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# WEIGHTED DISTORTION IN CONFORMAL MAPPING IN EUCLIDEAN, HYPERBOLIC AND ELLIPTIC GEOMETRY

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**Abstract.** Golusin-type inequalities for normalized univalent functions are combined with elementary monotonicity arguments to give quick and simple proofs for numerous sharp two-point distortion theorems for conformal maps from the unit disk into (i) the complex plane equipped with euclidean geometry, (ii) the unit disk equipped with hyperbolic geometry, and (iii) the real projective plane equipped with elliptic geometry.

## 1. Introduction

We consider three classes of conformal maps related to euclidean, hyperbolic and elliptic geometry:

(i) conformal maps from the unit disk $\mathbf{D}$ endowed with the hyperbolic distance $\mathrm{d}_{\mathbf{D}}$ into the complex plane $\mathbf{C}$ equipped with the euclidean distance $\mathrm{d}_{\mathbf{C}}$ (euclidean case);

(ii) conformal maps from the unit disk $\mathbf{D}$ with the hyperbolic distance $\mathrm{d}_{\mathbf{D}}$ into the unit disk $\mathbf{D}$ with the hyperbolic distance $\mathrm{d}_{\mathbf{D}}$ (hyperbolic case);

(iii) conformal maps from the unit disk $\mathbf{D}$ with the hyperbolic distance $\mathrm{d}_{\mathbf{D}}$ into the real projective plane $\mathbf{P}$ with the spherical distance $\mathrm{d}_{\mathbf{P}}$ (elliptic case).

If $f$ is a conformal map from the metric space $(\mathbf{D}, \mathrm{d}_{\mathbf{D}})$ into the metric space $(X, \mathrm{d}_X)$, where $X = \mathbf{C}$, $\mathbf{D}$ or $\mathbf{P}$, then the local length distortion of $f$ at a point $z \in \mathbf{D}$ is measured by the quantity

$$|\mathrm{D}_X f(z)| := \lim_{\xi \to z} \frac{\mathrm{d}_X\bigl(f(z), f(\xi)\bigr)}{\mathrm{d}_{\mathbf{D}}(z, \xi)}.$$

Two-point distortion theorems for conformal maps $f\colon (\mathbf{D}, \mathrm{d}_{\mathbf{D}}) \to (X, \mathrm{d}_X)$ provide sharp upper and lower bounds for the global length distortion $\mathrm{d}_X\bigl(f(z_1), f(z_2)\bigr)$ for two points $z_1, z_2 \in \mathbf{D}$ in terms of the local length distortions $|\mathrm{D}_X f(z_1)|$ and $|\mathrm{D}_X f(z_2)|$ at these two points as well as the hyperbolic distance $\mathrm{d}_{\mathbf{D}}(z_1, z_2)$ between $z_1$ and $z_2$. Kim and Minda [7] pointed out that the classical growth theorem of

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Koebe for normalized univalent functions[1] may be considered as the prototype for such a two-point distortion theorem in the euclidean case.

Recently, a multitude of two-point distortion theorems for univalent functions has been obtained using (a) differential geometric methods [1], [7], [9], [10], [11], (b) the general coefficient theorem [5], and (c) control theory [19], [20]. Here, we show how Golusin-type inequalities for univalent functions can effectively and systematically be combined with elementary monotonicity arguments to establish some new distortion theorems for conformal maps. As a byproduct, we also obtain quick proofs of distortion estimates due to Jenkins [5], [6] and Ma and Minda [10].

In particular, the results of the present paper for the euclidean case might be seen as completion of the work of Blatter [1], Kim and Minda [7], Jenkins [5] and others. We therefore include a brief survey on euclidean two-point distortion theorems in Section 2. As might be expected, the hyperbolic situation is more complicated and by now only some partial results have been obtained, see, for instance, [9], [6], [19], [20]. In this paper we prove some further distortion theorems for the hyperbolic case. In passing, we also obtain the correct version of a distortion estimate by Jenkins. Finally, we study the distortion properties of conformal maps from the unit disk into the real projective plane. These maps were introduced by Grunsky [3] (under the name elliptically schlicht functions) and seem to be the natural objects, when dealing with distortion theorems for univalent *meromorphic* functions. They embrace the class of spherically convex functions, which has been considered in [11].

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## 2. Results

**2.1. The euclidean case.** Let $f$ be a conformal map from the unit disk $\mathbf{D}$ into the complex plane $\mathbf{C}$. If we consider $f$ as a map from $\mathbf{D}$ equipped with the hyperbolic distance

$$\mathrm{d}_{\mathbf{D}}(z_1, z_2) = \operatorname{artanh}\left|\frac{z_1 - z_2}{1 - \overline{z_1} z_2}\right|,$$

induced by the line element

$$\lambda_{\mathbf{D}}(z)\,|dz| = \frac{|dz|}{1 - |z|^2}$$

into $\mathbf{C}$ equipped with the euclidean distance $\mathrm{d}_{\mathbf{C}} = |\cdot|$, then the infinitesimal length distortion of $f$ at a point $z \in \mathbf{D}$ is measured by the "hyperbolic-euclidean" derivative

$$|\mathrm{D}_{\mathbf{C}} f(z)| = \lim_{\xi \to z} \frac{|f(\xi) - f(z)|}{\mathrm{d}_{\mathbf{D}}(\xi, z)} = (1 - |z|^2)|f'(z)|.$$

[1] Sometimes also called the egg yolk principle, cf. [4, p. 93].

Two-point distortion theorems give upper and lower bounds for $|f(z_1)-f(z_2)|$ in terms of $d_{\mathbf{D}}(z_1,z_2)$ and the infinitesimal length distortion of $f$ at $z_1$ and $z_2$. Natural quantities to measure the local length distortion of a conformal map at the two points $z_1$ and $z_2$ are the $p$-means

$$\big(|\mathrm{D}_{\mathbf{C}}f(z_1)|^p+|\mathrm{D}_{\mathbf{C}}f(z_2)|^p\big)^{1/p},\qquad p\in\mathbf{R}, \tag{2.1}$$

of the hyperbolic-euclidean derivatives. These $p$-means had been used in [1], [5], [7], [10] to prove a number of sharp two-point distortion theorems for conformal maps.

One of the first two-point distortion theorems for conformal maps is due to Blatter [1] in 1978. He observed that the classical growth theorem of Koebe for normalized univalent functions $g(z)=z+a_2z^2+a_3z^3+\cdots$,

$$|g(z)|\geq\frac{|z|}{(1+|z|)^2},\qquad z\in\mathbf{D}, \tag{2.2}$$

is a necessary, but not sufficient criterion for the univalence of $g$. Blatter therefore asked for distortion theorems which are also sufficient for univalence. Using an ingenious mixture of differential geometry, comparison theorems for solutions of linear differential equations and coefficient estimates for normalized univalent functions, he proved the following beautiful result:

If $f\colon\mathbf{D}\to\mathbf{C}$ is a conformal map and $z_1$, $z_2\in\mathbf{D}$, then

$$|f(z_1)-f(z_2)|\geq\big(|\mathrm{D}_{\mathbf{C}}f(z_1)|^p+|\mathrm{D}_{\mathbf{C}}f(z_2)|^p\big)^{1/p}\frac{\sinh\big(2\,d_{\mathbf{D}}(z_1,z_2)\big)}{2\big(2\cosh\big(2p\,d_{\mathbf{D}}(z_1,z_2)\big)\big)^{1/p}} \tag{2.3}$$

for $p=2$. Moreover, equality holds in (2.3) for distinct points $z_1$ and $z_2$ if and only if $f$ maps the unit disk onto the complex plane slit along a ray on the line through the points $f(z_1)$ and $f(z_2)$.

Blatter noticed that (2.3) is in fact not only necessary but also sufficient for a nonconstant analytic function $f\colon\mathbf{D}\to\mathbf{C}$ to be univalent. Later Kim and Minda [7] extended Blatter's work and showed that (2.3) continues to hold for any $p\geq\frac{3}{2}$. They remarked (but did not prove) that the right side of (2.3) is a decreasing function of $p$ and also observed that the limiting case $p=\infty$,

$$|f(z_1)-f(z_2)|\geq\max\big\{|\mathrm{D}_{\mathbf{C}}f(z_1)|,|\mathrm{D}_{\mathbf{C}}f(z_2)|\big\}\frac{\sinh\big(2\,d_{\mathbf{D}}(z_1,z_2)\big)}{2\exp\big(2\,d_{\mathbf{D}}(z_1,z_2)\big)}, \tag{2.4}$$

is simply an invariant form of the Koebe estimate (2.2). In contrast to (2.2), inequality (2.4) is also sufficient for nonconstant analytic functions $f\colon\mathbf{D}\to\mathbf{C}$ to be univalent. Thus (2.4) provides an elementary answer to Blatter's question.

Using the general coefficient theorem, Jenkins [5] (see [19] for a different approach) proved the decisive result that (2.3) is valid for any $p \geq 1$, but not for $0 < p < 1$. Again, equality occurs in (2.3) for $p \geq 1$ and for distinct points $z_1$ and $z_2$ if and only if $f$ maps $\mathbf{D}$ onto $\mathbf{C}$ slit along a ray through $f(z_1)$ and $f(z_2)$.

Our first purpose is to point out the following analogous estimate to (2.3) for negative parameters $p$.

**Theorem 2.1.** *If $f$ is a conformal map on $\mathbf{D}$, then for any $z_1, z_2 \in \mathbf{D}$ and any $p \leq 0$,*

$$|f(z_1) - f(z_2)| \geq \big(|\mathrm{D}_{\mathbf{C}} f(z_1)|^p + |\mathrm{D}_{\mathbf{C}} f(z_2)|^p\big)^{1/p} \frac{\tanh\big(\mathrm{d}_{\mathbf{D}}(z_1, z_2)\big)}{2^{1/p}}. \tag{2.5}$$

*Equality holds for two distinct points $z_1$ and $z_2$ if and only if $f$ maps $\mathbf{D}$ onto $\mathbf{C}$ slit along one or two rays on the line $L$ which is perpendicular to the line joining $f(z_1)$ and $f(z_2)$ such that $f(z_1)$ and $f(z_2)$ are symmetric with respect to $L$. Conversely, if $f\colon \mathbf{D} \to \mathbf{C}$ is a nonconstant analytic function satisfying* (2.5) *for some $p \leq 0$ and all $z_1, z_2 \in \mathbf{D}$, then $f$ is univalent.*

The estimate (2.5) is actually an invariant form of a special case of the well-known Goluzin inequalities [2] for normalized univalent functions. In particular, Theorem 2.1 provides a necessary and sufficient condition for a nonconstant analytic function $f\colon \mathbf{D} \to \mathbf{C}$ to be univalent. This gives another elementary answer to Blatter's question.

**Remarks 2.2.** Both inequalities, (2.5) for $p \leq 0$ and (2.3) for $p \geq 1$ are sharp. Moreover, they characterize univalent functions up to constant functions. In some cases (e.g. if $f(z) = z/(1-z)^2$ and $z_2 = \overline{z_1} \notin \mathbf{R}$), (2.5) provides a better lower bound for $|f(z_1) - f(z_2)|$, whereas in other cases (e.g. if $f(z) = z/(1-z)^2$ and $z_1$ and $z_2$ are real) (2.3) gives a better estimate. We further note that the right side of (2.5) is an increasing function of $p$. Thus the limiting case $p \to 0$,

$$|f(z_1) - f(z_2)| \geq \sqrt{|\mathrm{D}_{\mathbf{C}} f(z_1)|\,|\mathrm{D}_{\mathbf{C}} f(z_2)|}\, \tanh\big(\mathrm{d}_{\mathbf{D}}(z_1, z_2)\big),$$

is the sharpest inequality contained in the one-parameter family (2.5). It is also noteworthy to mention the limiting case $p \to -\infty$ of (2.5),

$$|f(z_1) - f(z_2)| \geq \min\big\{|\mathrm{D}_{\mathbf{C}} f(z_1)|, |\mathrm{D}_{\mathbf{C}} f(z_2)|\big\} \tanh\big(\mathrm{d}_{\mathbf{D}}(z_1, z_2)\big), \tag{2.6}$$

which may be considered as a counterpart of the invariant form (2.4) of Koebe's growth theorem. Finally, the *sharp* inequality (2.6) may be compared with the following invariant version of the Koebe one-quarter theorem

$$|f(z_1) - f(z_2)| \geq \tfrac{1}{4} \max\big\{|\mathrm{D}_{\mathbf{C}} f(z_1)|, |\mathrm{D}_{\mathbf{C}} f(z_2)|\big\} \tanh(\mathrm{d}_{\mathbf{D}}(z_1, z_2)),$$

(see [17, Corollary 1.5]), which is, however, never sharp for $z_1$ and $z_2$ *inside* $\mathbf{D}$.

Our next result focuses on an upper bound for $|f(z_1) - f(z_2)|$.

**Theorem 2.3.** *If $f$ is a conformal map on $\mathbf{D}$, then for any $z_1, z_2 \in \mathbf{D}$ and any $p \leq 0$,*

$$|f(z_1) - f(z_2)| \leq \big(|\mathrm{D}_{\mathbf{C}} f(z_1)|^p + |\mathrm{D}_{\mathbf{C}} f(z_2)|^p\big)^{1/p} \frac{\sinh\big(2\, \mathrm{d}_{\mathbf{D}}(z_1, z_2)\big)}{2\big[2\cosh\big(2p\, \mathrm{d}_{\mathbf{D}}(z_1, z_2)\big)\big]^{1/p}}. \tag{2.7}$$

*If $p < 0$, then equality holds for two distinct points $z_1$ and $z_2$ if and only if $f$ maps $\mathbf{D}$ onto the complex plane slit along a ray on the line through $f(z_1)$ and $f(z_2)$.*

**Remarks 2.4.**

(a) Estimate (2.7) for $p = 0$ was found by Jenkins [5] (see [19] for a different proof). In fact, employing the general coefficient theorem, Jenkins even proved that every conformal map $f$ on $\mathbf{D}$ satisfies the inequality

$$|f(z_1) - f(z_2)| \leq \big(|\mathrm{D}_{\mathbf{C}} f(z_1)|^p + |\mathrm{D}_{\mathbf{C}} f(z_2)|^p\big)^{1/p} \frac{\sinh\big(2\, \mathrm{d}_{\mathbf{D}}(z_1, z_2)\big)}{2^{1+1/p}} \tag{2.8}$$

for any $p > 0$. Equality occurs in (2.8) for distinct points $z_1$ and $z_2$ if and only if $f$ maps $\mathbf{D}$ onto $\mathbf{C}$ slit symmetrically through the point at infinity on the line determined by $f(z_1)$ and $f(z_2)$.

(b) Note that for $p = 0$, (2.7) as well as (2.8) take the form

$$|f(z_1) - f(z_2)| \leq \sqrt{|\mathrm{D}_{\mathbf{C}} f(z_1)|\, |\mathrm{D}_{\mathbf{C}} f(z_2)|}\, \frac{\sinh\big(2\, \mathrm{d}_{\mathbf{D}}(z_1, z_2)\big)}{2}. \tag{2.9}$$

Using Blatter's method, Ma and Minda [10] proved (2.7) for $p \leq -1$. They also observed that the right side of (2.7) is a decreasing function of $p$, but did not offer a proof. In Section 4 we shall give a quick proof of this monotonicity for *all* $p \leq 0$, which implies that (2.7) is valid for any $p \leq 0$ and not only for $p \leq -1$. Moreover, we show that Jenkins's result (2.8) can be deduced from the same inequality of Golusin which we also use to prove Theorem 2.1.

(c) Again, both estimates, (2.7) and (2.8) are sharp. In some cases (e.g. if $f(z) = z/(1-z)^2$ and $z_1$ and $z_2$ are real), (2.7) gives a better bound, whereas in other cases (e.g. if $f(z) = z/(1+z^2)$ and $z_1$ and $z_2$ are real), (2.8) yields a better estimate.

(d) Theorem 2.3 and estimate (2.8) for $p > 0$ give the sharp lower bound for the $p$-means (2.1) for every $p \in \mathbf{R}$. On the other hand, Theorem 2.1 in conjunction with (2.3) for $p \geq 1$ provide the sharp upper bound for the $p$-means (2.1) for any $p \in \mathbf{R} \setminus (0,1)$. For $0 < p < 1$ the problem of finding the maximum of the functional (2.1) was solved by Jenkins [5]. In this case the extremal functions are no longer rational functions and cannot even be given in closed form. They map $\mathbf{D}$ onto the complex plane slit along a forked slit depending on the value of $p$. Consequently, the problem of maximizing and minimizing the $p$-means (2.1) for conformal maps $f\colon \mathbf{D} \to \mathbf{C}$ is now completely understood for any $p \in \mathbf{R}$.

(e) It is also of interest that the proofs of the distortion estimates (2.5) and (2.7) for $p < 0$ and (2.8) for $p > 0$ given in the present paper are completely elementary. In particular, a simple proof of Jenkins's inequality (2.8) is obtained without making use of the general coefficient theorem. In contrast, the distortion estimate (2.3) for $p \in [1, \infty)$ seems to lie deeper.

**2.2. The hyperbolic case.** It is not unexpected that the situation for bounded univalent functions is more involved. If $f\colon \mathbf{D} \to \mathbf{D}$ is a (bounded) conformal map, then it is natural to consider $f$ as a map $f\colon (\mathbf{D}, \mathrm{d}_{\mathbf{D}}) \to (\mathbf{D}, \mathrm{d}_{\mathbf{D}})$. Now, the local length distortion of $f$ at a point $z \in \mathbf{D}$ is given by

$$|\mathrm{D}_{\mathbf{D}} f(z)| = \lim_{\xi \to z} \frac{\mathrm{d}_{\mathbf{D}}\big(f(\xi), f(z)\big)}{\mathrm{d}_{\mathbf{D}}(\xi, z)} = \frac{1-|z|^2}{1-|f(z)|^2}|f'(z)|,$$

which might be called the "hyperbolic-hyperbolic" derivative of $f$ at $z \in \mathbf{D}$. Two-point distortion theorems for bounded univalent functions $f\colon \mathbf{D} \to \mathbf{D}$ aim at giving upper and lower bounds for the global length distortion of $f\colon (\mathbf{D}, \mathrm{d}_{\mathbf{D}}) \to (\mathbf{D}, \mathrm{d}_{\mathbf{D}})$ at two given points $z_1$ and $z_2$ in terms of the local length distortion at these points.

There are two more or less natural quantities to measure the local length distortion of a conformal map $f\colon (\mathbf{D}, \mathrm{d}_{\mathbf{D}}) \to (\mathbf{D}, \mathrm{d}_{\mathbf{D}})$ at two points $z_1$ and $z_2$. Ma and Minda [9] (see also [19]) employed the expression

$$\left(\left(\frac{|\mathrm{D}_{\mathbf{D}} f(z_1)|}{1-|\mathrm{D}_{\mathbf{D}} f(z_1)|}\right)^p + \left(\frac{|\mathrm{D}_{\mathbf{D}} f(z_2)|}{1-|\mathrm{D}_{\mathbf{D}} f(z_2)|}\right)^p\right)^{1/p}, \qquad p \in \mathbf{R}, \tag{2.10}$$

whereas Jenkins [6] used the quantity

$$\big(|\mathrm{D}_{\mathbf{D}} f(z_1)|^p + |\mathrm{D}_{\mathbf{D}} f(z_2)|^p\big)^{1/p} \tag{2.11}$$

to state sharp upper and lower bounds for the local length distortion of $f$. We would like to emphasize that in some cases (2.10) gives better results than (2.11), whereas in other cases (2.11) is more advantageous, cf. [20] for a discussion of this matter. Here we focus on quantity (2.11) and establish the following sharp upper and lower bounds for the local length distortion of $f$ in terms of (2.11) for negative parameters $p$.

**Theorem 2.5.** *If $f\colon \mathbf{D} \to \mathbf{D}$ is univalent, then for any $z_1, z_2 \in \mathbf{D}$ and any $p \leq 0$*

$$\tanh\big(\mathrm{d}_{\mathbf{D}}\big(f(z_1), f(z_2)\big)\big) \geq \big(|\mathrm{D}_{\mathbf{D}} f(z_1)|^p + |\mathrm{D}_{\mathbf{D}} f(z_2)|^p\big)^{1/p} \frac{\tanh\big(\mathrm{d}_{\mathbf{D}}(z_1, z_2)\big)}{2^{1/p}}. \tag{2.12}$$

*Equality holds for two distinct points* $z_1$ *and* $z_2$ *if and only if* $f$ *maps* $\mathbf{D}$ *onto* $\mathbf{D}$ *slit along two hyperbolic rays on the hyperbolic geodesic* $\gamma$ *which is perpendicular to the hyperbolic geodesic joining* $f(z_1)$ *and* $f(z_2)$ *and such that* $f(z_1)$ *and* $f(z_2)$ *are symmetric with respect to* $\gamma$. *Conversely, if* $f\colon \mathbf{D} \to \mathbf{D}$ *is a nonconstant analytic function satisfying* (2.12), *then* $f$ *is univalent.*

**Theorem 2.6.** *If* $f\colon \mathbf{D} \to \mathbf{D}$ *is univalent, then for any two distinct points* $z_1, z_2 \in \mathbf{D}$ *and any* $p \le 0$

$$\left(|\mathrm{D}_{\mathbf{D}} f(z_1)|^p + |\mathrm{D}_{\mathbf{D}} f(z_2)|^p\right)^{1/p} \ge \left(2\cosh\left(2p(\varrho' - \varrho)\right)\right)^{1/p} \frac{\sinh(2\varrho')}{\sinh(2\varrho)}, \tag{2.13}$$

*where* $\varrho$ *is the hyperbolic distance between* $z_1$ *and* $z_2$ *and* $\varrho'$ *is the hyperbolic distance between* $f(z_1)$ *and* $f(z_2)$. *For* $p < 0$ *equality holds for two distinct points* $z_1$ *and* $z_2$ *if and only if* $f$ *maps* $\mathbf{D}$ *onto* $\mathbf{D}$ *slit along a hyperbolic ray on the hyperbolic geodesic through* $f(z_1)$ *and* $f(z_2)$.

We notice that Jenkins [6] gave an estimate of the form

$$\left(|\mathrm{D}_{\mathbf{D}} f(z_1)|^p + |\mathrm{D}_{\mathbf{D}} f(z_2)|^p\right)^{1/p} \ge \frac{e^{2\varrho'}+1}{e^{2\varrho}+1}\left(\frac{e^{\varrho'}+1}{e^{\varrho}+1}\right)^3\left(\frac{\cosh(\varrho'/2)}{\cosh(\varrho/2)}\right)^4$$

for any conformal map $f\colon \mathbf{D} \to \mathbf{D}$ and any $p > 0$. Unfortunately, this formula is not quite correct. It has to be replaced by

$$\sinh\left(2\,\mathrm{d}_{\mathbf{D}}(f(z_1), f(z_2))\right) \le \left(|\mathrm{D}_{\mathbf{D}} f(z_1)|^p + |\mathrm{D}_{\mathbf{D}} f(z_2)|^p\right)^{1/p} \frac{\sinh\left(2\,\mathrm{d}_{\mathbf{D}}(z_1, z_2)\right)}{2^{1/p}}. \tag{2.14}$$

Equality occurs for fixed $p > 0$ if and only if $f$ maps $\mathbf{D}$ onto $\mathbf{D}$ slit along symmetric rays on the hyperbolic geodesic through $f(z_1)$ and $f(z_2)$.

Our proof of Theorem 2.6, Theorem 2.5 and inequality (2.14) given in Section 4.2 below is similar to the euclidean case, but instead of Goulzin's inequalities we employ Nehari's inequalities [14] for bounded univalent functions. We note that proving estimates analogous to (2.12) and (2.13), but the $p$-means (2.11) replaced by (2.10), requires a completely different argument. This is discussed in [20].

**2.3. The elliptic case.** We finally consider univalent *meromorphic* functions in the unit disk, i.e., conformal maps from $\mathbf{D}$ into the extended complex plane $\widehat{\mathbf{C}}$. Ma and Minda [11] observed that two-point distortion theorems for the class of all univalent meromorphic functions cannot exist. They studied the subclass of spherically convex functions and obtained sharp lower bounds for the global length distortion of these maps.

A wider class which contains all spherically convex functions is the set of conformal maps from the unit disk into the real projective plane $\mathbf{P}$. As usual we equip $\mathbf{P}$ with the conformal metric $|dz|/(1+|z|^2)$, which induces on $\mathbf{P}$ the spherical distance

$$\mathrm{d}_{\mathbf{P}}(z_1,z_2) = \arctan\left|\frac{z_1-z_2}{1+\overline{z_1}z_2}\right|.$$

If $f\colon \mathbf{D}\to\mathbf{P}$ is a conformal[2] map, then the limit

$$|\mathrm{D}_{\mathbf{P}}f(z)| := \lim_{\xi\to z}\frac{\mathrm{d}_{\mathbf{P}}\big(f(\xi),f(z)\big)}{\mathrm{d}_{\mathbf{D}}(\xi,z)}$$

exists and measures the local length distortion of $f\colon(\mathbf{D},\mathrm{d}_{\mathbf{D}})\to(\mathbf{P},\mathrm{d}_{\mathbf{P}})$. We call $|\mathrm{D}_{\mathbf{P}}f(z)|$ the hyperbolic-spherical derivative of $f$ at the point $z\in\mathbf{D}$.

The following two two-point distortion theorems for conformal maps $f\colon\mathbf{D}\to\mathbf{P}$ complement the euclidean and hyperbolic case discussed above in a natural way. Recall that for two points $w_1,w_2\in\mathbf{P}$ there are two perpendicular elliptic bisectors of $w_1$ and $w_2$, i.e., two elliptic geodesics $\gamma_1$ and $\gamma_2$ which are perpendicular to the elliptic geodesic joining $w_1$ and $w_2$ such that $w_1$ and $w_2$ are symmetric with respect to both $\gamma_1$ and $\gamma_2$.

**Theorem 2.7.** *If $f\colon\mathbf{D}\to\mathbf{P}$ is a conformal map, and $z_1$, $z_2$ are two distinct points in $\mathbf{D}$, then for any $p\le 0$*

$$\sin\big(2\mathrm{d}_{\mathbf{P}}\big(f(z_1),f(z_2)\big)\big) \ge 2\big(|\mathrm{D}_{\mathbf{P}}f(z_1)|^p+|\mathrm{D}_{\mathbf{P}}f(z_2)|^p\big)^{1/p}\frac{\tanh\big(\mathrm{d}_{\mathbf{D}}(z_1,z_2)\big)}{2^{1/p}}. \tag{2.15}$$

*If $\gamma_1$ and $\gamma_2$ denote the two perpendicular elliptic bisectors of $f(z_1)$ and $f(z_2)$, then equality holds in* (2.15) *precisely when $f$ maps $\mathbf{D}$ onto $\mathbf{P}$ slit along $\gamma_1$ and slit along an elliptic ray $\gamma$ on $\gamma_2$ such that $\gamma\cap\gamma_1\ne\emptyset$. Conversely, if $f\colon\mathbf{D}\to\widehat{\mathbf{C}}$ is a nonconstant meromorphic function satisfying* (2.15), *then $f\colon\mathbf{D}\to\mathbf{P}$ is a conformal map.*

**Theorem 2.8.** *If $f\colon\mathbf{D}\to\mathbf{P}$ is a conformal map, and $z_1$, $z_2$ are two distinct points in $\mathbf{D}$, then for any $p>0$*

$$\tan\big(\mathrm{d}_{\mathbf{P}}(f(z_1),f(z_2))\big) \le \big(|\mathrm{D}_{\mathbf{P}}f(z_1)|^p+|\mathrm{D}_{\mathbf{P}}f(z_2)|^p\big)^{1/p}\frac{\sinh\big(2\,\mathrm{d}_{\mathbf{D}}(z_1,z_2)\big)}{2^{1/p}}. \tag{2.16}$$

*If $\gamma_1$ is one of the perpendicular elliptic bisectors of $f(z_1)$ and $f(z_2)$, then equality holds if and only if $f$ maps $\mathbf{D}$ onto $\mathbf{P}$ slit along $\gamma_1$ and a ray $\gamma$ on the elliptic geodesic through $f(z_1)$ and $f(z_2)$ such that $\gamma\cap\gamma_1\ne\emptyset$ and $f(z_1)$ and $f(z_2)$ have the same spherical distance to $\gamma$.*

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[2] i.e. $f\colon\mathbf{D}\to\mathbf{P}$ is angle-preserving and injective. $\mathbf{P}$ is not orientable.

We deduce these two results from inequalities for elliptically schlicht functions due to Kühnau [8], see Section 5 below. We wish to emphasize that Kühnau [8] considered elliptically schlicht functions as conformal maps from the projective plane $(\mathbf{P}, \mathrm{d}_{\mathbf{P}})$ into the projective plane $(\mathbf{P}, \mathrm{d}_{\mathbf{P}})$, whereas we consider them as conformal maps from $(\mathbf{D}, \mathrm{d}_{\mathbf{D}})$ into $(\mathbf{P}, \mathrm{d}_{\mathbf{P}})$.

**Problem 2.9.** *Theorem* 2.7 *gives the sharp upper bound for the* $p$*-means*

$$\bigl(|\mathrm{D}_{\mathbf{P}} f(z_1)|^p + |\mathrm{D}_{\mathbf{P}} f(z_2)|^p\bigr)^{1/p}$$

*for any* $p \le 0$*, and Theorem* 2.8 *its sharp lower bound for any* $p > 0$*. What about the sharp upper bound for* $p > 0$ *and the sharp lower bound for* $p < 0$*? In particular, is there an analog of the Jenkins–Kim-Minda–Blatter distortion estimate* (2.3) *for conformal maps* $f\colon \mathbf{D} \to \mathbf{P}$*?*

## 3. Linear invariance

We first recall that the three distances $\mathrm{d}_{\mathbf{C}}$, $\mathrm{d}_{\mathbf{D}}$, $\mathrm{d}_{\mathbf{P}}$ are linearly invariant in the following sense. The euclidean distance $\mathrm{d}_{\mathbf{C}} = |\cdot|$ is invariant under euclidean motions $S$, i.e., $|S(z_1) - S(z_2)| = |z_1 - z_2|$, the hyperbolic distance $\mathrm{d}_{\mathbf{D}}$ is invariant under conformal automorphisms $T$ of $\mathbf{D}$, that is, $\mathrm{d}_{\mathbf{D}}\bigl(T(z_1), T(z_2)\bigr) = \mathrm{d}_{\mathbf{D}}(z_1, z_2)$, and the spherical distance $\mathrm{d}_{\mathbf{P}}$ is invariant under rotations $S$ of the Riemann sphere $\widehat{\mathbf{C}}$, i.e., $\mathrm{d}_{\mathbf{P}}\bigl(S(z_1), S(z_2)\bigr) = \mathrm{d}_{\mathbf{P}}(z_1, z_2)$.

Also, the differential operators $|\mathrm{D}_{\mathbf{C}} f|$, $|\mathrm{D}_{\mathbf{D}} f|$ and $|\mathrm{D}_{\mathbf{P}} f|$ are linearly invariant. If $f\colon \mathbf{D} \to \mathbf{C}$ is an analytic function, then

$$|\mathrm{D}_{\mathbf{C}}(S \circ f \circ T)| = |\mathrm{D}_{\mathbf{C}} f| \circ T \tag{3.1}$$

for every euclidean motion $S$ and every conformal automorphism $T$ of $\mathbf{D}$. It follows that if $f$ is replaced in one of the distortion inequalities (2.3), (2.5), (2.7) or (2.8) by $\tilde f = S \circ f \circ T$, where $S$ is a conformal automorphism of $\mathbf{C}$ and $T$ is a conformal automorphism of $\mathbf{D}$, then the new inequality has exactly the same form, except that $f$ is replaced by $\tilde f$, $z_1$ by $T^{-1}(z_1)$ and $z_2$ by $T^{-1}(z_2)$. Similarly, if $f\colon \mathbf{D} \to \mathbf{D}$ is a (bounded) analytic function, then $|\mathrm{D}_{\mathbf{D}} f|$ is invariant under pre- and postcomposition with conformal automorphisms $S$, $T$ of $\mathbf{D}$

$$|\mathrm{D}_{\mathbf{D}}(S \circ f \circ T)| = |\mathrm{D}_{\mathbf{D}} f| \circ T. \tag{3.2}$$

It therefore suffices to verify Theorem 2.5 and Theorem 2.6 for appropriately normalized conformal maps $f\colon \mathbf{D} \to \mathbf{D}$. Finally, if $f\colon \mathbf{D} \to \mathbf{P}$ is a conformal map, then the differential operator $|\mathrm{D}_{\mathbf{P}} f|$ is linearly invariant in the sense that $|\mathrm{D}_{\mathbf{P}}(S \circ f \circ T)| = |\mathrm{D}_{\mathbf{P}} f| \circ T$ for all conformal automorphisms $T$ of $\mathbf{D}$ and any rotation $S$ of $\mathbf{P}$.

**Remark 3.1.** The differential invariant $|\mathrm{D}_{\mathbf{C}} f|$ was introduced by Peschl [15]. It was later generalized to arbitrary conformal metrics by Minda [13].

## 4. Proofs (euclidean and hyperbolic case)

**4.1. Proof of Theorem 2.1, Theorem 2.3 and (2.8).** As we have already indicated in the introduction, Theorem 2.1 is a consequence of a well-known inequality of Golusin for normalized univalent functions. In order to state Golusin's inequality we need to introduce some notation.

Let $\Sigma$ denote the class of functions

$$G(z) = z + b_0 + \frac{b_1}{z} + \cdots$$

analytic and univalent in $\Delta := \{z : |z| > 1\}$. Also, let $G \in \Sigma$, let $\xi_1, \ldots, \xi_n$ be distinct points in $\Delta$ and let $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$. Then (see, e.g., [2, Chapter IV, Section 3, Theorem 1])

$$\operatorname{Re}\left( \sum_{j,k=1}^{n} \lambda_j \lambda_k \log \frac{G(\xi_j) - G(\xi_k)}{\xi_j - \xi_k} \right) \leq - \sum_{j,k=1}^{n} \lambda_j \overline{\lambda_k} \log \left( 1 - \frac{1}{\xi_j \overline{\xi_k}} \right). \tag{4.1}$$

Equality is only possible if $G$ maps $\Delta$ onto $\widehat{\mathbf{C}}$ slit along a system of arcs $w = w(t)$ satisfying

$$\operatorname{Re}\left( \sum_{k=1}^{n} \lambda_k \log\bigl(w - G(\xi_k)\bigr) \right) = \text{const}.$$

Next, let $\mathscr{S}$ be the set of functions

$$g(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

analytic and univalent in $\mathbf{D}$. For each $g \in \mathscr{S}$ the function $G(z) = g(z^{-1})^{-1}$ belongs to $\Sigma$ and satisfies $G(z) \neq 0$. Applying Golusin's inequality (4.1) for $n = 2$, $\lambda_1 = 1$, $\lambda_2 = -1$ and fixed distinct points $\xi_1 = 1/z_1$, $\xi_2 = 1/z_2 \in \Delta$ to $G(z) = g(z^{-1})^{-1}$, we obtain

$$|g(z_1) - g(z_2)|^2 \geq \left| \frac{z_1 - z_2}{1 - z_1 \overline{z_2}} \right|^2 (1 - |z_1|^2)|g'(z_1)|(1 - |z_2|^2)|g'(z_2)| \tag{4.2}$$

with equality possible only if $G$ maps $\Delta$ onto $\widehat{\mathbf{C}}$ slit along a system of arcs $w = w(t)$ satisfying

$$\left| \frac{w - G(\xi_1)}{w - G(\xi_2)} \right| = \text{const},$$

i.e., only if $g$ maps $\mathbf{D}$ onto $\mathbf{C}$ slit along one or two rays on the line $L$ which is perpendicular to the line joining $g(z_1)$ and $g(z_2)$ such that $g(z_1)$ and $g(z_2)$ are symmetric with respect to $L$. Note that (4.2) is already a linearly invariant estimate; it remains unchanged if we replace $g$ by $S \circ g$ with a conformal automorphism $S$ of $\mathbf{C}$.

**Proof of Theorem 2.1.** We first prove the if-part of the equality statement. If $f$ maps $\mathbf{D}$ onto $\mathbf{C}$ slit along one or two rays on a line which is perpendicular to the line joining $f(z_1)$ and $f(z_2)$ and such that $f(z_1)$ and $f(z_2)$ are symmetric with respect to this line, then it is easy to see that $f = S \circ k_c \circ T$ for some $-2 \le c \le 2$, where $S$ is a conformal automorphism of $\mathbf{C}$, $T$ is a conformal automorphism of $\mathbf{D}$ such that $T(z_1) = \overline{T(z_2)}$ and $k_c(z) = z/(1 + cz + z^2)$. We now employ the following simple, but crucial property of the extremal function $k_c$:

$$|\mathrm{D}_{\mathbf{C}} k_c(z)| = |\mathrm{D}_{\mathbf{C}} k_c(\bar{z})| = \frac{|k_c(z) - k_c(\bar{z})|}{\tanh \mathrm{d}_{\mathbf{D}}(z, \bar{z})}.$$

This shows that in (2.5) equality occurs for $f = k_c$, $z_1 = z$ and $z_2 = \bar{z}$. In view of the linear invariance (3.1) of the differential operator $|\mathrm{D}_{\mathbf{C}}|$, we have proved the if-part of the equality statement of Theorem 2.1. Next, we turn to the proof of the distortion estimate (2.5). For that fix two distinct points $z_1$ and $z_2$ in $\mathbf{D}$. Postcomposing $f$ with an appropriate conformal automorphism $S$ of $\mathbf{C}$ we may assume $g = S \circ f \in \mathscr{S}$. Hence, (4.2) implies

$$|f(z_1) - f(z_2)| \ge \tanh\bigl(\mathrm{d}_{\mathbf{D}}(z_1, z_2)\bigr) \sqrt{|\mathrm{D}_{\mathbf{C}} f(z_1)|\, |\mathrm{D}_{\mathbf{C}} f(z_2)|}\,.$$

Since

$$\left( \frac{|\mathrm{D}_{\mathbf{C}} f(z_1)|^p + |\mathrm{D}_{\mathbf{C}} f(z_2)|^p}{2} \right)^{1/p}$$

is an increasing function of $p \le 0$, we deduce that (2.5) holds for any $p \le 0$ with equality only if $f$ maps $\mathbf{D}$ onto $\mathbf{C}$ slit along one or two rays on the line $L$ which is perpendicular to the line joining $f(z_1)$ and $f(z_2)$ such that $f(z_1)$ and $f(z_2)$ are symmetric with respect to $L$. The fact that condition (2.5) is sufficient for univalence can be established exactly as in [7, pp. 144–145]. □

We now move on to the proof of Theorem 2.3. If we choose in Golusin's inequality (4.1) $n = 2$, $\lambda_1 = i$, $\lambda_2 = -i$, $\xi_1 = 1/z_1$ and $\xi_2 = 1/z_2$ in $\Delta$, and $G(z) = g(z^{-1})^{-1}$, where $g \in \mathscr{S}$, then we get

$$|g(z_1) - g(z_2)|^2 \le \frac{|g'(z_1)|\,|g'(z_2)|}{(1 - |z_1|^2)(1 - |z_2|^2)} |1 - z_1 \overline{z_2}|^2 |z_1 - z_2|^2 \tag{4.3}$$

with equality possible only if $G$ maps $\Delta$ onto $\widehat{\mathbf{C}}$ slit along a system of arcs satisfying

$$\arg\left( \frac{w - G(\xi_1)}{w - G(\xi_2)} \right) = \text{const.}$$

In other words, equality in (4.3) is possible only if $g$ maps $\mathbf{D}$ onto $\mathbf{C}$ slit along one or two rays on the line through $g(z_1)$ and $g(z_2)$.

**Proof of Theorem 2.3.** Let $z_1$ and $z_2$ be two distinct points in $\mathbf{D}$ and let $f\colon \mathbf{D} \to \mathbf{C}$ be a conformal map. Then $g = S \circ f \in \mathscr{S}$ for an appropriate conformal automorphism of $\mathbf{C}$ and (4.3) applied to $g$ yields

$$|f(z_1) - f(z_2)| \le \tfrac{1}{2}\sqrt{|\mathrm{D}_{\mathbf{C}} f(z_1)|\,|\mathrm{D}_{\mathbf{C}} f(z_2)|}\,\sinh\bigl(2\,\mathrm{d}_{\mathbf{D}}(z_1,z_2)\bigr).$$

Equality can hold only if $f$ maps $\mathbf{D}$ onto $\mathbf{C}$ slit along one or two rays on the line through $f(z_1)$ and $f(z_2)$. We shall show momentarily (in Lemma 4.1 below) that

$$\frac{\bigl(|\mathrm{D}_{\mathbf{C}} f(z_1)|^p + |\mathrm{D}_{\mathbf{C}} f(z_2)|^p\bigr)^{1/p}}{\bigl(2\cosh\bigl(2p\,\mathrm{d}_{\mathbf{D}}(z_1,z_2)\bigr)\bigr)^{1/p}}$$

is a decreasing function for $p \le 0$. Consequently,

$$|f(z_1) - f(z_2)| \le \bigl(|\mathrm{D}_{\mathbf{C}} f(z_1)|^p + |\mathrm{D}_{\mathbf{C}} f(z_2)|^p\bigr)^{1/p} \frac{\sinh\bigl(2\mathrm{d}_{\mathbf{D}}(z_1,z_2)\bigr)}{2\bigl(2\cosh\bigl(2p\,\mathrm{d}_{\mathbf{D}}(z_1,z_2)\bigr)\bigr)^{1/p}}$$

for every $p \le 0$, which proves (2.7). Moreover, equality for fixed $p < 0$ is only possible if $f$ maps $\mathbf{D}$ onto $\mathbf{C}$ slit along one or two rays on the line through $f(z_1)$ and $f(z_2)$. Let $f$ be such a conformal map. Replacing $f$ by $S \circ f \circ T$ with appropriate conformal automorphisms $S$ of $\mathbf{C}$ and $T$ of $\mathbf{D}$, we may assume $z_2 = 0$, $z_1 = r \in (0,1)$ and $f(z) = k_c(z)$ for some $c \in [-2,2]$. It is easy to check that for fixed $r \in (0,1)$ and fixed $p < 0$ the function

$$c \mapsto \frac{|k_c(r)|}{\bigl(|\mathrm{D}_{\mathbf{C}} k_c(r)|^p + |\mathrm{D}_{\mathbf{C}} k_c(0)|^p\bigr)^{1/p}}$$

takes on its maximal value in the interval $[-2,2]$ only for $c = -2$ and $c = 2$. Hence $f(z) = z/(1-z)^2$ or $f(z) = z/(1+z)^2$. Conversely, if $f$ has this form, then it is straightforward to verify that equality holds in (2.7) for all points $z_1, z_2 \in (-1,1)$. Therefore, equality holds in (2.7) for $p < 0$ and two distinct points $z_1$ and $z_2$ if and only if $f$ maps $\mathbf{D}$ onto $\mathbf{C}$ slit along a single ray on the line determined by $f(z_1)$ and $f(z_2)$. In order to complete the proof of Theorem 2.3, we are left to establish the following lemma.

**Lemma 4.1.** *If $f$ is a conformal map of $\mathbf{D}$, then for any $z_1, z_2 \in \mathbf{D}$, the expression*

$$\frac{\bigl(|\mathrm{D}_{\mathbf{C}} f(z_1)|^p + |\mathrm{D}_{\mathbf{C}} f(z_2)|^p\bigr)^{1/p}}{\bigl(2\cosh\bigl(2p\,\mathrm{d}_{\mathbf{D}}(z_1,z_2)\bigr)\bigr)^{1/p}}$$

*is a decreasing function for $p \le 0$.*

*Proof.* If we set $x = -p$, $a_1 = e^{2\,\mathrm{d}_{\mathbf{D}}(z_1,z_2)}$, $a_2 = e^{-2\,\mathrm{d}_{\mathbf{D}}(z_1,z_2)}$, $b_1 = |\mathrm{D}_{\mathbf{C}} f(z_1)|^{-1}$ and $b_2 = |\mathrm{D}_{\mathbf{C}} f(z_2)|^{-1}$, then we may assume $b_1 \geq b_2$ and have to prove that

$$\left(\frac{a_1^x + a_2^x}{b_1^x + b_2^x}\right)^{1/x}$$

is an increasing function for $x \geq 0$. By a result of Marshall, Olkin and Proschan [12] this is guaranteed if $b_1/a_1 \leq b_2/a_2$ or

$$|\mathrm{D}_{\mathbf{C}} f(z_2)| \leq e^{4\,\mathrm{d}_{\mathbf{D}}(z_1,z_2)} |\mathrm{D}_{\mathbf{C}} f(z_1)|.$$

However, the latter inequality is just an invariant form of the Koebe distortion theorem (see [17, p. 9]),

$$(4.4) \qquad |\mathrm{D}_{\mathbf{C}} f(z)| = (1-|z|^2)|f'(z)| \leq \left(\frac{1+|z|}{1-|z|}\right)^2 |f'(0)| = e^{4\,\mathrm{d}_{\mathbf{D}}(0,z)} |\mathrm{D}_{\mathbf{C}} f(0)|$$

as it is seen by precomposing $f$ with an appropriate conformal automorphism of $\mathbf{D}$. □

**Remarks 4.2.**

(a) The Koebe distortion theorem (4.4) can be easily deduced from (2.7) for $p = -\infty$, i.e., from

$$(4.5) \qquad \begin{aligned} |f(z_1) - f(z_2)| \leq \tfrac{1}{2} \exp\bigl(2\,\mathrm{d}_{\mathbf{D}}(z_1,z_2)\bigr) \sinh\bigl(2\,\mathrm{d}_{\mathbf{D}}(z_1,z_2)\bigr) \\ \times \min\bigl\{|\mathrm{D}_{\mathbf{C}} f(z_1)|, |\mathrm{D}_{\mathbf{C}} f(z_2)|\bigr\}. \end{aligned}$$

In fact, applying (4.5) to $f(z) = z + a_2 z^2 + \cdots$ with $z_1 = 0$ and $z_2 = z$ yields

$$|f(z)| \leq \frac{|z|}{(1-|z|)^2} \min\{1, |\mathrm{D}_{\mathbf{C}} f(z)|\} \leq \frac{|z|}{(1-|z|)^2}.$$

This clearly implies $|a_2| \leq 2$, which in turn leads to (4.4) in the usual way (see, for instance, [17, pp. 8–9]). Thus the one-parameter family (2.7) of distortion estimates can be deduced from the strongest inequality (the case $p = 0$) and the weakest inequality (the case $p = -\infty$) using a general monotonicity result for ratios of means.

(b) The argument employed in the above proof of Lemma 4.1 can also be used to show that the right side of the Blatter–Kim-Minda–Jenkins inequality (2.3) is a decreasing function of $p$ on $[1, +\infty)$. To see this we recall that the case $p = \infty$ in (2.3) is simply the invariant version of the Koebe growth theorem (2.2). Now, the Koebe growth theorem is equivalent to the Koebe one-quarter theorem, which easily gives (4.4). The estimate (4.4) and the result of Marshall, Olkin and Proschan [12] now imply that the right side in

(2.3) is a decreasing function for $p \geq 1$. Thus, the one-parameter family (2.3) can easily be deduced from the special cases $p = 1$ and $p = \infty$.

(c) Jenkins's one-parameter family of inequalities (2.8) can also be verified quickly in the following way. Note, the estimate (2.8) for $p > 0$ follows immediately from (2.9) by monotonicity. Furher, we already know (see the proof of Theorem 2.3) that equality in (2.8) for two distinct points $z_1$, $z_2$ and $p = 0$ (and therefore also for $p > 0$) is only possible if $f$ maps $\mathbf{D}$ onto $\mathbf{C}$ slit along one or two rays on the line through $f(z_1)$ and $f(z_2)$. Let $f$ be such a conformal map. We may assume $z_1 = -r$ and $z_2 = r$ for some $r \in (0,1)$ and $f \in \mathscr{S}$ with $f(r)$, $f(-r)$ real. Hence $f(z) = k_c(z)$ for some $c \in [-2,2]$. Now, for fixed $p > 0$ and $r \in (0,1)$, the function

$$c \mapsto \frac{|k_c(r) - k_c(-r)|}{\bigl(|\mathrm{D}_{\mathbf{C}} k_c(r)|^p + |\mathrm{D}_{\mathbf{C}} k_c(-r)|^p\bigr)^{1/p}}$$

attains its maximal value only for $c = 0$, as one can quickly verify. Therefore, $f(z) = z/(1+z^2)$. On the other hand, it is easily seen that equality holds in (2.8) for $f(z) = z/(1+z^2)$ and $z_1 = -z_2 \in (0,1)$. In other words, equality holds in (2.8) for $p > 0$ and two distinct points $z_1$ and $z_2$ if and only if $f$ maps $\mathbf{D}$ onto $\mathbf{C}$ slit symmetrically through the point at infinity on the line determined by $f(z_1)$ and $f(z_2)$.

**4.2. Proofs of Theorem 2.5 and Theorem 2.6.** We first recall the Nehari inequalities for bounded univalent functions. Let $f\colon \mathbf{D} \to \mathbf{D}$ be a univalent function, $z_1, \dots, z_n$ points in $\mathbf{D}$ and $\lambda_1, \dots, \lambda_n$ complex numbers such that $\lambda_1 + \dots + \lambda_n = 0$. Then (see [14])

$$\operatorname{Re}\Biggl(\sum_{j,k=1}^{n} \lambda_j \lambda_k \log \frac{f(z_k) - f(z_j)}{z_k - z_j}\Biggr) \leq -\sum_{j,k=1}^{n} \lambda_j \overline{\lambda_k} \log\Biggl(\frac{1 - f(z_j)\overline{f(z_k)}}{1 - z_j \overline{z_k}}\Biggr), \tag{4.6}$$

with equality possible only if $f$ maps $\mathbf{D}$ onto $\mathbf{D}$ slit along a system of arcs $w = w(t)$ satisfying

$$\operatorname{Re}\Biggl(\sum_{k=1}^{n} \lambda_k \log\bigl(w - f(z_k)\bigr) - \overline{\lambda_k} \log\bigl(1 - \overline{f(z_k)} w\bigr)\Biggr) = 0.$$

**Proof of Theorem 2.5.** We begin with the if-part of the equality statement. Note that for fixed $\mu \in (0,1]$ and fixed $c \in [-2,2]$ the equation

$$k_c\bigl(P_{\mu,c}(z)\bigr) = \mu k_c(z), \qquad z \in \mathbf{D}, \tag{4.7}$$

defines a conformal map $P_{\mu,c}\colon \mathbf{D} \to \mathbf{D}$, which maps $\mathbf{D}$ onto $\mathbf{D}$ slit along two (possibly degenerate) segments $(-1, l_{\mu,c}]$ and $[r_{\mu,c}, 1)$ on the real axis. By construction, the points $l_{\mu,c} \in [-1,0)$ and $r_{\mu,c} \in (0,1]$ can be arbitrarily prescribed by varying $\mu$ and $c$. The hyperbolic derivative of $P_{\mu,c}$ is given by

$$|\mathrm{D}_{\mathbf{D}} P_{\mu,c}(z)| = \mu \frac{1-|z|^2}{1-|P_{\mu,c}(z)|^2} \frac{|k_c'(z)|}{\big|k_c'\big(P_{\mu,c}(z)\big)\big|}. \tag{4.8}$$

Subtracting (4.7) for $\bar{z}$ from (4.7) for $z$, using $P_{\mu,c}(\bar{z}) = \overline{P_{\mu,c}(z)}$ and

$$|k_c(z) - k_c(\bar{z})| = (1-|z|^2)|k_c'(z)| \frac{|z-\bar{z}|}{|1-z^2|},$$

yields

$$\frac{|1-z^2|}{|1-P_{\mu,c}(z)^2|} \left| \frac{P_{\mu,c}(z) - \overline{P_{\mu,c}(z)}}{z-\bar{z}} \right| = \mu \frac{1-|z|^2}{1-|P_{\mu,c}(z)|^2} \frac{|k_c'(z)|}{\big|k_c'\big(P_{\mu,c}(z)\big)\big|}.$$

Combining this and (4.8) gives

$$\begin{aligned} |\mathrm{D}_{\mathbf{D}} P_{\mu,c}(z)| &= |\mathrm{D}_{\mathbf{D}} P_{\mu,c}(\bar{z})| \\ &= \left| \frac{P_{\mu,c}(z) - \overline{P_{\mu,c}(z)}}{z-\bar{z}} \right| \frac{|1-z^2|}{|1-P_{\mu,c}(z)^2|} = \frac{\tanh\big(\mathrm{d}_{\mathbf{D}}\big(P_{\mu,c}(z), \overline{P_{\mu,c}(z)}\big)\big)}{\tanh\big(\mathrm{d}_{\mathbf{D}}(z,\bar{z})\big)}. \end{aligned} \tag{4.9}$$

Consequently, equality holds in (2.12) for $f = P_{\mu,c}$ and $z_1 = \overline{z_2}$. Now, if $f$ maps $\mathbf{D}$ conformally onto $\mathbf{D}$ slit along two hyperbolic rays on the hyperbolic geodesic $\gamma$ which is perpendicular to the hyperbolic geodesic joining $f(z_1)$ and $f(z_2)$ and such that $f(z_1)$ and $f(z_2)$ are symmetric with respect to $\gamma$, then it is easy to see that $f = S \circ P_{\mu,c} \circ T$ for some $\mu \in (0,1]$ and $c \in [-2,2]$, where $S$ and $T$ are conformal automorphisms of $\mathbf{D}$ such that $T(z_1) = \overline{T(z_2)}$. This proves the if-part of the equality statement of Theorem 2.5. The distortion estimate (2.12) for $p = 0$ follows immediately from Nehari's inequality (4.6) for $n = 2$, $\lambda_1 = 1$ and $\lambda_2 = -1$, which is equivalent to

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right|^2 \geq \frac{1-|z_1|^2}{1-|f(z_1)|^2} |f'(z_1)| \frac{1-|z_2|^2}{1-|f(z_2)|^2} |f'(z_2)| \left| \frac{z_1 - z_2}{1 - \overline{z_1} z_2} \right|^2,$$

that is

$$\tanh\big(\mathrm{d}_{\mathbf{D}}\big(f(z_1), f(z_2)\big)\big) \geq \sqrt{|\mathrm{D}_{\mathbf{D}} f(z_1)|\, |\mathrm{D}_{\mathbf{D}} f(z_2)|}\, \tanh\big(\mathrm{d}_{\mathbf{D}}(z_1, z_2)\big).$$

Equality is only possible if $f$ maps onto $\mathbf{D}$ slit along two hyperbolic rays on the hyperbolic geodesic $\gamma$ which is perpendicular to the hyperbolic geodesic joining $f(z_1)$ and $f(z_2)$ and such that $f(z_1)$ and $f(z_2)$ are symmetric with respect to $\gamma$. Since $(|\mathrm{D}_{\mathbf{D}} f(z_1)|^p + |\mathrm{D}_{\mathbf{D}} f(z_2)|^p)^{1/p}/2^{1/p}$ is an increasing function of $p \leq 0$, we deduce the only-if part of the equality statement and that (2.12) holds for every $p \leq 0$. Conversely, if $f\colon \mathbf{D} \to \mathbf{D}$ is a nonconstant analytic function satisfying (2.12) for some $p \leq 0$, then the Kim–Minda argument [7, pp. 144–145] shows that $f$ is univalent. □

**Proof of Theorem 2.6.** By applying (4.6) with $n = 2$, $\lambda_1 = i$ and $\lambda_2 = -i$, we get

$$\frac{|1-\overline{f(z_1)}f(z_2)|^2|f(z_1)-f(z_2)|^2}{\big(|1-|f(z_1)|^2\big)\big(1-|f(z_2)|^2\big)} \le |f'(z_1)|\,|f'(z_2)|\frac{|1-\overline{z_1}z_2|^2|z_1-z_2|^2}{(|1-|z_1|^2)(1-|z_2|^2)},$$

or

$$\sinh(2\varrho') \le \sqrt{|\mathrm{D}_{\mathbf{D}}f(z_1)|\,|\mathrm{D}_{\mathbf{D}}f(z_2)|}\,\sinh(2\varrho), \tag{4.10}$$

with $\varrho' = \mathrm{d}_{\mathbf{D}}\big(f(z_1), f(z_2)\big)$ and $\varrho = \mathrm{d}_{\mathbf{D}}(z_1, z_2)$. Equality is possible only if $f$ maps $\mathbf{D}$ onto $\mathbf{D}$ slit along one or two rays on the hyperbolic geodesic through $f(z_1)$ and $f(z_2)$. We next observe that

$$\left(\frac{|\mathrm{D}_{\mathbf{D}}f(z_1)|^p + |\mathrm{D}_{\mathbf{D}}f(z_2)|^p}{2\cosh\big(2p(\varrho'-\varrho)\big)}\right)^{1/p}$$

is a decreasing function for $p \le 0$. This follows analogously as in the euclidean case (Lemma 4.1) using again the monotonicity result [12]. Instead of the Koebe estimate (4.4) we now have to use its analog for bounded univalent functions, which is

$$|\mathrm{D}_{\mathbf{D}}f(z_2)| \le |\mathrm{D}_{\mathbf{D}}f(z_1)|e^{-4\varrho'}e^{4\varrho}. \tag{4.11}$$

Consequently, (2.13) holds for any $p \le 0$ and equality is only possible if $f$ maps $\mathbf{D}$ onto $\mathbf{D}$ slit along one or two rays on the hyperbolic geodesic through $f(z_1)$ and $f(z_2)$. If $f$ is such a conformal map, then replacing $f$ by $S \circ f \circ T$ with conformal automorphisms $S, T$ of $\mathbf{D}$, we may assume $z_2 = 0$, $z_1 = r \in (0, 1)$ and $f(z) = P_{\mu,c}(z)$ for some $c \in [-2, 2]$. As in the euclidean case it is a straightforward calculation to check that for $p < 0$ the expression

$$c \mapsto \left(\frac{|\mathrm{D}_{\mathbf{D}}P_{\mu,c}(z_2)|^p + |\mathrm{D}_{\mathbf{D}}P_{\mu,c}(z_1)|^p}{2\cosh 2p\big(\mathrm{d}_{\mathbf{D}}\big(P_{\mu,c}(z_2), P_{\mu,c}(z_1)\big)\big)}\right)^{1/p} \frac{1}{\sinh\big(\mathrm{d}_{\mathbf{D}}\big(P_{\mu,c}(z_2), P_{\mu,c}(z_1)\big)\big)}$$

attains its minimal value in the interval $[-2, 2]$ only for $c = 2$ or $c = -2$. Thus $f(z) = P_{\mu,-2}(z)$ or $f(z) = P_{\mu,2}(z)$, so $f$ maps $\mathbf{D}$ onto $\mathbf{D}$ slit along a single ray on the real axis. Conversely, if $f(z) = P_{\mu,-2}(z)$ or $f(z) = P_{\mu,2}(z)$ then equality holds in (2.12) for $z_2 = 0$ and $z_1 \in (0, 1)$. This proves the equality statement of Theorem 2.6. ◻

**Remarks 4.3.**

(a) A quick proof of (4.11) runs as follows. We may assume $z_1 = 0$, $z_2 = z$, $f(0) = 0$ and $f(z) > 0$. Then

$$g(z) = \frac{1}{f'(0)}\frac{f(z)}{\big(1-f(z)\big)^2}$$

belongs to $\mathscr{S}$, so $|g'(z)| \le (1+|z|)/(1-|z|)^3$, which is equivalent to

$$|f'(z)| \le |f'(0)| \frac{1+|z|}{(1-|z|)^3} \left| \frac{\big(1-f(z)\big)^3}{1+f(z)} \right| = |f'(0)| \frac{1+|z|}{(1-|z|)^3} \frac{\big(1-|f(z)|\big)^3}{1+|f(z)|}.$$

Thus

$$\begin{aligned} |\mathrm{D}_{\mathbf{D}} f(z)| &= (1-|z|^2) \frac{|f'(z)|}{1-|f(z)|^2} \le |f'(0)| \left( \frac{1-|f(z)|}{1+|f(z)|} \right)^2 \left( \frac{1+|z|}{1-|z|} \right)^2 \\ &= |\mathrm{D}_{\mathbf{D}} f(0)| e^{-4\,\mathrm{d}_{\mathbf{D}}(f(z),f(0))} e^{4\,\mathrm{d}_{\mathbf{D}}(z,0)}, \end{aligned}$$

which proves (4.11).

(b) The distortion estimate (2.14) for $p > 0$ follows immediately from (4.10) by monotonicity. The discussion of the case of equality is similar to the euclidean case (see Remark 4.2(c)) and will be omitted.

(c) As in the euclidean case the one-parameter family (2.13) of distortion estimates can be deduced from the inequalities (2.13) for $p = 0$ and $p = -\infty$ combined with a monotonicity argument. For this, as indicated in the above proof of Theorem 2.6, it remains to show that inequality (4.11) can be derived from (2.13) for $p = -\infty$.

If we set $z_1 = z$, $z_2 = 0$ and assume $f(0) = 0$ then (2.13) for $p = -\infty$ takes the form

$$\min\left\{ \frac{1-|z|^2}{1-|f(z)|^2} |f'(z)|, |f'(0)| \right\} \ge \frac{|f(z)|}{|z|} \frac{(1-|z|)^2}{\big(1-|f(z)|\big)^2}.$$

This is a well-known estimate for normalized bounded univalent functions due to Robinson [18]. From

$$|f'(0)| \ge \frac{|f(z)|}{|z|} \frac{(1-|z|)^2}{\big(1-|f(z)|\big)^2}$$

we immediately obtain for $f(z) = a_1 z + a_2 z^2 + \cdots$ Pick's coefficient inequality [16]

$$|a_2| \le 2|a_1|(1-|a_1|).$$

If $f\colon \mathbf{D} \to \mathbf{D}$ is a not necessarily normalized bounded univalent function, then Pick's inequality $|g''(0)| \le 4|g'(0)|\big(1-|g'(0)|\big)$ for

$$g(\xi) := \frac{f\left( \dfrac{\xi+z}{1+\bar{z}\xi} \right) - f(z)}{1 - \overline{f(z)} f\left( \dfrac{\xi+z}{1+\bar{z}\xi} \right)} \qquad \text{for fixed } z \in \mathbf{D}$$

is equivalent to

$$\left|\frac{f''(z)(1-|z|^2)}{1-|f(z)|^2}-2\bar{z}\frac{f'(z)}{1-|f(z)|^2}+2\overline{f(z)}\frac{f'(z)^2(1-|z|^2)}{(1-|f(z)|^2)^2}\right| \\ \leq \frac{4|f'(z)|}{1-|f(z)|^2}\left(1-\frac{|f(z)|(1-|z|^2)}{1-|f(z)|^2}\right).$$

We integrate from 0 to $|z|$ to get

$$\left|\log\frac{f'(z)}{f'(0)}+\log(1-|z|^2)-2\log(1-|f(z)|^2)\right| \leq 2\log\left(\frac{1+|z|}{1-|z|}\right)+\log\left(\frac{1-|f(z)|}{1+|f(z)|^3}\right),$$

and by exponentiation we finally obtain

$$\left|\frac{f'(z)}{f'(0)}\right|\frac{1-|z|^2}{1-|f(z)|^2} \leq \left(\frac{1+|z|}{1-|z|}\right)^2\left(\frac{1-|f(z)|}{1+|f(z)|}\right)^2,$$

which is nothing else than (4.11).

## 5. Proofs (elliptic case)

Conformal maps into the projective plane are closely related to the class of elliptically schlicht functions introduced by Grunsky in [3]. We recall that a function $g$ meromorphic in the unit disk $\mathbf{D}$ is called elliptically schlicht if it maps $\mathbf{D}$ univalently onto a domain on the Riemann sphere $\widehat{\mathbf{C}}$ which contains no pair of antipodal points. In other words, if we identify on the Riemann sphere antipodal points, i.e., if we consider the elliptic plane $\mathbf{P}=\{\{z,-1/\bar{z}\}:z\in\widehat{\mathbf{C}}\}$, then we may identify elliptically schlicht functions $g\colon\mathbf{D}\to\widehat{\mathbf{C}}$ with conformal maps $f\colon\mathbf{D}\to\mathbf{P}$ as follows.

The map $\pi\colon\widehat{\mathbf{C}}\to\mathbf{P}$, $\pi(z)=\{z,-1/\bar{z}\}$, is a two-sheeted covering of $\mathbf{P}$ and a local isometry from $(\widehat{\mathbf{C}},\mathrm{d}_{\mathbf{P}})$ onto $(\mathbf{P},\mathrm{d}_{\mathbf{P}})$. If $f\colon\mathbf{D}\to\mathbf{P}$ is a conformal (i.e. angle-preserving and injective) map, then there exist two uniquely determined lifts $g\colon\mathbf{D}\to\widehat{\mathbf{C}}$ and $\tilde{g}\colon\mathbf{D}\to\widehat{\mathbf{C}}$ of $f$. Since both maps $g\colon(\mathbf{D},\mathrm{d}_{\mathbf{D}})\to(\widehat{\mathbf{C}},\mathrm{d}_{\mathbf{P}})$ and $\tilde{g}\colon(\mathbf{D},\mathrm{d}_{\mathbf{D}})\to(\widehat{\mathbf{C}},\mathrm{d}_{\mathbf{P}})$ are angle-preserving and $\pi\big(g(z)\big)=\pi\big(\tilde{g}(z)\big)$, $z\in\mathbf{D}$, they are related via

$$g(z)=-\frac{1}{\overline{\tilde{g}(z)}},\qquad z\in\mathbf{D}.$$

Thus $g\colon\mathbf{D}\to\widehat{\mathbf{C}}$ is analytic and $\tilde{g}\colon\mathbf{D}\to\widehat{\mathbf{C}}$ is anti-analytic, or vice versa, so we may assume $g\colon\mathbf{D}\to\widehat{\mathbf{C}}$ is analytic and call $g$ the analytic lift of $f$. It follows by construction that $g$ is elliptically schlicht. Conversely, if $g\colon\mathbf{D}\to\widehat{\mathbf{C}}$ is elliptically schlicht, then $f:=\pi\circ g$ is a conformal map from $\mathbf{D}$ into $\mathbf{P}$ and $g$ is its analytic lift. We obtain this way a one-to-one correspondence between elliptically schlicht functions $g\colon \mathbf{D} \to \widehat{\mathbf{C}}$ and conformal maps $f\colon \mathbf{D} \to \mathbf{P}$. Clearly, if $f\colon \mathbf{D} \to \mathbf{P}$ is a conformal map and $g\colon \mathbf{D} \to \widehat{\mathbf{C}}$ is its analytic lift, then

$$\begin{aligned}|\mathrm{D}_{\mathbf{P}} f(z)| &= \lim_{\xi\to z} \frac{\mathrm{d}_{\mathbf{P}}\bigl(f(\xi), f(z)\bigr)}{\mathrm{d}_{\mathbf{D}}(\xi, z)} = \lim_{\xi\to z} \frac{\mathrm{d}_{\mathbf{P}}\bigl(g(\xi), g(z)\bigr)}{\mathrm{d}_{\mathbf{D}}(\xi, z)} \\ &= \frac{1-|z|^2}{1+|g(z)|^2}|g'(z)| = |\mathrm{D}_{\mathbf{P}} g(z)|.\end{aligned}$$

Kühnau [8] proved that if $g\colon \mathbf{D} \to \widehat{\mathbf{C}}$ is an elliptically schlicht function with $g(-r) = -w$ and $g(r) = w$ for some real numbers $0 < w \le r < 1$, then

$$\left(\frac{w(1+w^2)(1-r^2)}{r(1+r^2)(1-w^2)}\right)^2 \le |g'(r)|\,|g'(-r)| \le \left(\frac{w(1-w^2)(1+r^2)}{r(1-r^2)(1+w^2)}\right)^2. \tag{5.1}$$

Equality holds in the left (right) inequality if and only if $g$ maps $\mathbf{D}$ onto $\mathbf{D}$ slit along one or two rays on the real (imaginary) axis. (5.1) is equivalent to

$$\frac{(1-r^2)^2}{r(1+r^2)}\frac{w}{1-w^2} \le \sqrt{|\mathrm{D}_{\mathbf{P}} g(r)|\,|\mathrm{D}_{\mathbf{P}} g(-r)|} \le \frac{1+r^2}{r}\frac{w(1-w^2)}{(1+w^2)^2},$$

or

$$\frac{\tan \mathrm{d}_{\mathbf{P}}\bigl(g(r), g(-r)\bigr)}{\sinh\bigl(2\,\mathrm{d}_{\mathbf{D}}(r,-r)\bigr)} \le \sqrt{|\mathrm{D}_{\mathbf{P}} g(r)|\,|\mathrm{D}_{\mathbf{P}} g(-r)|} \le \frac{\sin\bigl(2\mathrm{d}_{\mathbf{P}}\bigl(g(r), g(-r)\bigr)\bigr)}{2\tanh\bigl(\mathrm{d}_{\mathbf{D}}(r,-r)\bigr)}. \tag{5.2}$$

Now, Theorem 2.7 and Theorem 2.8 follow quite easily from the inequalities (5.2). We indicate the proof of Theorem 2.7 and omit the proof of Theorem 2.8.

**Proof of Theorem 2.7.** If $g\colon \mathbf{D} \to \widehat{\mathbf{C}}$ is an elliptically schlicht function, then (5.2) implies by linear invariance

$$\sin\bigl(2\mathrm{d}_{\mathbf{P}}\bigl(g(z_1), g(z_2)\bigr)\bigr) \ge 2\left(\frac{|\mathrm{D}_{\mathbf{P}} g(z_1)|^p + |\mathrm{D}_{\mathbf{P}} g(z_2)|^p}{2}\right)^{1/p} \tanh\bigl(\mathrm{d}_{\mathbf{D}}(z_1, z_2)\bigr) \tag{5.3}$$

for any $p \le 0$ and any pair of points $z_1$, $z_2$ in $\mathbf{D}$. Moreover, equality for two distinct points $z_1$ and $z_2$ is only possible if $g$ maps onto a hemisphere slit along one or two rays on the great circle $C$ such that $g(z_1)$ and $g(z_2)$ are symmetric with respect to $C$. Conversely, if $g$ is such a conformal map, then $|\mathrm{D}_{\mathbf{P}} g(z_1)| = |\mathrm{D}_{\mathbf{P}} g(z_2)|$ and equality holds in (5.3) for any $p \le 0$. Thus, if $f\colon \mathbf{D} \to \mathbf{P}$ is a conformal map, then (2.15) holds for any $p \le 0$ and any pair of points $z_1$ and $z_2$. Also, equality holds for distinct $z_1$ and $z_2$ if and only if $f$ maps $\mathbf{D}$ onto $\mathbf{P}$ slit along $\gamma_1$ and slit along a ray $\gamma$ on $\gamma_2$ such that $\gamma \cap \gamma_1 \ne \emptyset$. Here $\gamma_1$ and $\gamma_2$ are the two perpendicular elliptic bisectors of $f(z_1)$ and $f(z_2)$.

If $f\colon \mathbf{D} \to \widehat{\mathbf{C}}$ is a nonconstant meromorphic function satisfying (2.15), then $f\colon \mathbf{D} \to \widehat{\mathbf{C}}$ is univalent, see again [7, pp. 144–145] for the required argument. $f$ is also elliptically schlicht, since $\mathrm{d}_{\mathbf{P}}\bigl(f(z_1), f(z_2)\bigr) = \frac{1}{2}\pi$ in (2.15) implies $\mathrm{d}_{\mathbf{D}}(z_1, z_2) = 0$. Thus $f(\mathbf{D})$ contains no pair of antipodal points. □

**Remark 5.1.** There are also Golusin-type theorems for elliptically schlicht functions [8, pp. 85–90], which lead to two-point distortion theorems for conformal maps $f\colon \mathbf{D} \to \mathbf{P}$ as in the euclidean and hyperbolic case. These two-point distortion theorems, however, are not as sharp as (2.15) and (2.16).

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