zeta-z} \log\big(\min\{|O(\zeta)|,1\}\big)\frac{|d\zeta|}{2\pi}\right].$$

Then it is easy to see that both O_- and $1/O_+$ belong to H^∞ and $O = O_-O_+$.

Recently, we proved the following quotient decompositions of the $Q_p^\#$ and $Q_{p,0}^\#$ functions.

Theorem A ([AW]). *Let $f = IO/J \in T_0$, where I, J are inner functions whose greatest common divisor is 1 and O is outer having two cut-off outer functions O_+ and O_-. Let $f_1 = IO_-$ and $f_2 = J/O_+$. For $0 < p < 1$, if $f = f_1/f_2$ belongs to $Q_p^\#$, respectively $Q_{p,0}^\#$, then both f_1 and f_2 lie in $H^\infty \cap Q_p$, respectively $H^\infty \cap Q_{p,0}$.*

We should mention that the converse of Theorem A is not true.

Theorem 1. *There are two functions $f_1, f_2 \in H^\infty \cap Q_p$ but $f = f_1/f_2 \notin Q_p^\#$ for $0 < p < \infty$.*

Proof. Let $0 < \beta < \frac{1}{2}$ and take the sequences $\{z_n^{(1)}\} = \{1-\beta^n\}$ and $\{z_n^{(2)}\} = \{1-\beta^n-\beta^{2n}\}$. Note that

$$(1-|z_{n+1}^{(i)}|^2) \le 2\beta(1-|z_n^{(i)}|^2), \qquad n \ge 1,\ i = 1,2.$$

Consequently, the sequences $\{z_n^{(i)}\}$ $(i = 1,2)$ are uniformly separated. Consider the Blaschke products B_i associated with the sequences $\{z_n^{(i)}\}$:

$$B_i(z) = \prod_{n=1}^{\infty} \frac{z_n^{(i)} - z}{1 - z_n^{(i)} z}, \qquad i = 1,2.$$

Clearly, $B_i \in H^\infty \cap Q_p$ $(i = 1, 2)$ if $1 \le p < \infty$.

Take now $p \in (0, 1)$. By a simple computation, we see that

$$\sum_{j=k+1}^{\infty} (1 - |z_j^{(i)}|^2)^p \le \frac{2^p \beta^p}{1 - (2\beta)^p} (1 - |z_k^{(i)}|^2)^p, \qquad k = 1, 2, \ldots, \ i = 1, 2.$$

Then using the results of [RT], we deduce that $d\mu_i(z) = \sum_{n=1}^{\infty} (1 - |z_n^{(i)}|^2)^p \delta_{z_n^{(i)}}$ is a bounded p-Carleson measure and then, by [EX], $B_i \in Q_p \cap H^\infty$, $i = 1, 2$.

Finally, $B_1/B_2 \notin Q_p^\#$ for any $p > 0$ because B_1/B_2 is not a normal function. Indeed, we have

$$\left| \frac{z_j^{(2)} - z_j^{(1)}}{1 - z_j^{(1)} z_j^{(2)}} \right| \le \frac{\beta^j}{2 - \beta^{2j}} \to 0, \qquad \text{as } j \to \infty,$$

and then it follows that $\{z_n^{(1)}\} \cup \{z_n^{(2)}\}$ is not interpolating which, using Theorem 2 of [CC], implies B_1/B_2 is not a normal function as announced.

Now we turn to the main result in this paper.

Theorem 2. *Let $p \in (0, \infty)$. Let $f = IO/J \in T_0$, where I, J are inner functions whose greatest common divisor is 1 and O is outer having the two cut-off outer functions O_+ and O_-. Let $f_1 = IO_-$ and $f_2 = J/O_+$. Then $f = f_1/f_2 \in Q_p^\#$ if and only if both f_1 and f_2 lie in $H^\infty \cap Q_p$ and*

$$\inf_{z \in \Delta} \big(|f_1(z)|^2 + |f_2(z)|^2 \big) > 0.$$

Proof. Our first observation is the following identity:

$$\begin{aligned} T(1, f_1/f_2) = & \frac{1}{4\pi} \int_0^{2\pi} \log\big(|f_1(e^{i\theta})|^2 + |f_2(e^{i\theta})|^2\big)\, d\theta \\ & - \frac{1}{2} \log\big(|f_1(0)|^2 + |f_2(0)|^2\big). \end{aligned} \tag{2.3}$$

In fact, let $J = B_J S_J$, where B_J is a Blaschke product and S_J is a singular inner function. By Lemma 4.2 in [Ya2] we have

$$\begin{aligned} T(1, f_1/f_2) &= \lim_{r \to 1} \frac{1}{4\pi} \int_0^{2\pi} \log\big(1 + |(IO/J)(re^{i\theta})|^2\big)\, d\theta \\ &\quad - \frac{1}{2} \log\big(|B_J(0)|^2 + |IO(0)/S_J(0)|^2\big) \\ &= \lim_{r \to 1} \frac{1}{4\pi} \int_0^{2\pi} \log\big(|(J/O_+)(re^{i\theta})|^2 + |(IO_-)(re^{i\theta})|^2\big)\, d\theta \\ &\quad - \lim_{r \to 1} \frac{1}{4\pi} \int_0^{2\pi} \log\big|(B_J S_J/O_+)(re^{i\theta})\big|^2\, d\theta \\ &\quad - \frac{1}{2} \log\big(|B_J(0)|^2 + |IO(0)/S_J(0)|^2\big). \end{aligned}$$

Note that every function $f \in T_0$ with $f \neq 0$ has a non-tangential limit at $e^{i\theta}$ a.e. on $\partial\Delta$, denoted by $f(e^{i\theta})$, and $\log|f(e^{i\theta})|$ belongs to $L^1[0,2\pi]$. Bearing in mind that $\log|S_J/O_+|$ is harmonic and that

$$\lim_{r\to 1}\int_0^{2\pi}\log|B_J(re^{i\theta})|\,d\theta = 0,$$

we have

$$\begin{aligned} T(1, f_1/f_2) &= \lim_{r\to 1}\frac{1}{4\pi}\int_0^{2\pi}\log\bigl(|(J/O_+)(re^{i\theta})|^2 + |(IO_-)(re^{i\theta})|^2\bigr)\,d\theta \\ &\quad - \frac{1}{2}\log|S_J(0)/O_+(0)|^2 - \frac{1}{2}\log\bigl(|B_J(0)|^2 + |IO(0)/S_J(0)|^2\bigr) \\ &= \frac{1}{4\pi}\int_0^{2\pi}\log\bigl(|f_1(e^{i\theta})|^2 + |f_2(e^{i\theta})|^2\bigr)\,d\theta - \frac{1}{2}\log\bigl(|f_1(0)|^2 + |f_2(0)|^2\bigr). \end{aligned}$$

Thus (2.3) is proved. Next, putting $f_\varrho(z) = f(\varrho z)$ for $0 < \varrho \le 1$ in (2.3) we get

$$\begin{aligned} T(\varrho, f_1/f_2) &= \frac{1}{4\pi}\int_0^{2\pi}\log\bigl(|f_1(\varrho e^{i\theta})|^2 + |f_2(\varrho e^{i\theta})|^2\bigr)\,d\theta \\ &\quad - \frac{1}{2}\log\bigl(|f_1(0)|^2 + |f_2(0)|^2\bigr). \end{aligned} \tag{2.4}$$

For $a \in \Delta$, by replacing f in (2.4) by $f\circ\varphi_a$, we obtain

$$\begin{aligned} T(\varrho, f_1\circ\varphi_a/f_2\circ\varphi_a) &= \frac{1}{4\pi}\int_0^{2\pi}\log\bigl(|f_1\circ\varphi_a(\varrho e^{i\theta})|^2 + |f_2\circ\varphi_a(\varrho e^{i\theta})|^2\bigr)\,d\theta \\ &\quad - \frac{1}{2}\log\bigl(|f_1(a)|^2 + |f_2(a)|^2\bigr). \end{aligned} \tag{2.5}$$

Proof of necessity. Suppose $f_1/f_2 \in Q_p^\#$.

Case 1. If $0 < p \le 1$, by Theorem A and the fact that $H^\infty \subset Q_1$ we know that both f_1 and f_2 belong to $H^\infty \cap Q_p$. Since $Q_p^\# \subset \mathrm{UBC}$ for $0 < p \le 1$, $f_1/f_2 \in \mathrm{UBC}$, which is equivalent to

$$\sup_{a\in\Delta} T(1, f_1\circ\varphi_a/f_2\circ\varphi_a) < \infty.$$

Because the first term in the right side of (2.5) is increasing on ϱ, we have

$$\begin{aligned} 0 &< \sup_{a\in\Delta}\lim_{\varrho\to 1}\frac{1}{4\pi}\int_0^{2\pi}\log\bigl(|f_1\circ\varphi_a(\varrho e^{i\theta})|^2 + |f_2\circ\varphi_a(\varrho e^{i\theta})|^2\bigr)\,d\theta \\ &\le \frac{1}{2}\log\bigl(\|f_1\|_\infty^2 + \|f_2\|_\infty^2\bigr) < \infty. \end{aligned}$$

Hence

$$\sup_{a\in\Delta} T(1, f_1\circ\varphi_a/f_2\circ\varphi_a) < \infty \quad \Longleftrightarrow \quad \inf_{a\in\Delta}\big(|f_1(a)|^2+|f_2(a)|^2\big) > 0.$$

Case 2. If $1 < p < \infty$, we have that $f_1/f_2 \in N$ since $Q_p^\# = N$ for all $p \in (1,\infty)$. It is clear that both f_1 and f_2 belong to $H^\infty \cap Q_p$ since $f_1, f_2 \in H^\infty$ which is contained in Q_p for all $p \in (1,\infty)$. To complete the proof of necessity we claim that the following statement is true.

Statement. *Let f be a meromorphic function on Δ. Then $f \in N$ if and only if there exists a $\varrho \in (0,1)$ such that* $\sup_{a\in\Delta} T(\varrho, f\circ\varphi_a) < \infty$.

Once the statement is proved just as in the case 1 above we have that $f_1/f_2 \in N$ implies that

$$\inf_{a\in\Delta}\big(|f_1(a)|^2+|f_2(a)|^2\big) > 0.$$

Now we give the proof of the statement. Assume first that $f \in N$. Then for a fixed $\varrho \in (0,1)$ we have

$$\begin{aligned} T(\varrho, f\circ\varphi_a) &= \frac{1}{\pi}\int_0^\varrho \frac{dt}{t}\iint_{|z|<t} (f\circ\varphi_a)^\#(z)^2\,dA(z) \\ &= \frac{1}{\pi}\int_0^\varrho \frac{dt}{t}\iint_{\Delta(a,t)} \big(f^\#(z)\big)^2\,dA(z) \\ &\le \frac{1}{\pi}\|f\|_N^2 \int_0^\varrho \frac{dt}{t}\iint_{\Delta(a,t)} \frac{dA(z)}{(1-|z|^2)^2} \\ &= \frac{1}{2}\|f\|_N^2 \log\frac{1}{1-\varrho^2}, \end{aligned}$$

which shows that

$$\sup_{a\in\Delta} T(\varrho, f\circ\varphi_a) < \infty.$$

Conversely, suppose that $\varrho \in (0,1)$ and $\sup_{a\in\Delta} T(\varrho, f\circ\varphi_a) < \infty$. Choose ϱ_0 with $0 < \varrho_0 < \varrho$ such that

$$\sup_{a\in\Delta} T(\varrho, f\circ\varphi_a)\left(\log\frac{\varrho}{\varrho_0}\right)^{-1} < 1.$$

Now,

$$\begin{aligned} T(\varrho, f\circ\varphi_a) &\ge \frac{1}{\pi}\int_{\varrho_0}^\varrho \frac{dt}{t}\iint_{\Delta(a,t)} \big(f^\#(z)\big)^2\,dA(z) \\ &\ge \frac{1}{\pi}\left(\log\frac{\varrho}{\varrho_0}\right)\iint_{\Delta(a,\varrho_0)} \big(f^\#(z)\big)^2\,dA(z). \end{aligned}$$

It follows that

$$\sup_{a\in\Delta}\iint_{\Delta(a,\varrho_0)}\big(f^\#(z)\big)^2\,dA(z)\le\sup_{a\in\Delta}T(\varrho,f\circ\varphi_a)\pi\biggl(\log\frac{\varrho}{\varrho_0}\biggr)^{-1}<\pi,$$

which shows that $f\in N$ by Lemma 3.2 in [Ya2].

Proof of sufficiency. Let

$$\delta:=\inf_{z\in\Delta}\big(|f_1(z)|^2+|f_2(z)|^2\big)>0,$$

and let $f_1,f_2\in H^\infty\cap Q_p$ with $0<p<\infty$. We obtain

$$\begin{aligned}
&\sup_{a\in\Delta}\iint_\Delta\big((f_1/f_2)^\#(z)\big)^2\big(g(z,a)\big)^p\,dA(z)\\
&\qquad=\sup_{a\in\Delta}\iint_\Delta\frac{|f_1'(z)f_2(z)-f_1(z)f_2'(z)|^2}{\big(|f_1(z)|^2+|f_2(z)|^2\big)^2}\big(g(z,a)\big)^p\,dA(z)\\
&\qquad\le\frac{2}{\delta^2}(\|f_1\|_\infty^2+\|f_2\|_\infty^2)\sup_{a\in\Delta}\iint_\Delta\big(|f_1'(z)|^2+|f_2'(z)|^2\big)\big(g(z,a)\big)^pdA(z)\\
&\qquad<\infty,
\end{aligned}$$

which shows that $f_1/f_2\in Q_p^\#$. The proof of Theorem 2 is complete.

We have the following result, similar to Theorem 2.

Theorem 3. *Let $f=IO/J\in T_0$, where I, J are inner functions whose greatest common divisor is* 1 *and O is outer having the two cut-off outer functions O_+ and O_-. Let $f_1=IO_-$ and $f_2=J/O_+$. Then, for $0<p<1$, the following are equivalent.*

(a) $f=f_1/f_2\in Q_{p,0}^\#$.

(b) $\inf_{z\in\Delta}\big(|f_1(z)|^2+|f_2(z)|^2\big)>0$ *and both f_1 and f_2 lie in $H^\infty\cap Q_{p,0}$.*

Bearing in mind that $Q_{p,0}\subset\mathrm{VMOA}$ if $0<p\le1$ and that the only Blaschke products in VMOA are the finite Blaschke products [Se] and using Theorem 3, we obtain the following result.

Corollary 1. *If $0<p\le1$, B_1 and B_2 are two Blaschke products without common zeros, then $B_1/B_2\in Q_{p,0}^\#$ if and only if $B_i\in Q_{p,0}$, $i=1,2$.*

3. Note that $Q_p^\#=N$ for all $p\in(1,\infty)$ and $Q_1^\#=\mathrm{UBC}\subsetneqq N$. It is natural to ask whether there exists a class X of meromorphic functions on Δ such that $\mathrm{UBC}\subsetneqq X\subsetneqq N$. We describe such a class X which in fact also gives an affirmative answer to the question in [Wu] (see Remark 3.1).

For $0<p<\infty$, let X_p be the family of meromorphic functions f on Δ such that

$$\sup_{a\in\Delta}\iint_\Delta\big(f^\#(z)\big)^p(1-|z|^2)^{p-2}g(z,a)\,dA(z)<\infty. \tag{3.1}$$

Obviously, $X_p=Q^\#(p,1)$ (cf. [Wu]).

Theorem 4. *We have* UBC $\subsetneqq X_p \subsetneqq N$ *for* $2<p<\infty$.

Proof. By Theorem 3.2.1(ii) in [Wu] we need only show that there exists a normal function f such that $f\notin X_p$. By [LX] there are two functions f_1 and f_2 in N such that

$$M_0 := \inf_{z\in\Delta}(1-|z|^2)\big(f_1^{\#}(z)+f_2^{\#}(z)\big)>0.$$

Therefore,

$$\begin{aligned}
\sup_{a\in\Delta}&\iint_\Delta \big\{(f_1^{\#}(z))^p+(f_2^{\#}(z))^p\big\}(1-|z|^2)^{p-2}g(z,a)\,dA(z)\\
&\ge \sup_{a\in\Delta}2^{-p}\iint_\Delta\big(f_1^{\#}(z)+f_2^{\#}(z)\big)^p(1-|z|^2)^{p-2}g(z,a)\,dA(z)\\
&\ge \sup_{a\in\Delta}\frac{M_0^p}{2^p}\iint_\Delta(1-|z|^2)^{-2}g(z,a)\,dA(z)\\
&= \sup_{a\in\Delta}\frac{M_0^p}{2^p}\iint_\Delta(1-|w|^2)^{-2}\log\frac{1}{|w|}\,dA(w)=\infty.
\end{aligned}$$

Hence, $f_1\notin X_p$ or $f_2\notin X_p$. Thus, $f_1\in N\setminus X_p$ or $f_2\in N\setminus X_p$.

4. Dyakonov and Girela [DG] gave a new characterization of Q_p functions as follows.

Theorem B ([DG]). *Let* $0<p<1$. *An analytic function* f *belongs to* Q_p *if and only if*

$$\sup_{a\in\Delta}\iint_\Delta |f'(z)|^2\frac{\big(1-|\varphi_a(z)|^2\big)^p}{|\varphi_a(z)|^{2p}}\,dA(z)<\infty. \tag{4.1}$$

In fact, Theorem B can be generalized.

Theorem 5. *Let* $0<p<\infty$ *and* $0\le s<1$. *An analytic function* f *belongs to* Q_p *if and only if*

$$\sup_{a\in\Delta}\iint_\Delta |f'(z)|^2\frac{\big(1-|\varphi_a(z)|^2\big)^p}{|\varphi_a(z)|^{2s}}\,dA(z)<\infty. \tag{4.2}$$

Note that if $s=0$ it has been proved in [ASX] and if $0<p=s<1$ it is just Theorem B. The following is the counterpart of Theorem 5 for the meromorphic case and we omit the proof of Theorem 5 here since it is similar to that of Theorem 6 below.

Theorem 6. *Let $0 < p < \infty$ and $0 < s < 1$. Then a meromorphic function f belongs to $Q_p^\#$ if and only if*

$$\sup_{a\in\Delta}\iint_\Delta \big(f^\#(z)\big)^2 \frac{\big(1-|\varphi_a(z)|^2\big)^p}{|\varphi_a(z)|^{2s}}\,dA(z) < \infty. \tag{4.3}$$

Proof. We suppose that $f \in Q_p^\#$. By Theorem 2.2.2 in [Wu] we know that $f \in N \cap M_p^\#$, where

$$M_p^\# = \Big\{ f : f \text{ meromorphic on } \Delta,\ \sup_{a\in\Delta}\iint_\Delta \big(f^\#(z)\big)^2\big(1-|\varphi_a(z)|^2\big)^p\,dA(z) < \infty \Big\}.$$

Thus for a fixed r, $0 < r < 1$ and $0 < s < 1$, we have

$$\begin{aligned}\iint_\Delta \big(f^\#(z)\big)^2 \frac{\big(1-|\varphi_a(z)|^2\big)^p}{|\varphi_a(z)|^{2s}}\,dA(z) &= \iint_{\Delta(a,r)} \big(f^\#(z)\big)^2 \frac{\big(1-|\varphi_a(z)|^2\big)^p}{|\varphi_a(z)|^{2s}}\,dA(z)\\ &\quad + \iint_{\Delta\setminus\Delta(a,r)} \big(f^\#(z)\big)^2 \frac{\big(1-|\varphi_a(z)|^2\big)^p}{|\varphi_a(z)|^{2s}}\,dA(z)\\ &\le \iint_{\Delta(a,r)} \big(f^\#(z)\big)^2 \frac{\big(1-|\varphi_a(z)|^2\big)^p}{|\varphi_a(z)|^{2s}}\,dA(z)\\ &\quad + \frac{1}{r^{2s}}\iint_{\Delta\setminus\Delta(a,r)} \big(f^\#(z)\big)^2\big(1-|\varphi_a(z)|^2\big)^p\,dA(z).\end{aligned} \tag{4.4}$$

Since $f \in M_p^\#$, the supremum over all $a \in \Delta$ of the second term in the right-hand side of (4.4) is finite. The change of variables $w = \varphi_a(z)$ in the first term of (4.4) yields

$$\begin{aligned}\sup_{a\in\Delta}&\iint_{\Delta(a,r)} \big(f^\#(z)\big)^2 \frac{\big(1-|\varphi_a(z)|^2\big)^p}{|\varphi_a(z)|^{2s}}\,dA(z)\\ &\le \|f\|_N^2 \iint_{\Delta(a,r)} (1-|z|^2)^{-2}\frac{\big(1-|\varphi_a(z)|^2\big)^p}{|\varphi_a(z)|^{2s}}\,dA(z)\\ &= \|f\|_N^2 \iint_{|w|<r} (1-|w|^2)^{p-2}|w|^{-2s}\,dA(w)\\ &= 2\pi\|f\|_N^2 \int_0^r (1-t^2)^{p-2}\, t^{1-2s}\,dt < \infty.\end{aligned}$$

Hence, (4.3) follows.

Conversely, suppose that the supremum in (4.3) is K with $0 < K < \infty$. It is easy to see that this implies that $f \in M_p^\#$ since $|\varphi_a(z)| < 1$. Fix an $r \in (0,1)$ such that $r^{2s}K/(1-r^2)^p < \frac{1}{2}\pi$. Thus we obtain

$$\begin{aligned}\iint_{\Delta(a,r)} \big(f^\#(z)\big)^2 d\,A(z) &\le \frac{r^{2s}}{(1-r^2)^p} \iint_{\Delta(a,r)} \big(f^\#(z)\big)^2 \frac{\big(1-|\varphi_a(z)|^2\big)^p}{|\varphi_a(z)|^{2s}}\, dA(z)\\ &\le \frac{r^{2s}K}{(1-r^2)^p} < \frac{\pi}{2}.\end{aligned}$$

By Lemma 3.2 in [Ya2] we obtain that $f \in N$, and thus $f \in N \cap M_p^\#$. By Theorem 2.2.2 in [Wu] again we conclude that $f \in Q_p^\#$. The proof is now complete.

5. An example in [AWZ] shows that for $s = 0$ Theorem 6 is not true. That is, unlike the analytic case, for a meromorphic function f, $(f^\#(z))^2$ may not be subharmonic in general and the Green function $g(z,a)$ in the definition of $Q_p^\#$, sometimes, can not be replaced by the expression $1-|\varphi_a(z)|^2$. Moreover, we have

Theorem 7. *Let $u \ge 0$ be subharmonic on Δ, $0 < p < \infty$ and $0 \le s < 1$. Then the following are equivalent.*

(i)
$$\sup_{a\in\Delta} \iint_\Delta u(z)\big(g(z,a)\big)^p\, dA(z) < \infty.$$

(ii)
$$\sup_{a\in\Delta} \iint_\Delta u(z) \frac{\big(1-|\varphi_a(z)|^2\big)^p}{|\varphi_a(z)|^{2s}}\, dA(z) < \infty.$$

Corollary 2. *Let $u \ge 0$ be subharmonic in Δ and $p > 1$. Then*

$$\sup_{z\in\Delta} u(z)(1-|z|^2)^2 < \infty$$

if and only if

$$\sup_{a\in\Delta} \iint_\Delta u(z)\big(g(z,a)\big)^p\, dA(z) < \infty.$$

Remark. Theorem 7 and Corollary 2 generalize some results in [AL] and [ASX] since $|f'(z)|^2$ is subharmonic in Δ for an analytic function f.

Acknowledgements. This research is supported in part by a grant from the National Natural Science Foundation of China (No. 10371069) and a grant from the NSFG (No. 010446) of China. I would like to thank the referee for helpful comments.

References

[Al] Aleman, A.: A conjecture of L. Carleson and applications. - Michigan Math J. 39, 1992, 537–549.

[AL] Aulaskari, R., and P. Lappan: Criteria for an analytic function to be Bloch and a harmonic or meromorphic function to be normal. - In: Complex Analysis and its Applications, Pitman Res. Notes Math. Ser. 305, Longman Scientific & Technical, Harlow, 1994, pp. 136–146.

[ASX] Aulaskari, R., D. Stegenga, and J. Xiao: Some subclasses of BMOA and their characterization in terms of Carleson measures. - Rocky Mountain J. Math. 26, 1996, 485–506.

[AW] Aulaskari, R., and H. Wulan: Outer functions in $Q_p^\#$, $Q_{p,0}^\#$. - J. Funct. Spaces Appl. 2, 2004, 17–24.

[AXZ] Aulaskari, R., J. Xiao, and R. Zhao: On subspaces and subsets of BMOA and UBC. - Analysis 15, 1995, 101–121.

[Ca] Carleson, L.: On a class of meromorphic functions and its associated exceptional sets. - Thesis, University of Uppsala, 1950.

[CC] Cima, J. A., and P. Colwell: Blaschke quotients and normality. - Proc. Amer. Math. Soc. 19, 1968, 796–798.

[DG] Dyakonov, K. M., and D. Girela: On Q_p spaces and pseudoanalytic extension. - Ann. Acad. Sci. Fenn. Math. 25, 2000, 477–486.

[EX] Essén, M., and J. Xiao: Some results on Q_p-spaces, $0 < p < 1$. - J. Reine Angew. Math. 485, 1997, 173–195.

[LX] Lappan, P., and J. Xiao: $Q_\alpha^\#$-bounded composition maps on normal classes. - Note Mat. 20, 2000/01, 65–72.

[Ne] Nevanlinna, R.: Analytic Functions. - Springer-Verlag, Berlin–Heidelberg–New York, 1970.

[RT] Reséndiz, O., and L. M. Tovar: Carleson measures, Blaschke products and Q_p spaces. In: Complex Analysis and Differential Equations (Uppsala, 1997), pp. 296–305, Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist. 64, Uppsala Univ., Uppsala, 1999.

[Se] Stegenga, D.: Bounded Toeplitz operators on H^1 and applications of the duality between H^1 and the functions of bounded mean osillation. - Amer. J. Math. 98, 1976, 573–589.

[Wu] Wulan, H.: On some classes of meromorphic functions. - Ann. Acad. Sci. Fenn. Math. Diss. 116, 1998, 1–57.

[Xi] Xiao, J.: Biholomorphically invariant families amongst Carleson class . - Indag. Math. (N.S.) 12, 2001, 597–607.

[Ya1] Yamashita, S.: Criteria for a Blaschke quotient to be of uniformly bounded characteristic. - Proc. Amer. Math. Soc. 93, 1985, 618–620.

[Ya2] Yamashita, S.: Functions of uniformly bounded characteristic . - Ann. Acad. Sci. Fenn. Ser. A I Math. 7, 1982, 349–367.

Received 27 May 2003