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CALORIC MEASURE AND REIFENBERG FLATNESS

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Abstract. In this paper we study a (two-phase) free boundary regularity problem for caloric measure in parabolic δ_0-Reifenberg flat domains $\Omega\subset\mathbf{R}^{n+1}$. In particular for such a domain we define $\Omega^1=\Omega\subset\mathbf{R}^{n+1}$, $\Omega^2=\mathbf{R}^{n+1}\setminus\overline{\Omega}$ and we let, for $i\in\{1,2\}$, $\omega^i(\widehat{X}^i,\hat{t}^i,\cdot\,)$ be the caloric measure at $(\widehat{X}^i,\hat{t}^i)\in\Omega^i$ defined with respect to Ω^i. If $\hat{t}^2<\hat{t}^1$ we assume that $\omega^2(\widehat{X}^2,\hat{t}^2,\cdot\,)$ is absolutely continuous with respect to $\omega^1(\widehat{X}^1,\hat{t}^1,\cdot\,)$ on $\partial\Omega$ and we denote by $k(\widehat{X}^1,\hat{t}^1,\widehat{X}^2,\hat{t}^2,\cdot\,)=d\omega^2(\widehat{X}^2,\hat{t}^2,\cdot\,)/d\omega^1(\widehat{X}^1,\hat{t}^1,\cdot\,)$ the Radon–Nikodym derivative. Our main result states that there exists $\delta_n>0$ such that if $\delta_0<\delta_n$ and if

$$\log k(\widehat{X}^1,\hat{t}^1,\widehat{X}^2,\hat{t}^2,\cdot\,)\in \mathrm{VMO}(d\omega^1(\widehat{X}^1,\hat{t}^1,\cdot\,))$$

then $C_r(X,t)\cap\partial\Omega$ is Reifenberg flat with vanishing constant whenever $(X,t)\in\partial\Omega$ and $\hat{t}^2>t+4r^2$.

1. Introduction

In this paper we study a free boundary regularity problem for caloric measure below the continous threshold. We consider unbounded domains $\Omega\subset\mathbf{R}^{n+1}$ assuming that $\partial\Omega$ is δ_0-Reifenberg flat in the parabolic sense (this notion is defined below). As is described below the bounded continuous Dirichlet problem for the heat equation always has a unique solution in this type of domains. Let (X,t), $X=(x_0,\dots,x_{n-1})$, $t\in\mathbf{R}$ denote a point in $\mathbf{R}^{n+1}$ and for given $r>0$ set $C_r(X,t)=\{(Y,s):|Y-X|<r,\ |t-s|<r^2\}$. For fixed $(\widehat{X},\hat{t})\in\Omega$ we let $\omega(\widehat{X},\hat{t},\cdot\,)$ denote the parabolic measure (in this paper this measure is referred to as the caloric measure) for the heat equation obtained from the maximum principle and the Riesz representation theorem. Let $\Delta(X,t,r)=C_r(X,t)\cap\partial\Omega$ whenever $(X,t)\in\partial\Omega$ and $r>0$. Given $(\widehat{X},\hat{t})\in\Omega$ let $(X,t)\in\partial\Omega$ and suppose $|X-\widehat{X}|^2\le A(\hat{t}-t)$ for some $A\ge 2$. In [HLN2] it is proven that $\omega(\widehat{X},\hat{t},\cdot\,)$ is, in the setting of Reifenberg flat domains as well as in the more general setting of parabolic NTA domains, a doubling measure in the sense that there exists $a_1=a_1(n,A)$ such that if $\hat{t}-t\ge 8r^2$ then

$$\omega\bigl(\widehat{X},\hat{t},\Delta(X,t,2r)\bigr)\le a_1\,\omega\bigl(\widehat{X},\hat{t},\Delta(X,t,r)\bigr).$$

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We let $\Omega^1 = \Omega \subset \mathbf{R}^{n+1}$, $\Omega^2 = \mathbf{R}^{n+1} \setminus \overline{\Omega}$. We also let $(\widehat{X}^i, \hat{t}^i) \in \Omega^i$, for $i \in \{1,2\}$, $\hat{t}^2 < \hat{t}^1$ and define $\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)$ and $\omega^2(\widehat{X}^2, \hat{t}^2, \cdot\,)$ to be the caloric measures defined with respect to Ω^1 and Ω^2, respectively. In the following we will assume that $\omega^2(\widehat{X}^2, \hat{t}^2, \cdot\,)$ is absolutely continuous with respect to $\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)$ on $\partial\Omega$ and that the Radon–Nikodym derivative

$$k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot\,) = d\omega^2(\widehat{X}^2, \hat{t}^2, \cdot\,)/d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)$$

is such that $\log k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot\,) \in \mathrm{VMO}(d\omega^1)$.

$$\mathrm{VMO}(d\omega^1) = \mathrm{VMO}\big(d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)\big)$$

is the space of functions of vanishing mean oscillation defined with respect to the measure $\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)$. This space is defined in the bulk of the paper.

To formulate our main theorem we need to properly introduce the notion of δ_0-Reifenberg flat domains.

Definition 1. If Ω is a connected open set in $\mathbf{R}^{n+1}$ then we say that $\partial\Omega$ separates $\mathbf{R}^{n+1}$ and is δ_0-Reifenberg flat, $0 < \delta_0 \leq 1/10$, if given any $(X,t) \in \partial\Omega$, $R > 0$, there exists an n-dimensional plane $\widehat{P} = \widehat{P}(X,t,R)$, containing (X,t) and a line parallel to the t-axis, having unit normal $\hat{n} = \hat{n}(X,t,R)$ such that

$$\begin{aligned}
&\{(Y,s) + r\hat{n} \in C_R(X,t) : (Y,s) \in \widehat{P},\ r > \delta_0 R\} \subset \Omega,\\
&\{(Y,s) - r\hat{n} \in C_R(X,t) : (Y,s) \in \widehat{P},\ r > \delta_0 R\} \subset \mathbf{R}^{n+1} \setminus \Omega.
\end{aligned}$$

For short we say that $\partial\Omega$ separates $\mathbf{R}^{n+1}$ when the last two conditions hold for some δ_0.

Note that if $\partial\Omega$ separates $\mathbf{R}^{n+1}$ in the sense of Definition 1, then a line segment drawn parallel to $\hat{n}$ and with endpoints in each of the sets stated in the definition also intersects $\partial\Omega$. We will often refer to Ω as being a δ_0-Reifenberg flat domain if $\partial\Omega$ is δ_0-Reifenberg flat. We pose one more definition.

Definition 2. Let Ω be a connected open set in $\mathbf{R}^{n+1}$, $(X,t) \in \partial\Omega$, and $r > 0$. We say that $C_r(X,t) \cap \partial\Omega$ is Reifenberg flat with vanishing constant in the parabolic sense, if for each $\varepsilon > 0$, there exists $\varrho_0 = \varrho_0(\varepsilon) > 0$ with the following property. If $(\widetilde{X}, \tilde{t}) \in C_r(X,t) \cap \partial\Omega$ and $0 < \varrho \leq \varrho_0$, then there exists a plane $P'(\widetilde{X}, \tilde{t}, \varrho)$ containing a line parallel to the t axis such that the statement in Definition 1 holds with R, δ_0, $\widehat{P}$ replaced by ϱ, ε and P'.

We can now formulate our main result.

Theorem 1. *Let $\Omega \subset \mathbf{R}^{n+1}$ be a δ_0-Reifenberg flat domain and define $\Omega^1 = \Omega \subset \mathbf{R}^{n+1}$, $\Omega^2 = \mathbf{R}^{n+1} \setminus \overline{\Omega}$. Let $(\widehat{X}^i, \hat{t}^i) \in \Omega^i$, for $i \in \{1,2\}$, $\hat{t}^2 < \hat{t}^1$ and* *assume that* $\omega^2(\widehat{X}^2,\hat{t}^2,\cdot\,)$ *is absolutely continuous with respect to* $\omega^1(\widehat{X}^1,\hat{t}^1,\cdot\,)$ *on* $\partial\Omega$ *and that the Radon–Nikodym derivative*

$$k(\widehat{X}^1,\hat{t}^1,\widehat{X}^2,\hat{t}^2,\cdot\,)=d\omega^2(\widehat{X}^2,\hat{t}^2,\cdot\,)/d\omega^1(\widehat{X}^1,\hat{t}^1,\cdot\,)$$

is such that

$$\log k(\widehat{X}^1,\hat{t}^1,\widehat{X}^2,\hat{t}^2,\cdot\,)\in \mathrm{VMO}\big(d\omega^1(\widehat{X}^1,\hat{t}^1,\cdot\,)\big).$$

Then there exists $\delta_n>0$ *such that if* $\delta_0<\delta_n$ *then* $C_r(X,t)\cap\partial\Omega$ *is Reifenberg flat with vanishing constant whenever* $(X,t)\in\partial\Omega$ *and* $\hat{t}^2>t+4r^2$.

In [KT3], Kenig and Toro consider the elliptic version of the two-phase problem stated in Theorem 1. In particular, they prove ([KT3, Corollary 4.1]) that if $\Omega\subset\mathbf{R}^n$ is δ_0-Reifenberg flat (in the elliptic sense) for small $\delta_0>0$ and if $k=d\omega^2/d\omega^1$, $\log k\in \mathrm{VMO}(d\omega^1)$, then $\partial\Omega$ is Reifenberg flat with vanishing constant. In the elliptic setting questions of this type have previous been addressed, from a slightly different point of view, in the case $n=2$, i.e., in the plane, through the works of Bishop, Carleson, Garnett and Jones. See [B], [BCGJ] and [BJ]. We are not aware of any results of this type in the parabolic setting.

The rest of the paper is organized as follows. In Section 2 we put the problem considered in this paper in perspective and state that our main result, i.e. Theorem 1, can be seen as part of a program focusing on the understanding of certain parabolic one-phase and two-phase problems in parabolic Reifenberg flat domains. In Section 3 we in Section 3.1 list some basic estimates for solutions to the heat or adjoint heat equation in parabolic NTA domains. These estimates are then complemented, in Section 3.2, by a set of what we refer to as refined estimates, the latter being based on δ_0-Reifenberg flatness and an exploration of the condition $\log k(\widehat{X}^1,\hat{t}^1,\widehat{X}^2,\hat{t}^2,\cdot\,)\in \mathrm{VMO}\big(d\omega^1(\widehat{X}^1,\hat{t}^1,\cdot\,)\big)$. In Section 3.3 we clarify the notion of Green function with pole at infinity and the associated caloric measure. In Section 4, which is at the heart of the matter, our regularity assumption on the kernel $k(\widehat{X}^1,\hat{t}^1,\widehat{X}^2,\hat{t}^2,\cdot\,)$ is explored in a blow-up argument. In the limit we encounter the problem of classification of what we refer to as global solutions to a specific two-phase free boundary problem. The section ends with a theorem giving us the appropriate classification and finally it is shown that Theorem 1 is a consequence of that classification theorem.

2. One and two-phase free boundary problems below the continous threshold

In this section we put Theorem 1 into perspective and briefly discuss how this result can be seen as part of a program focusing on the understanding of certain parabolic one-phase and two-phase problems in parabolic Reifenberg flat domains.

Given a Borel set $F\subset\mathbf{R}^{n+1}$ we let $\overline{F}$, ∂F denote the closure and the boundary of F respectively and define $\sigma(F)=\int_F d\sigma_t\,dt$ where $d\sigma_t$ is $n-1$-dimensional Hausdorff measure on the time slice $F\cap(\mathbf{R}^n\times\{t\})$. Let Ω be a connected open set in $\mathbf{R}^{n+1}$.

Definition 3. We say that $\partial\Omega$ satisfies a (M,R) Ahlfors condition, $M \geq 4$, if for all $(X,t) \in \partial\Omega$ and $0 < r \leq R$,

$$\sigma\big(\partial\Omega \cap C_r(X,t)\big) \leq M r^{n+1}.$$

Combining the notion of Reifenberg flatness and the Ahlfors condition, the fact that Hausdorff measure does not increase under a projection we deduce that for $0 < r \leq R$, $(X,t) \in \partial\Omega$,

$$(r/2)^{n+1} \leq \sigma\big(\partial\Omega \cap C_r(X,t)\big) \leq M r^{n+1},$$

whenever $\partial\Omega$ separates $\mathbf{R}^{n+1}$ and satisfies a (M,R) Ahlfors condition.

Let $\Omega \subset \mathbf{R}^{n+1}$ be a δ_0-Reifenberg flat domain and define $\Omega^1 = \Omega \subset \mathbf{R}^{n+1}$, $\Omega^2 = \mathbf{R}^{n+1} \setminus \overline{\Omega}$. As above we let $(\widehat{X}^i, \hat{t}^i) \in \Omega^i$, for $i \in \{1,2\}$, $\hat{t}^2 < \hat{t}^1$ and define $\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)$ and $\omega^2(\widehat{X}^2, \hat{t}^2, \cdot\,)$ to be the caloric measures defined with respect to Ω^1 and Ω^2, respectively. Based on this notation there are several problems of free boundary type that one can pose. As discussed in this paper we can assume that $\omega^2(\widehat{X}^2, \hat{t}^2, \cdot\,)$ is absolutely continuous with respect to $\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)$ on $\partial\Omega$ and that the Radon–Nikodym derivative $k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot\,) = d\omega^2(\widehat{X}^2, \hat{t}^2, \cdot\,)/d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)$ is such that $\log k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot\,) \in \mathrm{VMO}(d\omega^1)$. The question is then what these conditions imply on the geometry and regularity of $\partial\Omega$. Assuming that $\partial\Omega$ is δ_0-Reifenberg flat and satisfies a (M,R) Ahlfors condition it is also relevant to study the implication of similar conditions phrased in terms of the Poisson kernel. I.e., we could assume at least one of the caloric measures $\omega^i(\widehat{X}^i, \hat{t}^i, \cdot\,)$ to be absolutely continuous with respect to σ on $\partial\Omega$ and hence define a Poisson kernel as $\tilde{k}^i(\widehat{X}^i, \hat{t}^i, \cdot\,) = d\omega^i(\widehat{X}^i, \hat{t}^i, \cdot\,)/d\sigma$. Using this notation the following natural problems, in the spirit of the one considered in this paper, can be formulated. $\mathrm{VMO}(d\sigma)$ is the space of functions of vanishing mean oscillation, defined with respect to $d\sigma$, defined in the bulk of the paper.

1. (One-phase problem). Assume that $\log \tilde{k}^1(\widehat{X}^1, \hat{t}^1, \cdot\,) \in \mathrm{VMO}(d\sigma)$. What implications does this condition have on $\partial\Omega$?
2. (Two-phase problem). Assume that $\log \tilde{k}^1(\widehat{X}^1, \hat{t}^1, \cdot\,) \in \mathrm{VMO}(d\sigma)$ and that $\log \tilde{k}^2(\widehat{X}^2, \hat{t}^2, \cdot\,) \in \mathrm{VMO}(d\sigma)$. What implications do these conditions have on $\partial\Omega$?

In the following we briefly describe recent developments on these kind of problems in the geometric settings of this paper. We first discuss the development in the elliptic situation before we focus on the parabolic situation.

2.1. Elliptic theory. In [D] B. Dahlberg showed that in a Lipschitz domain Ω the harmonic measure with respect to a fixed point, $d\omega$, and surface measure, $d\sigma$, are mutually absolutely continuous. In fact if $\tilde{k} = d\omega/d\sigma$, then Dahlberg showed that $\tilde{k}$ is in a certain L^2 reverse Hölder class from which it follows that

$\log \tilde{k} \in \mathrm{BMO}(d\sigma)$, the functions of bounded mean oscillation with respect to surface area on $\partial\Omega$. Jerison and Kenig [JK] showed for a C^1 domain that $\log \tilde{k} \in \mathrm{VMO}(d\sigma)$, the functions in $\mathrm{BMO}(d\sigma)$ of vanishing mean oscillation. In [KT] this result was generalized to 'chord arc domains with vanishing constant'. Concerning reverse conclusions, i.e., elliptic free boundary problems a classical result of Alt–Caffarelli states (for the definition of all the concepts we refer to [AC] and [KT2]) that if $\Omega \subset \mathbf{R}^n$ is δ_0-Reifenberg flat with an Ahlfors regular boundary and if $\log \tilde{k} \in C^{0,\beta}(\partial\Omega)$ for some $\beta \in (0,1)$, then Ω is a $C^{1,\alpha}$-domain for some $\alpha \in (0,1)$ which depends on β and n. In [J] Jerison proved that $\alpha = \beta$. The conclusion is that the oscillation of the logarithm of the Poisson kernel controls the geometry and in particular the 'flatness' or the oscillation of the unit normal. Furthermore in [J] Jerison treated a case beyond the $C^{0,\beta}$ situation for $\beta > 0$ under the assumption that the domain is locally given as the graph of a Lipschitz function and assuming that the normal derivative is continuous instead of having just vanishing mean oscillation. In the setting of domains not locally given by graphs, in [KT2], Kenig and Toro were able to prove the following theorem which is the analogue of the result of [AC] assuming vanishing oscillation of the logarithm of the Poisson kernel in an integral sense ($\mathrm{VMO}(d\sigma)$) instead of in the classical pointwise sense.

Theorem 2. *Assume that $\Omega \subset \mathbf{R}^n$ is δ_0-Reifenberg flat for some small enough $\delta_0 > 0$ and assume that $\partial\Omega$ is Ahlfors regular. If $\log \tilde{k} \in \mathrm{VMO}(d\sigma)$ then Ω is a chord arc domain with vanishing constant, i.e., the measure theoretical normal $\vec{n}$ is in $\mathrm{VMO}(d\sigma)$.*

This theorem can be seen as an answer to the elliptic one-phase type problem stated as Problem 1 above. In [KT3], Kenig and Toro consider the elliptic version of the two-phase problem we consider in Theorem 1. In particular, they prove ([KT3, Corollary 4.1]) that if $\Omega \subset \mathbf{R}^n$ is a δ_0-Reifenberg flat (in the elliptic sense) for some small enough $\delta_0 > 0$ and if $k = d\omega^1/d\omega^2$, $\log k \in \mathrm{VMO}(d\omega^1)$, then $\partial\Omega$ is Reifenberg flat with vanishing constant. In the elliptic setting questions of this type have previous been addressed in the case $n = 2$, i.e., in the plane, through the works of Bishop, Carleson, Garnett and Jones; see [B], [BCGJ] and [BJ].

Assuming that $\Omega \subset \mathbf{R}^n$ is a two-sided chord arc domain (meaning that Ω^1 and Ω^2 are NTA-domains and that $\partial\Omega$ is Ahlfors) they also prove ([KT3, Corollary 5.2]) that if $\log \tilde{k}^1 \in \mathrm{VMO}(d\sigma)$ and $\log \tilde{k}^2 \in \mathrm{VMO}(d\sigma)$ then firstly $\partial\Omega$ is Reifenberg flat with vanishing constant and secondly Ω is a chord arc domain with vanishing constant, i.e., the measure theoretical normal $\vec{n}$ is in $\mathrm{VMO}(d\sigma)$. This result can be seen as an answer to the elliptic two-phase type problem stated as Problem 2 above. One interesting aspect of the last result is that by imposing the two-phase condition $\log \tilde{k}^1 \in \mathrm{VMO}(d\sigma)$ and $\log \tilde{k}^2 \in \mathrm{VMO}(d\sigma)$ the conclusion of Theorem 2 remains true without an assumption on Reifenberg flatness. I.e., the two-phase condition serves as a replacement for flatness.

2.2. Parabolic theory. Through the works in [LM], [HL] it has become clear that from the perspective of parabolic singular integrals and caloric measure the parabolic analogue of the notion of Lipschitz domains, explored in elliptic partial differential equations, is graph domains $\Omega = \{(X,t) \in \mathbf{R}^{n+1} : x_0 > \psi(x,t)\}$ where $\psi = \psi(x,t)$: $\mathbf{R}^n \to \mathbf{R}$ has compact support and satisfies

$$|\psi(x,t) - \psi(y,t)| \le b_1|x-y|, \quad x,y \in \mathbf{R}^{n-1},\ t \in \mathbf{R}, \tag{1}$$

$$D^t_{1/2}\psi \in \mathrm{BMO}(\mathbf{R}^n), \quad \|D^t_{1/2}\psi\|_* \le b_2 < \infty. \tag{2}$$

Here $D^t_{1/2}\psi(x,t)$ denotes the $1/2$ derivative in t of $\psi(x,\cdot\,)$, x fixed. This half derivative in time can be defined by way of the Fourier transform or by

$$D^t_{1/2}\psi(x,t) \equiv \hat{c}\int_{\mathbf{R}} \frac{\psi(x,s)-\psi(x,t)}{|s-t|^{3/2}}\,ds$$

for properly chosen $\hat{c}$. $\|\cdot\|_*$ denotes the norm in parabolic $\mathrm{BMO}(\mathbf{R}^n)$ (for a definition of this space see [HLN2]). One can prove that the conditions in (1) and (2) imply that $\psi(x,t)$ is parabolically Lipschitz in the following sense,

$$|\psi(x,t)-\psi(y,s)| \le \beta(|x-y|+|t-s|^{1/2}) \quad x,y \in \mathbf{R}^n\ t,s \in \mathbf{R}.$$

Under the smoothness assumptions on ψ stated in (1) and (2) it was proven in [LM] that the parabolic Poisson kernel is in a certain L^p reverse Hölder class for some $p > 1$. In particular $\omega(\widehat{X},\hat{t},\cdot\,)$ is an A^∞ weight (with respect to σ). The result of [LM] was later shown to be sharp in [HL] where examples are given of graph domains, as in [LM], with p arbitrarily close to 1. In [HL] the relevant L^2-result was established. Finally we note that examples of [KW] and [LS] show that caloric and adjoint caloric measure need not be absolutely continous with respect to the surface measure σ in graph $\mathrm{Lip}(1,1/2)$ domain.

In [HLN2] the parabolic Poisson kernel was analyzed in domains not locally given by graphs. In this situation the geometry was controlled by a certain geometric square function, the boundedness of which implied that on every scale the boundary contained 'big pieces of graph', graph with the regularity stated in (1) and (2) (see [HLN1]). A fundamental assumption in [HLN2] is that $\partial\Omega$ is δ_0-Reifenberg flat and satisfies a (M,R) Ahlfors condition (as defined above) but to properly formulate the result in [HLN2] we need to introduce some more notation and concepts.

Let

$$d(F_1,F_2) = \inf\{|X-Y|+|s-t|^{1/2} : (X,t) \in F_1,\ (Y,s) \in F_2\}$$

denote the parabolic distance between the sets F_1, F_2 and for Ω (such that $\partial\Omega$ separates $\mathbf{R}^{n+1}$ and satisfies a (M,R) Ahlfors condition) we set

$$\gamma(Z,\tau,r) = \inf_P\biggl[r^{-n-3}\int_{\partial\Omega\cap C_r(Z,\tau)} d(\{(Y,s)\},P)^2\,d\sigma(Y,s)\biggr].$$

Here the infimum is taken over all n-dimensional planes P containing a line parallel to the t axis. Let

$$d\nu(Z,\tau,r)=\gamma(Z,\tau,r)\,d\sigma(Z,\tau)\,r^{-1}dr.$$

We say that ν is a Carleson measure on $[\partial\Omega\cap C_R(Y,s)]\times(0,R)$ if there exists $M_1<\infty$ such that whenever $(X,t)\in\partial\Omega$ and $C_\varrho(X,t)\subset C_R(Y,s)$, we have

$$\nu\big([C_\varrho(X,t)\cap\partial\Omega]\times(0,\varrho)\big)\le M_1\,\varrho^{n+1}. \tag{3}$$

The smallest such M_1 is called the Carleson norm of ν on $[\partial\Omega\cap C_R(Y,s)]\times(0,R)$ and we write $\|\nu\|_+$ for the Carleson norm of ν if the inequality in (3) holds for all $\varrho>0$. The following two definitions can be found in [HLN1] and [HLN2].

Definition 4. $\partial\Omega$ is said to be uniformly rectifiable (in the parabolic sense) if $\|\nu\|_+<\infty$ and (3) holds for all $R>0$. If furthermore $\partial\Omega$ separates $\mathbf{R}^{n+1}$ and is uniformly rectifiable, then Ω is called a parabolic regular domain.

Definition 5. Ω is called a chord arc domain with vanishing constant if Ω is a parabolic regular domain and

$$\sup_{\substack{(X,t)\in\partial\Omega\\0<\varrho\le r}}\Big[\varrho^{-(n+1)}\,\nu\big([C_\varrho(X,t)\cap\partial\Omega]\times(0,\varrho)\big)\Big]\to0\quad\text{as } r\to0. \tag{4}$$

In [HLN2] it is proven that if Ω is a parabolic regular domain with Reifenberg constant $\delta_0=\delta_0(M,\|\nu\|_+)$, sufficiently small, then ω is an A^∞ weight. Furthermore if Ω is a chord arc domain with vanishing constant and $\tilde k(\widehat X,\hat t,\cdot\,)=d\omega(\widehat X,\hat t,\cdot\,)/d\sigma$, then $\log\tilde k(\widehat X,\hat t,\cdot\,)\in\mathrm{VMO}(d\sigma)$.

To formulate the result in [HLN2] which is more relevant to the discussions in this paper let $a=a\big(\Delta(X,t,\varrho),f\big)$ denote the average of $f=\log\tilde k(\widehat X,\hat t,\cdot\,)$ on $\Delta(X,t,\varrho)$ with respect to σ. Then we say that $f\in\mathrm{VMO}(d\sigma)$ provided for each compact $K\subset\partial\Omega\cap\{(Y,s):s<\hat t\}$,

$$\lim_{r\to0}\sup_{\substack{(X,t)\in K\\0<\varrho\le r}}\sigma\big(\Delta(X,t,\varrho)\big)^{-1}\int_{\Delta(X,t,\varrho)}|f(Y,s)-a|\,d\sigma=0.$$

Let $(X,t)\in\partial\Omega$, and $r,\varrho>0$ and put

$$\Delta(X,t,r,\varrho)=\{(Y,s)\in\partial\Omega:|Y-X|<r,\ |s-t|<\varrho^2\}.$$

In this notation $\Delta(X,t,r)=\Delta(X,t,r,r)$. We say that $\omega(\widehat X,\hat t,\cdot\,)$ is asymptotically optimal doubling if, whenever $K\subset\partial\Omega\cap\{(Y,s):s<\hat t\}$ is compact and $0<\tau_1,\tau_2<1$, we have

$$\lim_{r\to0}\sup_{(X,t)\in K}\frac{\omega\big(\Delta(X,t,\tau_1r,\tau_2r)\big)}{\omega\big(\Delta(X,t,r)\big)}=\lim_{r\to0}\inf_{(X,t)\in K}\frac{\omega\big(\Delta(X,t,\tau_1r,\tau_2r)\big)}{\omega\big(\Delta(X,t,r)\big)}=\tau_1^{n-1}\tau_2^2.$$

In [HLN2] the following theorem is proven.

Theorem 3. *Let* Ω *be a parabolic regular domain and put* $\tilde{k}(\widehat{X},\hat{t},\cdot\,) = d\omega(\widehat{X},\hat{t},\cdot\,)/d\sigma$. *If* $\omega(\widehat{X},\hat{t},\cdot\,)$ *is asymptotically optimal doubling,* $\log\tilde{k}(\widehat{X},\hat{t},\cdot\,) \in \mathrm{VMO}(d\sigma)$ *and* $\|\nu\|_+$ *is small enough then* (4) *holds with* $\partial\Omega$ *replaced by any compact subset,* $F\subset\partial\Omega\cap\{(Y,s): s<\hat{t}\}$.

Note that this result is weaker than the result proved in [KT2] as Theorem 3 uses the assumption that $\omega(\widehat{X},\hat{t},\cdot\,)$ is asymptotically optimal doubling. In fact the proof in [KT2] uses the important result of [AC] for elliptic pde, whose potential generalization to the heat equation is currently unknown. In fact these 'free boundary' type problems appear harder in the caloric case. Similar problems have been considered in [ACS], [ACS1] under stronger assumptions.

Summarizing we can conclude that to a large extent the answers to Problem 1 and 2 in the parabolic setting remain unclear but that the main theorem of this paper, Theorem 1, gives a perfect parabolic analogue to the corresponding elliptic result ([KT3, Corollary 4.1]).

3. Estimates of caloric functions in parabolic NTA-domains

Recall from [LM, Chapter 3, Section 6] that $\Omega\subset\mathbf{R}^{n+1}$ is an unbounded parabolic nontangentially accessible domain if $\partial\Omega$ separates $\mathbf{R}^{n+1}$ and if the following conditions are satisfied for some $\lambda,\gamma\ge 100$. Given $(X,t)\in\partial\Omega$ and $r>0$ there exist

$$\begin{aligned}
\underline{A}_r^1(X,t) &= \big(U_1(X,t),t_1(X,t)\big) = (U_1,t_1)\in\Omega\cap C_r(X,t),\\
\bar{A}_r^1(X,t) &= \big(U_2(X,t),t_2(X,t)\big) = (U_2,t_2)\in\Omega\cap C_r(X,t),\\
\underline{A}_r^2(X,t) &= \big(N_1(X,t),\tau_1(X,t)\big) = (N_1,\tau_1)\in(\mathbf{R}^{n+1}\setminus\overline{\Omega})\cap C_r(X,t),\\
\bar{A}_r^2(X,t) &= \big(N_2(X,t),\tau_2(X,t)\big) = (N_2,\tau_2)\in(\mathbf{R}^{n+1}\setminus\overline{\Omega})\cap C_r(X,t),
\end{aligned}$$

such that

$$\begin{aligned}
&\lambda^{-1}r^2\le\min(t_2-t,t-t_1)\le\lambda r^2,\\
&\lambda^{-1}r^2\le\min(\tau_2-t,t-\tau_1)\le\lambda r^2,\\
&\quad r/\lambda\le\min\big[d\big(\{(N_i,\tau_i)\},\partial\Omega\big),d\big(\{(U_i,t_i)\},\partial\Omega\big)\big].
\end{aligned}$$

Here $d(\,\cdot\,,\cdot\,)$ denotes the parabolic distance defined in the previous section. As in [JK1] we refer to these conditions as the *corkscrew condition.* Next suppose $(U_i,s_i)\in\Omega$, $i=1,2$, with $(s_2-s_1)^{1/2}>\gamma^{-1}d\big(\{(U_1,s_1)\},\{(U_2,s_2)\}\big)$. We say as in [JK1] that $\{C_{r_i}(X_i,t_i)\}_1^l$ is a *Harnack chain* from (U_1,s_1) to (U_2,s_2) with constant γ provided there exists $c(\gamma)\ge 1$ such that

- $(U_1,s_1)\in C_{r_1}(X_1,t_1)$, $(U_2,s_2)\in C_{r_l}(X_l,t_l)$, and for $i=1,2,\dots,l-1$, $C_{r_{i+1}}(X_{i+1},t_{i+1})\cap C_{r_i}(X_i,t_i)\neq\emptyset$,
- $c(\gamma)^{-1}d\big(\{(X_i,t_i)\},\partial\Omega\big)\le r_i\le c(\gamma)d\big(\{(X_i,t_i)\},\partial\Omega\big)$, when $i=1,2,\dots,l$,
- $t_{i+1}-t_i\ge c(\gamma)^{-1}r_i^2$, for $i=1,2,\dots,l$,

- $l \leq c(\gamma) \log\left(2 + \dfrac{d\big(\{(U_1, s_1)\}, \{(U_2, s_2)\}\big)}{\min\big[d\big(\{(U_1, s_1)\}, \partial\Omega\big), d\big(\{(U_2, s_2)\}, \partial\Omega\big)\big]}\right).$

l is referred to as the length of the Harnack chain. Our first lemma states that δ-Reifenberg flat domains are examples of parabolic NTA-domains. This lemma is also proven in [HLN2].

Lemma 4. *Let Ω be δ_0-Reifenberg flat. If $\delta_0 > 0$ is sufficiently small, then Ω is a parabolic NTA domain for $\lambda = 100$ and any $\gamma \geq 100$.*

Proof. Let $(X,t) \in \partial\Omega$ and $r > 0$. The definition of Reifenberg flatness (Definition 1) implies that for $\delta_0 > 0$ sufficiently small and with $\lambda = 100$, the corkscrew conditions are fulfilled with

$$\begin{aligned}
\underline{A}_r^1(X,t) &= (U_1, t_1) = (X + r\hat{n}, t - r^2), \quad & \bar{A}_r^1(X,t) &= (U_2, t_2) = (X + r\hat{n}, t + r^2),\\
\underline{A}_r^2(X,t) &= (N_1, \tau_1) = (X - r\hat{n}, t - r^2), \quad & \bar{A}_r^2(X,t) &= (N_2, \tau_2) = (X - r\hat{n}, t + r^2).
\end{aligned}$$

Here $\hat{n} = \hat{n}(X,t,r)$. To prove, for any $\gamma \geq 100$, the existence of Harnack chains we follow [KT1] and for $(U_1, s_1), (U_2, s_2) \in \Omega$, as above, we choose points $(P_1, t_1), (P_2, t_2) \in \partial\Omega$ with

$$\varrho_i = d\big(\{(P_i, t_i)\}, \{(U_i, t_i)\}\big) = d\big(\{(U_i, t_i)\}, \partial\Omega\big), \quad i = 1, 2.$$

If $\varrho = d\big(\{(U_1, t_1)\}, \{(U_2, t_2)\}\big)$, then using that Ω is δ_0-Reifenberg flat for $\delta_0 > 0$ small, we can choose a Harnack chain of length

$$l \leq c(\gamma) \log\left(2 + \frac{\varrho}{\min\{\varrho_1, \varrho_2\}}\right)$$

joining (U_1, s_1) to (U_2, s_2). If for example $\varrho > 1000 \max(\varrho_1, \varrho_2)$ and l_0 is the smallest positive integer greater than $\log(\varrho/\varrho_1)$, then from the Reifenberg flatness it follows that we can choose $\{(X_i, t_i)\}$ with $(X_i, t_i) \in C_{e^i \varrho_1}(P_1, t_1)$ for $2 \leq i \leq l_0$ and then $(X_i, t_i) \in C_{e^{l_0 - i}\varrho}(P_2, t_2)$ for $l_0 + 1 \leq i \leq l_0 + l_1$, where $l_1 \leq \log(2\varrho/\varrho_2)$. ◻

In this section we will assume, in order to ensure that Lemma 4 is valid, that Ω is a δ_0-Reifenberg flat with small constant. For $(X,t) \in \partial\Omega$ and $r > 0$ we define the following points located in Ω,

$$\underline{A}_r(X,t) = (X + r\hat{n}, t - r^2), \quad A_r(X,t) = (X + r\hat{n}, t), \quad \bar{A}_r(X,t) = (X + r\hat{n}, t + r^2).$$

Again here $\hat{n} = \hat{n}(X,t,r)$. The existence of these points is a consequence of the δ_0-Reifenberg flatness and we will make use of these points throughout the section.

3.1. Basic estimates. In this section we state some basic estimates for certain solutions to the heat and adjoint heat equation in parabolic NTA domains. An outline of the proofs of these lemmas valid in the current situation can be found in [LM, Chapter 3, Section 6] and [HLN2]. Apart from these references many of the relevant ideas used in the proofs can also be found in [FS], [FSY] and [N]. In particular, in [N] all relevant estimates are stated and proved, in $\mathrm{Lip}(1,1/2)$-domains, in the general setting of second order parabolic equations in divergence form.

Note that the characteristics of a parabolic NTA-domain is described by the parameters λ and γ and hence basically all constants appearing below will depend on these two parameters. I.e., below $c = c(\lambda,\gamma)$ but the constants often also depend on other parameters and we will not always indicate the dependence on λ and γ.

We start by a lemma on Hölder decay at the boundary of non-negative solutions vanishing on the boundary. The lemma is proved using standard comparison arguments and the fact that the complement of Ω is uniformly 'fat'.

Lemma 5. *Let $\Omega \subset \mathbf{R}^{n+1}$ be a parabolic NTA-domain with parameters λ and γ. Let $(X,t) \in \partial\Omega$ and suppose that u is a non-negative solution to either the heat or the adjoint heat equation in $\Omega \cap C_{2r}(X,t)$ which vanishes continuously on $\partial\Omega \cap C_{2r}(X,t)$. Then there exists $\alpha = \alpha(\lambda,\gamma)$, $0 < \alpha < \frac{1}{2}$, and $c = c(\lambda,\gamma) \geq 1$ such that whenever $(Y,s) \in \Omega \cap C_r(X,t)$*

$$u(Y,s) \leq c\left[\frac{d\big(\{(Y,s)\},\{(X,t)\}\big)}{r}\right]^{\alpha} \sup_{(Z,\tau)\in\Omega\cap C_r(X,t)} u(Z,\tau).$$

The next lemma is a standard Carleson type lemma.

Lemma 6. *Let u, Ω and (X,t) be as in the previous lemma. If $(Y,s) \in \Omega \cap C_{r/2}(X,t)$, then*

$$u(Y,s) \leq cu\big(\bar{A}_r(X,t)\big)$$

when u is a solution to the heat equation while

$$u(Y,s) \leq cu\big(\underline{A}_r(X,t)\big)$$

when u is a solution to the adjoint heat equation in $C_{2r}(X,t) \cap \Omega$.

Given $(Y,s) \in \Omega$, let $G(\,\cdot\,,Y,s)$ denote Green's function for the heat equation in Ω with pole at (Y,s). That is

$$\frac{\partial}{\partial t}G(X,t,Y,s) - \Delta G(X,t,Y,s) = \delta\big((X,t)-(Y,s)\big) \quad \text{in } \Omega \text{ and } G \equiv 0 \text{ on } \partial\Omega.$$

Here δ denotes the Dirac delta function and Δ is the Laplacian in X. We note that $G(Y,s,\cdot\,)$ is Green's function for the adjoint heat equation with pole at $(Y,s) \in \Omega$

(i.e. $-(\partial/\partial t)G(Y,s,\cdot\,)-\Delta G(Y,s,\cdot)=\delta\big(\cdot-(Y,s)\big)$. Let ω, $\hat\omega$ be the corresponding caloric and adjoint caloric measures for the heat/adjoint heat equation in Ω. We note that $\omega(Y,s,\cdot\,)$, $\hat\omega(Y,s,\cdot\,)$ are the Riesz measures associated with $G(\cdot,Y,s)$, $G(Y,s,\cdot\,)$ by way of the Riesz representation theorem for sub caloric/adjoint caloric functions in $\mathbf{R}^{n+1}\setminus\{(Y,s)\}$ (see [Do]). From this theorem we have that

$$\int_{\partial\Omega}\phi\,d\omega(Y,s,\cdot\,)=\int_{\Omega}G(Y,s,\cdot\,)\left(\Delta\phi-\frac{\partial\phi}{\partial s}\right)dZ\,d\tau$$

for all $\phi\in C_0^\infty\big(\mathbf{R}^{n+1}\setminus\{(Y,s)\}\big)$. A similar formula holds for $\hat\omega$. Estimates for caloric/adjoint caloric measure in terms of Green's function and vice versa are given by the following lemma. The proof follows by standard arguments.

Lemma 7. *Let Ω and (X,t) be as in the previous lemma. Let $A\ge 100$ and assume that $(Y,s)\in\Omega$ with $|Y-X|^2\le A|s-t|$ and $|s-t|\ge 4r^2$. There exists $c=c(A)\ge 1$ such that if $s>t$, then*

$$c^{-1}r^nG\big(Y,s,\bar A_r(X,t)\big)\le\omega\big(Y,s,\,\Delta(X,t,r/2)\big)\le cr^nG\big(Y,s,\underline{A}_r(X,t)\big)$$

while if $s<t$,

$$c^{-1}r^nG\big(\underline{A}_r(X,t),Y,s\big)\le\hat\omega\big(Y,s,\Delta(X,t,r/2)\big)\le cr^nG\big(\bar A_r(X,t),Y,s\big).$$

Next we have the following backward Harnack inequality.

Lemma 8. *Let Ω and (X,t) be as in the previous lemma. Let $A\ge 100$ and assume that $|Y-X|^2\le A|s-t|$ and $|s-t|\ge 5r^2$. There exists $c=c(A)\ge 1$ such that*

$$G\big(Y,s,\underline{A}_r(X,t)\big)\le cG\big(Y,s,\bar A_r(X,t)\big)$$

when $s>t$ while if $s<t$, then

$$G\big(\underline{A}_r(X,t),Y,s\big)\le cG\big(\bar A_r(X,t),Y,s\big).$$

Combining the previous two lemmas the doubling property of caloric/adjoint caloric measure can be proven.

Lemma 9. *Let Ω, (X,t), (Y,s) and A be as in the previous lemma. Then*

$$\omega^*\big(Y,s,\Delta(X,t,r)\big)\le c(A)\omega^*\big(Y,s,\Delta(X,t,r/2)\big)$$

where $\omega^*=\omega$ when $s>t$ while $\omega^*=\hat\omega$ when $s<t$.

Let $(X,t)\in\partial\Omega$, $\varrho>0$ and $R>0$. $u>0$ is said to satisfy a strong Harnack inequality in $C_R(X,t)\cap\Omega$ provided that u is a solution to either the heat or adjoint heat equation in $C_R(X,t)\cap\Omega$ and

$$u(\widehat{X},\hat{t})\le\tilde{\lambda}u(\widetilde{X},\tilde{t})\text{ whenever }(\widehat{X},\hat{t}),(\widetilde{X},\tilde{t})\in C_\varrho(Z,\tau)\\ \text{and}\quad C_{2\varrho}(Z,\tau)\subset C_R(X,t)\cap\Omega.$$

Here $\tilde{\lambda}, 1\le\tilde{\lambda}<\infty$, is independent of $C_{2\varrho}(Z,\tau)\subset C_R(X,t)\cap\Omega$. For (X,t), ϱ as above and $A>0$ we define

$$\Gamma_A^+(X,t,\varrho)=\Omega\cap\{(Y,s):|Y-X|^2\le A|s-t|,\ |s-t|\ge5\varrho^2,\ s>t\},$$
$$\Gamma_A^-(X,t,\varrho)=\Omega\cap\{(Y,s):|Y-X|^2\le A|s-t|,\ |s-t|\ge5\varrho^2,\ s<t\}.$$

Using Lemmas 7, 8 and 9 one can prove that if $(Y,s)\in\Gamma_A^+(X,t,R)$ then $G(Y,s,\cdot\,)$ satisfies a strong Harnack inequality in $C_R(X,t)\cap\Omega$ while if $(Y,s)\in\Gamma_A^-(X,t,R)$ then $G(\,\cdot\,,Y,s)$ satisfies a strong Harnack inequality in $C_R(X,t)\cap\Omega$. Moreover, $\tilde{\lambda}$ depends only on A once the NTA-constants λ and γ have been chosen. Using the notion of strong Harnack inequality the following two comparison lemmas can be proven.

Lemma 10. *Let $u,v>0$ be continuous in $\overline{C}_{2r}(X,t)\cap\overline{\Omega}$, $u=v=0$ on $\Delta(X,t,2r)$ and assume that u and v both are solutions either to the heat or the adjoint heat equation in $C_{2r}(X,t)\cap\Omega$. If u, v satisfy a strong Harnack inequality in $C_{2r}(X,t)\cap\Omega$ for some $\tilde{\lambda}\ge1$, then*

$$\frac{u(Y,s)}{v(Y,s)}\le c(\tilde{\lambda})\frac{u(\widehat{U})}{v(\widehat{U})}\quad\text{in } C_{r/2}(X,t)\cap\overline{\Omega}.$$

Here $\widehat{U}=\bar{A}_r(X,t)$ when u, v are solutions to the heat equation while $\widehat{U}=\underline{A}_r(X,t)$ when u,v are solutions to the adjoint heat equation in $\Omega\cap C_{2r}(X,t)$.

Lemma 11. *Under the same hypotheses as in Lemma 10 there exists $\tilde{\gamma}=\tilde{\gamma}(\tilde{\lambda})$, $0<\tilde{\gamma}\le1/2$, and $c=c(\tilde{\lambda})\ge1$, such that whenever $0<\varrho\le r/2$ then*

$$\left|\frac{u(Z,\tau)v(Y,s)}{v(Z,\tau)u(Y,s)}-1\right|\le c\,(\varrho/r)^{\tilde{\gamma}}\quad\text{for } (Z,\tau),(Y,s)\in C_\varrho(X,t)\cap\Omega.$$

3.2. Refined estimates. Let $\Omega\subset\mathbf{R}^{n+1}$ be a δ_0-Reifenberg flat domain and define $\Omega^1=\Omega\subset\mathbf{R}^{n+1}$, $\Omega^2=\mathbf{R}^{n+1}\setminus\overline{\Omega}$. We assume that $\delta_0>0$ is small. Let $(\widehat{X}^i,\hat{t}^i)\in\Omega^i$, for $i\in\{1,2\}$, $\hat{t}^2<\hat{t}^1$ and define $\omega^1(\widehat{X}^1,\hat{t}^1,\cdot\,)$ and $\omega^2(\widehat{X}^2,\hat{t}^2,\cdot\,)$ to be the caloric measures defined with respect to Ω^1 and Ω^2, respectively. In the following we will assume that $\omega^2(\widehat{X}^2,\hat{t}^2,\cdot\,)$ is absolutely continuous with respect to $\omega^1(\widehat{X}^1,\hat{t}^1,\cdot\,)$ on $\partial\Omega$ and that the Radon–Nikodym derivative $k(\widehat{X}^1,\hat{t}^1,\widehat{X}^2,\hat{t}^2,\cdot\,)=$

$d\omega^2(\widehat{X}^2, \hat{t}^2, \cdot\,)/d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)$ is such that $\log k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot\,)$ is in the space $\mathrm{VMO}\big(d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)\big)$. To properly define the space $\mathrm{VMO}\big(d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)\big)$ we let $a = a(\Delta(X,t,\varrho), f)$ denote the average of $f = \log k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot\,)$ on $\Delta(X,t,\varrho)$ with respect to $d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)$. f is said to be in $\mathrm{VMO}\big(d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)\big)$ provided for each compact $K \subset \partial\Omega \cap \{(Y,s) : s < \hat{t}^2\}$,

$$\lim_{r\to 0} \sup_{\substack{(X,t)\in K\\ 0<\varrho\le r}} \omega^1\big(\widehat{X}^1, \hat{t}^1, \Delta(X,t,\varrho)\big)^{-1} \int_{\Delta(X,t,\varrho)} |f(Y,s) - a|\, d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,) = 0.$$

We start by exploring the information contained in the condition

$$\log k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot\,) \in \mathrm{VMO}\big(d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)\big).$$

Lemma 12. *Let $\Omega \subset \mathbf{R}^{n+1}$ be a δ_0-Reifenberg flat domain and define $\Omega^1 = \Omega \subset \mathbf{R}^{n+1}$, $\Omega^2 = \mathbf{R}^{n+1} \setminus \overline{\Omega}$. Assume furthermore that $\omega^2(\widehat{X}^2, \hat{t}^2, \cdot\,)$ is absolutely continuous with respect to $\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)$ on $\partial\Omega$ and that the Radon–Nikodym derivative*

$$k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot\,) = d\omega^2(\widehat{X}^2, \hat{t}^2, \cdot\,)/d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)$$

is such that

$$\log k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot\,) \in \mathrm{VMO}\big(d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)\big).$$

Then there exists $\alpha \in (0,1)$ and a constant $C = C(n, \delta_0, A)$ such that the following is true. If $(X,t) \in \partial\Omega$, $r < r_0$, $|X - \widehat{X}^i|^2 \le A(\hat{t}^i - t)$ for some $A \ge 2$ and for $i \in \{1,2\}$, $\min\{\hat{t}^1, \hat{t}^2\} - t \ge 8r^2$ and $E \subset \Delta(X,t,r)$, then

$$\frac{\omega^2(\widehat{X}^2, \hat{t}^2, E)}{\omega^2\big(\widehat{X}^2, \hat{t}^2, \Delta(X,t,r)\big)} \le C\bigg(\frac{\omega^1(\widehat{X}^1, \hat{t}^1, E)}{\omega^1\big(\widehat{X}^1, \hat{t}^1, \Delta(X,t,r)\big)}\bigg)^{\alpha}.$$

Proof. Let $E \subset \Delta(X,t,r)$ and $\Delta(X,t,r)$ be as in the statement of the lemma. Note that the restrictions on the points (X,t) and $(\widehat{X}^i, \hat{t}^i)$ imply uniform bounds on the doubling constants of the caloric measures under consideration. The inequality stated in the lemma is therefore a standard consequence of the fact that if $\log k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot\,) \in \mathrm{VMO}\big(d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)\big)$ then $k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot\,)$ is an A_∞ weigth with respect to the measure $\omega^1(\widehat{X}^1, \hat{t}^1, \cdot\,)$. For more on the relation between VMO and A_p-weights we refer to [S]. □

Lemma 13. *Let Ω be a δ_0-Reifenberg flat domain with $\delta_0 > 0$ small enough. Let $(X,t) \in \partial\Omega$, $r > 0$ and let $u > 0$ be a solution to either the heat or the adjoint heat equation in $C_{2r}(X,t) \cap \Omega$, continous in $\overline{C}_{2r}(X,t) \cap \overline{\Omega}$ and such that $u = 0$ on $\Delta(X,t,2r)$. If u satisfies a strong Harnack inequality in $C_{2r}(X,t) \cap \Omega$ for* *some* $\tilde\lambda \ge 1$, *then, given* $\varepsilon > 0$, *there exists* $\hat\delta_0 = \hat\delta_0(n,\varepsilon,\tilde\lambda) > 0$ *and a constant* $C = C(n,\varepsilon,\tilde\lambda)$ *such that if* $\delta_0 \le \hat\delta_0$ *and* $\hat r \le r$ *then*

$$C^{-1}\left(\frac{\hat r}{r}\right)^{1+\varepsilon} u\bigl(A_r(X,t)\bigr) \le u\bigl(A_{\hat r}(X,t)\bigr) \le C\left(\frac{\hat r}{r}\right)^{1-\varepsilon} u\bigl(A_r(X,t)\bigr).$$

Proof. By scaling and translation we can without loss of generality assume that $r = 1$, $(X,t) = (0,0)$. We can furthermore assume that $u\bigl(A_1(0,0)\bigr) = 1$. Based on these simplifications we want to prove that given $\varepsilon > 0$, there exists $\hat\delta_0 = \hat\delta_0(n,\varepsilon,\tilde\lambda) > 0$ and a constant $C = C(n,\varepsilon,\tilde\lambda)$ such that if $\delta_0 \le \hat\delta_0$ and $\hat r \le 1$ then

$$C^{-1}\hat r^{1+\epsilon} \le u\bigl(A_{\hat r}(0,0)\bigr) \le C\hat r^{1-\varepsilon}.$$

In order to prove this inequality we will make use of a number of auxiliary sets and functions. Recall that by definition of the δ_0-Reifenberg flatness there exists an n-dimensional plane $\widehat P = \widehat P(0,0,2)$, containing $(0,0)$ and a line parallel to the t axis, having unit normal $\hat n = \hat n(0,0,2)$ such that

$$\bigl\{(\widehat Z,\hat\tau) + r\hat n \in C_2(0,0) : (\widehat Z,\hat\tau) \in \widehat P,\ r > 2\delta_0\bigr\} \subset \Omega,$$
$$\bigl\{(\widehat Z,\hat\tau) - r\hat n \in C_2(0,0) : (\widehat Z,\hat\tau) \in \widehat P,\ r > 2\delta_0\bigr\} \subset \mathbf{R}^{n+1}\setminus\Omega.$$

Based on this we introduce

$$S^- = S^-(\delta_0) = \bigl\{(\widehat Z,\hat\tau) + r\hat n : (\widehat Z,\hat\tau) \in \widehat P,\ r > -4\delta_0\bigr\},$$
$$S^+ = S^+(\delta_0) = \bigl\{(\widehat Z,\hat\tau) + r\hat n : (\widehat Z,\hat\tau) \in \widehat P,\ r > 4\delta_0\bigr\}.$$

Then $S^+ \subset S^-$ and S^+ is a translation of the halfspace S^-. Consider the sets $S^-\cap C_2(0,0)$ and $S^+\cap C_2(0,0)$. The parabolic boundary of $S^-\cap C_2(0,0)$ consists of two pieces Γ_1^- and Γ_2^- where

$$\Gamma_1^- = C_2(0,0)\cap\bigl\{(\widehat Z,\hat\tau) - 4\delta_0\hat n : (\widehat Z,\hat\tau) \in \widehat P\bigr\},$$
$$\Gamma_2^- = \partial_p C_2(0,0)\cap\bigl\{(\widehat Z,\hat\tau) + r\hat n : (\widehat Z,\hat\tau) \in \widehat P,\ r > -4\delta_0\bigr\}.$$

Here $\partial_p C_2(0,0)$ is the parabolic boundary of $C_2(0,0)$. Similarly the parabolic boundary of $S^+\cap C_2(0,0)$ consists of two pieces Γ_1^+ and Γ_2^+. We define two auxilary functions $\tilde u^-$ and $\tilde u^+$. $\tilde u^-$ is a caloric function defined in $S^-(\delta_0)\cap C_2(0,0)$ satisfying $\tilde u^-(Y,s) = 0$ if $(Y,s) \in \Gamma_1^-$ and $\tilde u^-(Y,s) = 1$ if $(Y,s) \in \Gamma_2^-$. Similarly $\tilde u^+$ is a caloric function defined in $S^+(\delta_0)\cap C_2(0,0)$ satisfying $\tilde u^+(Y,s) = 0$ if $(Y,s) \in \Gamma_1^+$ and $\tilde u^+(Y,s) = 1$ if $(Y,s) \in \Gamma_2^+$. Finally we also define

$$\tilde v^-\bigl((\widehat Z,\hat\tau) + r\hat n\bigr) := r + 4\delta_0, \quad \tilde v^+\bigl((\widehat Z,\hat\tau) + r\hat n\bigr) := r - 4\delta_0$$

for all $(\widehat{Z},\hat{\tau}) \in \widehat{P}$ and $r \in \mathbf{R}$. I.e., the last two caloric functions are independent of the time variable and grow linearly in the direction of the normal $\hat{n}$.

Based on this notation we will now prove the right-hand side inequality. By the maximum principle we have, by construction, that

$$u(Y,s) \le \tilde{u}^-(Y,s)$$

for all $(Y,s) \in \Omega \cap C_1(0,0)$. Note that if $\delta_0 \le \hat{\delta}_0$ then $\tilde{u}^-$ will satisfy, as $\tilde{u}^-$ essentially is the caloric measure in $S^-(\delta_0)\cap C_2(0,0)$ of the set $S^-(\delta_0)\cap\partial_p C_2(0,0)$, a strong Harnack inequality in $S^-(\delta_0)\cap C_{1/2}(0,0)$ with a universal $\tilde{\lambda}$. Let in the following $(\widehat{Z},\hat{\tau}) \in \widehat{P}$. For all $(Y,s) = (\widehat{Z},\hat{\tau}) + r\hat{n} \in C_{1/2}(0,0)\cap\Omega$, $r \in \mathbf{R}$, we get, using Lemma 10, that

$$\tilde{u}^-(Y,s) \le C\tilde{v}^-(Y,s) = C(r + 4\delta_0).$$

Therefore if $(Y,s) = (\widehat{Z},\hat{\tau}) + r\hat{n} \in C_\theta(0,0)\cap\Omega$ for $\theta < 1/2$ an elementary argument implies that $\tilde{u}^-(Y,s) \le C(\theta+\delta_0)$ and hence $u(Y,s) \le C(\theta+\delta_0)$. Iteratively $u(Y,s) \le [C(\theta+\delta_0)]^k$ for $(Y,s) = (\widehat{Z},\hat{\tau}) + r\hat{n} \in C_{\theta^k}(0,0)\cap\Omega$. In particular, we can conclude that if δ_0 is small enough, then $u(Y,s) \le [2C\delta_0]^k$ for all $(Y,s) = (\widehat{Z},\hat{\tau}) + r\hat{n} \in C_{\delta_0^k}(0,0)\cap\Omega$. For given $\varepsilon > 0$ small, let δ_0 be such that $2C\delta_0 \le \delta_0^{1-\varepsilon}$. If we let k be determined through $\delta_0^k = \hat{r}$, then $u\big(A_{\hat{r}}(0,0)\big) \le C\hat{r}^{1-\varepsilon}$ and the proof is complete in one direction.

Left is to prove the inequality in the other direction. To start with we in this case let $\hat{u}$ be a caloric function defined in $\Omega\cap C_2(0,0)$ satisfying $\hat{u}(Y,s) = 0$ if $(Y,s) \in \partial\Omega\cap C_2(0,0)$ and $\hat{u}(Y,s) = 1$ if $(Y,s) \in \Omega\cap\partial_p C_2(0,0)$. Again if $\delta_0 \le \hat{\delta}_0$ then $\hat{u}$ will satisfy a strong Harnack inequality in $\Omega\cap C_1(0,0)$ with a universal $\tilde{\lambda}$. Applying Lemma 10 it follows that there exists a constant C such that

$$C^{-1}\frac{u\big(A_1(0,0)\big)}{\hat{u}\big(A_1(0,0)\big)} \le \frac{u\big(A_{\tilde{r}}(0,0)\big)}{\hat{u}\big(A_{\tilde{r}}(0,0)\big)} \le C\frac{u\big(A_1(0,0)\big)}{\hat{u}\big(A_1(0,0)\big)}$$

for all $0 < \tilde{r} \le 1$. Therefore, as $u\big(A_1(0,0)\big) = 1 \sim \hat{u}\big(A_1(0,0)\big)$, we have $u\big(A_{\tilde{r}}(0,0)\big) \sim \hat{u}\big(A_{\tilde{r}}(0,0)\big)$ for all $0 < \tilde{r} \le 1$. Furthermore, by construction, we have by the maximum principle that

$$\tilde{u}^+(Y,s) \le \hat{u}(Y,s)$$

for all $(Y,s) \in S^+(\delta_0)\cap C_1(0,0)$. Applying Lemma 10 once again we can also conclude that

$$\frac{\tilde{u}^+\big(A_{32\delta_0}(0,0)\big)}{\tilde{v}^+\big(A_{32\delta_0}(0,0)\big)} \sim \frac{\tilde{u}^+\big(A_1(0,0)\big)}{\tilde{v}^+\big(A_1(0,0)\big)} \sim \frac{\tilde{u}^+\big(A_1(0,0)\big)}{1-4\delta_0}.$$

This implies that

$$\tilde{u}^+\big(A_{32\delta_0}(0,0)\big) \geq C\delta_0 \frac{\tilde{u}^+\big(A_1(0,0)\big)}{1-4\delta_0} \geq C\delta_0 \tilde{u}^+\big(A_1(0,0)\big).$$

By iteration

$$\tilde{u}^+\big(A_{(32\delta_0)^k}(0,0)\big) \geq (C\delta_0)^k \tilde{u}^+\big(A_1(0,0)\big).$$

Choosing δ_0 so small that $C\delta_0 \geq (32\delta_0)^{1+\varepsilon}$ we can conclude that if

$$\hat{r} \in \big[(32\delta_0)^{k+1}, (32\delta_0)^k\big]$$

then by the Harnack principle

$$\begin{aligned}\tilde{u}^+\big(A_{\hat{r}}(0,0)\big) &\geq C\tilde{u}^+\big(A_{(32\delta_0)^k}(0,0)\big)\\ &\geq C(32\delta_0)^{k(1+\varepsilon)}\tilde{u}^+\big(A_1(0,0)\big) \geq C\hat{r}^{(1+\varepsilon)}\tilde{u}^+\big(A_1(0,0)\big).\end{aligned}$$

As $\tilde{u}^+\big(A_1(0,0)\big) \sim \hat{u}\big(A_1(0,0)\big)$ and as $\tilde{u}^+ \leq \hat{u}$ on $S^+(\delta_0) \cap C_2(0,0)$ we can, by a straightforward comparison argument, conclude that

$$\hat{u}\big(A_{\hat{r}}(0,0)\big) \geq C\hat{r}^{1+\varepsilon}\hat{u}\big(A_1(0,0)\big).$$

As $\hat{u}\big(A_1(0,0)\big) \sim u\big(A_1(0,0)\big) = 1$ we can put all the estimates together and finally conclude that

$$u\big(A_{\hat{r}}(0,0)\big) \geq C\hat{r}^{1+\varepsilon}u\big(A_1(0,0)\big) = C\hat{r}^{1+\varepsilon}.$$

This completes the proof of the lemma. □

3.3. The Green function and caloric measure at infinity. In this section we will clarify the notion of Green function with pole at infinity and the associated caloric measure.

Lemma 14. *Let Ω be a δ_0-Reifenberg flat domain with $\delta_0 > 0$ small. Then there exists a unique function u (unique modulo a constant) such that u is a non-negative solution to the adjoint heat in Ω and such that u vanishes continuously on $\partial\Omega$.*

In fact a similar result holds for the heat equation. The function u, in the statement of the lemma, should be referred to as the Green function with pole at $+$infinity. By $+$ we refer to the 'infinity' in the positive direction of time.

Proof. There are two steps in the proof, the uniqueness and the existence. We start by proving the existence. We let $(X,t) \in \partial\Omega$ and let $R > 0$ be a large positive number. Assume that $(\widehat{X},\hat{t}) \in \Gamma_A^+(X,t,100R)$ and let $K \subset \mathbf{R}^{n+1}$ be a fixed compact set. Assume that R is so large that $K \cap \Omega \subset C_R(X,t)$. Using

Lemma 6, the fact that if $(\widehat{X},\hat{t}) \in \Gamma_A^+(X,t,100R)$ then $G(\widehat{X},\hat{t},\cdot\,)$ satisfies a strong Harnack inequality in $C_R(X,t)\cap\Omega$ and Lemma 10 it follows that if $(Z,\tau)\in K\cap\Omega$, then

$$G(\widehat{X},\hat{t},Z,\tau) \le C_{K,n,A} G\big(\widehat{X},\hat{t},A_1(X,t)\big).$$

In particular this implies that if $(\widehat{X},\hat{t}) \in \Gamma_A^+(X,t,100R)$ then

$$\sup_{(Z,\tau)\in K\cap\Omega} \frac{G(\widehat{X},\hat{t},Z,\tau)}{G\big(\widehat{X},\hat{t},A_1(X,t)\big)} \le C_{K,n,A}.$$

Let $(\widehat{X}_j,\hat{t}_j) \in \Gamma_A^+(X,t,2^jR)$ for $j=1,2,\dots$ and define for $(Z,\tau)\in C_R(X,t)\cap\Omega$

$$u_j(Z,\tau) = \frac{G(\widehat{X}_j,\hat{t}_j,Z,\tau)}{G\big(\widehat{X}_j,\hat{t}_j,A_1(X,t)\big)}.$$

Then $\{u_j\}$ is a set of positive adjoint caloric functions in $C_R(X,t)\cap\Omega$ vanishing on $\partial\Omega$. Furthermore, we can assume that $\{u_j\}$ is a uniformly bounded set of functions on $\Omega\cap C_R(X,t)$. By the Arzela–Ascoli theorem there exists a subsequence $\{\tilde{j}_k\}$ such that $\{u_{\tilde{j}_k}\}$ converges to a non-negative solution $\tilde{u}=\tilde{u}_R$ to the adjoint heat equation in $\Omega\cap C_R(X,t)$. If we choose a sequence of numbers R_i such that $R_i\to\infty$ and pick a diagonal subsequence we can conclude that there exists a subsequence $\{j_k\}$ such that $\{u_{j_k}\}$ converges to a non-negative solution u_∞ to the adjoint heat equation in Ω, uniformly on compact sets of Ω. Furthermore, u_∞ vanishes continuously on $\partial\Omega$ and $u_\infty\big(A_1(X,t)\big)=1$.

Left is to prove uniqueness. Let u and v be two functions fulfilling the statement of the lemma and assume that $u\big(A_1(X,t)\big) = v\big(A_1(X,t)\big)$ for some point $(X,t)\in\partial\Omega$. Under these assumptions we want to prove that $u\equiv v$. Let ϱ and R be fixed numbers such that $0<\varrho\le R/2$. Using the same argument as in the proof of Lemma 8 (see the proof of Lemma 3.11 in [HLN2]) one can prove that u and v satisfy a strong Harnack inequality in $C_R(X,t)\cap\Omega$ with a constant $\tilde{\lambda}$ which only depends on the Reifenberg constant δ_0 and the dimension n. Using Lemma 11 we therefore get that whenever $(Z,\tau),(Y,s)\in C_\varrho(X,t)\cap\Omega$ then

$$\left|\frac{u(Z,\tau)v(Y,s)}{v(Z,\tau)u(Y,s)} - 1\right| \le c\,(\varrho/R)^\gamma.$$

Hence if we put $(Y,s)=A_1(X,t)$ then

$$\left|\frac{u(Z,\tau)}{v(Z,\tau)} - 1\right| \le C(\varrho/R)^\gamma$$

whenever $(Z,\tau)\in C_\varrho(X,t)\cap\Omega$. Letting $R\to\infty$ completes the proof. □

Lemma 15. *Let Ω be a δ_0-Reifenberg flat domain with $\delta_0 > 0$ small. Let $(X,t) \in \partial\Omega$. Then there exists a unique doubling Radon measure ω such that $\omega\big(\Delta(X,t,1)\big) = 1$ and a non-negative solution u to the adjoint heat in Ω vanishing continuously on $\partial\Omega$ such that for all $\phi \in C_0^\infty(\mathbf{R}^{n+1})$*

$$\int_{\partial\Omega} \phi(Y,s)\, d\omega(Y,s) = \int_\Omega u(Y,s)(\Delta - \partial_s)\phi(Y,s)\, dY\, ds.$$

ω is referred to as the caloric measure for Ω at $+$ infinity and normalized at (X,t).

Proof. Again there are two steps in the proof, the uniqueness and the existence. In this case we start by proving the uniqueness. I.e., we assume that there exist two measures ω_1 and ω_2 as in the statement of the lemma and such that $\omega_1\big(\Delta(X,t,1)\big) = \omega_2\big(\Delta(X,t,1)\big) = 1$ for a point $(X,t) \in \partial\Omega$. We want to prove that $\omega_1 \equiv \omega_2$. Let u_1 and u_2 be related to ω_1 respectively ω_2 according to the statement of the lemma. Using Lemma 14 we can conclude that there exist constants α_1 and α_2 as well as a function u such that $u_i = \alpha_i u$. Here u is a non-negative solution to the adjoint heat in Ω such that u vanishes continuously on $\partial\Omega$. I.e., for all $\phi \in C_0^\infty(\mathbf{R}^{n+1})$

$$\int_{\partial\Omega} \phi(Y,s)\, d\omega_i(Y,s) = \alpha_i \int_\Omega u(Y,s)(\Delta - \partial_s)\phi(Y,s)\, dY\, ds.$$

From this we can we conclude that

$$\alpha_1^{-1} \int_{\partial\Omega} \phi(Y,s)\, d\omega_1(Y,s) = \alpha_2^{-1} \int_{\partial\Omega} \phi(Y,s)\, d\omega_2(Y,s).$$

Choosing ϕ as the indicator function of $\Delta(X,t,1)$ and using the normalization of ω_1 and ω_2 we get that $\alpha_1 = \alpha_2$. Therefore $u_1 \equiv u_2$ and $\omega_1 \equiv \omega_2$.

To prove the existence we argue as in the proof of Lemma 14. We let $(X,t) \in \partial\Omega$ and define $R > 0$ to be a large positive number. Let $(\widehat{X}_j, \hat{t}_j) \in \Gamma_A^+(X,t,2^jR)$ for $j = 1,2,\dots$ and define for $(Z,\tau) \in C_R(X,t) \cap \Omega$

$$u_j(Z,\tau) = \frac{G(\widehat{X}_j, \hat{t}_j, Z, \tau)}{G\big(\widehat{X}_j, \hat{t}_j, A_1(X,t)\big)}.$$

Let $\phi \in C_0^\infty\big(C_R(X,t)\big)$ and let as usual $\omega(\widehat{X}_j, \hat{t}_j, \cdot\,)$ be the caloric measure defined with respect to $(\widehat{X}_j, \hat{t}_j)$. Then

$$\int_{\partial\Omega} \phi(Z,\tau) G\big(\widehat{X}_j, \hat{t}_j, A_1(X,t)\big)^{-1}\, d\omega(\widehat{X}_j, \hat{t}_j, Z, \tau)$$
$$= \int_\Omega u_j(Z,\tau)(\Delta - \partial_\tau)\phi(Z,\tau)\, dZ\, d\tau.$$

Defining measures

$$d\mu_j(Z,\tau) = G\big(\widehat{X}_j,\hat{t}_j,A_1(X,t)\big)^{-1}\,d\omega(\widehat{X}_j,\hat{t}_j,Z,\tau)$$

we can conclude that

$$\int_{\partial\Omega}\phi(Z,\tau)\,d\mu_j(Z,\tau) = \int_{\Omega} u_j(Z,\tau)(\Delta-\partial_\tau)\phi(Z,\tau)\,dZ\,d\tau$$

for all $\phi\in C_0^\infty\big(C_R(X,t)\big)$. Using Lemma 7 and the fact that $G(\widehat{X}_j,\hat{t}_j,\cdot\,)$ satisfies a strong Harnack inequality in $C_R(X,t)\cap\Omega$ we have that

$$\begin{aligned}\mu_j\big(\Delta(X,t,R)\big) &= \frac{\omega\big(\widehat{X}_j,\hat{t}_j,\Delta(X,t,R)\big)}{G\big(\widehat{X}_j,\hat{t}_j,A_1(X,t)\big)} \sim \frac{R^n G\big(\widehat{X}_j,\hat{t}_j,A_R(X,t)\big)}{G\big(\widehat{X}_j,\hat{t}_j,A_1(X,t)\big)}\\ &\sim R^n u_j(Z,\tau)\end{aligned}$$

for all $(Z,\tau)\in C_R(X,t)\cap\Omega$. As in the proof of Lemma 14, $\{u_j\}$ is a uniformly bounded set of functions on $\Omega\cap C_R(X,t)$. Hence the sequence $\{\mu_j\}$ is a uniformly bounded set of measures on $C_R(X,t)\cap\partial\Omega$ and therefore there exists a subsequence $\{\mu_{\tilde{j}_k}\}$ and a Radon measure μ such that,

$$\int_{\partial\Omega}\phi(Z,\tau)\,d\mu_{j_k}(Z,\tau)\to\int_{\partial\Omega}\phi(Z,\tau)\,d\mu(Z,\tau),$$

for all $\phi\in C_0^\infty\big(C_R(X,t)\big)$ as $k\to\infty$. If we again choose a sequence of numbers R_i such that $R_i\to\infty$ and pick a diagonal subsequence we can therefore conclude that there exists a subsequence $\{j_k\}$ such that $\{\mu_{j_k}\}$ converges to a Radon measure μ such that for all $\phi\in C_0^\infty(\mathbf{R}^{n+1})$,

$$\int_{\partial\Omega}\phi(Z,\tau)\,d\mu_{j_k}(Z,\tau)\to\int_{\partial\Omega}\phi(Z,\tau)\,d\mu(Z,\tau),$$

as $k\to\infty$. Repeating the argument of Lemma 14 we can also conclude that $\{u_{j_k}\}$ converges, uniformly on compacts subsets, to a non-negative solution u_∞ and for all $\phi\in C_0^\infty(\mathbf{R}^{n+1})$

$$\int_{\partial\Omega}\phi(Z,\tau)\,d\mu(Z,\tau) = \int_{\Omega} u_\infty(Z,\tau)(\Delta-\partial_\tau)\phi\,dZd\tau.$$

Define $\omega_\infty = \mu/\mu\big(\Delta(X,t,1)\big)$ and $u_\infty = u_\infty/\mu\big(\Delta(X,t,1)\big)$. Then for all $\phi\in C_0^\infty(\mathbf{R}^{n+1})$

$$\int_{\partial\Omega}\phi(Z,\tau)\,d\omega_\infty(Z,\tau) = \int_{\Omega} u_\infty(Z,\tau)(\Delta-\partial_\tau)\phi\,dZ\,d\tau.$$

This completes the existence part of the proof. Left is to prove that ω_∞ is a doubling measure. But if $(\widetilde{X},\tilde{t})\in\partial\Omega$ and $r>0$ it follows from Lemma 9 that

$$\begin{aligned}\omega_\infty\big(\Delta(\widetilde{X},\tilde{t},2r)\big)&\le\liminf_{j_k\to\infty}\frac{\omega\big(\widehat{X}_{j_k},\hat{t}_{j_k},\Delta(\widetilde{X},\tilde{t},2r)\big)}{\mu\big(\Delta(X,t,1)\big)G\big(\widehat{X}_{j_k},\hat{t}_{j_k},A_1(X,t)\big)}\\&\le C\liminf_{j_k\to\infty}\frac{\omega\big(\widehat{X}_{j_k},\hat{t}_{j_k},\Delta(\widetilde{X},\tilde{t},r/2)\big)}{\mu\big(\Delta(X,t,1)\big)G\big(\widehat{X}_{j_k},\hat{t}_{j_k},A_1(X,t)\big)}\\&\le C\omega_\infty\big(\Delta(\widetilde{X},\tilde{t},r)\big).\end{aligned}$$

This completes the proof. □

4. A blow-up argument and the classification of global solutions

In the following we let Ω be δ_0-Reifenberg flat. Let $(X_j,t_j)\in\partial\Omega\to(\widetilde{X},\tilde{t})\in\partial\Omega$ and assume that $(\widetilde{X},\tilde{t})=(0,0)$. For a sequence $\{r_j\}$ of real numbers tending to zero we define,

$$\Omega_j=\big\{\big(r_j^{-1}(X-X_j),r_j^{-2}(t-t_j)\big):(X,t)\in\Omega\big\}.$$

This section is devoted to the analysis of these blow-ups by making use of our assumption on the kernel $k(\widehat{X}^1,\hat{t}^1,\widehat{X}^2,\hat{t}^2,\cdot)$ and therefore we will also assume that $t_j<\hat{t}^2<\hat{t}^1$ for all j.

4.1. Blow-ups. Recall that the parabolic distance between the two sets F_1,F_2 is defined as

$$d(F_1,F_2)=\inf\{|X-Y|+|s-t|^{1/2}:(X,t)\in F_1,\ (Y,s)\in F_2\}.$$

Based on this we introduce the parabolic Hausdorff distance between two sets F_1, F_2 as

$$D(F_1,F_2)=\sup\{d(x,F_2):x\in F_1\}+\sup\{d(F_1,y):y\in F_2\}.$$

In the following we will consider uniform Hausdorff convergence (in the metric induced by the parabolic Hausdorff distance) on compact sets. To define this properly we consider a sequence of closed sets $\{F_j\}_j$, $F_j\subset\mathbf{R}^{n+1}$. We say that F_j converges to a closed set $F\subset\mathbf{R}^{n+1}$ in the parabolic Hausdorff distance sense, uniformly on compact subsets of $\mathbf{R}^{n+1}$, if for any compact set $K\subset\mathbf{R}^{n+1}$ and any $\varepsilon>0$ there exists $j_0\ge1$ so that if $j\ge j_0$ then

$$D(F_j\cap K,F\cap K)<\varepsilon.$$

Furthermore a sequence of open sets $\{E_j\}_j$, $E_j\subset\mathbf{R}^{n+1}$, is said to converge to an open set $E\subset\mathbf{R}^{n+1}$ in the parabolic Hausdorff distance sense uniformly on compact subsets of $\mathbf{R}^{n+1}$ if $\mathbf{R}^{n+1}\setminus E_j \to \mathbf{R}^{n+1}\setminus E$ in the parabolic Hausdorff distance sense uniformly on compact subsets of $\mathbf{R}^{n+1}$.

Let $\omega(\widehat{X},\hat{t},\cdot)$ and $G(\widehat{X},\hat{t},\cdot,\cdot)$ be the caloric measure and Green function defined with Ω as domain of reference. For arbitrary Borel sets $E\subset \mathbf{R}^{n+1}$ we define,

$$\omega_j(E)=\frac{\omega\bigl(\widehat{X},\hat{t},\bigl\{(Z,\tau)\in\partial\Omega:\bigl((Z-X_j)/r_j,(\tau-t_j)/r_j^2\bigr)\in E\bigr\}\bigr)}{\omega\bigl(\widehat{X},\hat{t},\Delta(X_j,t_j,r_j)\bigr)}.$$

We furthermore define

$$u_j(Z,\tau)=\frac{G(\widehat{X},\hat{t},X_j+r_jZ,t_j+r_j^2\tau)}{\omega\bigl(\widehat{X},\hat{t},\Delta(X_j,t_j,r_j)\bigr)}r_j^n$$

whenever $(Z,\tau)\in\Omega_j$. Then u_j is adjoint caloric in Ω_j outside of its pole and it is zero on $\partial\Omega_j$.

We will start by proving the following two lemmas.

Lemma 16. *Let Ω be δ_0-Reifenberg flat domain with $\delta_0>0$ small. Let $(X_j,t_j)\in\partial\Omega\to(\widetilde{X},\tilde{t})\in\partial\Omega$ and assume that $(\widetilde{X},\tilde{t})=(0,0)$. For a sequence $\{r_j\}$ of real numbers tending to zero we define,*

$$\Omega_j=\bigl\{\bigl(r_j^{-1}(X-X_j),r_j^{-2}(t-t_j)\bigr):(X,t)\in\Omega\bigr\}.$$

Then $\Omega_j\to\Omega_\infty$ and $\partial\Omega_j\to\partial\Omega_\infty$ in the parabolic Hausdorff distance sense uniformly on compact subsets of $\mathbf{R}^{n+1}$ as $j\to\infty$. Furthermore, Ω_∞ is a $4\delta_0$-Reifenberg flat domain.

Lemma 17. *Let Ω_j and Ω_∞ be as in Lemma 16. Then $u_j\to u_\infty$ uniformly on compact subsets, u_∞ is a positive adjoint caloric function in Ω_∞ and $u_\infty=0$ on $\partial\Omega_\infty$. Moreover $\omega_j\to\omega_\infty$ weakly as Radon measures and for all $\phi\in C_0^\infty(\mathbf{R}^{n+1})$*

$$\int_{\partial\Omega}\phi(Y,s)\,d\omega_\infty(Y,s)=\int_\Omega u_\infty(Y,s)(\Delta-\partial_s)\phi(Y,s)\,dY\,ds.$$

ω_∞ is the caloric measure of Ω_∞ at infinity.

We start by proving Lemma 16.

Proof. Note that for each $j\ge 1$, $(0,0)\in\partial\Omega_j$ and $C_1(0,0)\cap\Omega_j\ne\emptyset$. Using this we can conclude that given a compact set $K\subset\mathbf{R}^{n+1}$ there exists a subsequence $\{\tilde{j}_m\}_m$ such that $K\cap\partial\Omega_{\tilde{j}_m}$ and $K\cap\Omega_{\tilde{j}_m}$ converge in the parabolic Hausdorff distance sense. We can therefore exhaust $\mathbf{R}^{n+1}$ by a sequence of compact sets in order to ensure that there exists a subsequence $\{j_m\}_m$ such that $\partial\Omega_{j_m}$

and Ω_{j_m} converge in the parabolic Hausdorff distance sense uniformly on compact sets. Hence by an appropriate relabeling we can conclude that as $j \to \infty$, $\Omega_j \to \Omega_\infty$, $\partial\Omega_j \to \Sigma_\infty$ in the parabolic Hausdorff distance sense uniformly on compact subsets of $\mathbf{R}^{n+1}$. In analogy with the proof of Theorem 4.1 in [KT2] we want to prove that $\partial\Omega_\infty = \Sigma_\infty$ and that $\partial\Omega_\infty$ is $4\delta_0$-Reifenberg flat.

Since ω is a doubling measure we have that for any compact set $K \subset \mathbf{R}^{n+1}$, $\sup_{j\ge1}\omega_j(K) \le C_K$. Hence arguing as in the proof of Lemma 15 there exists a subsequence (which we relabel) such that $\omega_j \to \omega_\infty$ in the sense that

$$\int \phi\, d\omega_j \to \int \phi\, d\omega_\infty$$

for all $\phi \in C_0^\infty(\mathbf{R}^{n+1})$. We start by proving that the support of ω_∞ equals Σ_∞. To do this we let $(\widehat{Z},\hat\tau) \in \Sigma_\infty$. By construction there exists a sequence (Z_j,τ_j) such that $(Z_j,\tau_j) \in \partial\Omega$ and

$$(\widetilde{Z}_j,\tilde\tau_j) := \big((Z_j - X_j)/r_j, (\tau_j - t_j)/r_j^2\big) \to (\widehat{Z},\hat\tau).$$

Furthermore for every $r \in (0,1)$ there exists $j_0 \ge 1$ such that for $j \ge j_0$, $d\big((\widetilde{Z}_j,\tilde\tau_j),(\widehat{Z},\hat\tau)\big) < r/2$ and $d\big((Z_j,\tau_j),(X_j,t_j)\big) < Cr_j$ for some large $C = C\big((\widehat{Z},\hat\tau)\big)$. Hence we have that

$$\begin{aligned}\omega_j\big(C_r(\widehat{Z},\hat\tau)\big) &= \frac{\omega\Big(\widehat{X},\hat t,\big\{(Z,\tau)\in\partial\Omega : \big((Z-X_j)/r_j,(\tau-t_j)/r_j^2\big)\in C_r(\widehat{Z},\hat\tau)\big\}\Big)}{\omega\big(\widehat{X},\hat t,\Delta(X_j,t_j,r_j)\big)}\\ &\ge \frac{\omega\big(\widehat{X},\hat t, C_{rr_j/2}(Z_j,\tau_j)\big)}{\omega\big(\widehat{X},\hat t,\Delta(X_j,t_j,r_j)\big)} \ge \frac{\omega\big(\widehat{X},\hat t, C_{r_j/2}(Z_j,\tau_j)\big)}{\omega\big(\widehat{X},\hat t, C_{2Cr_j}(Z_j,\tau_j)\big)} \ge \widetilde{C}(r,C)\end{aligned}$$

as ω is a doubling measure. Hence this implies that if $(\widehat{Z},\hat\tau) \in \Sigma_\infty$ then $(\widehat{Z},\hat\tau)$ is in the support of ω_∞. Left is to prove the other inclusion. I.e., in this case we start by assuming that $(\widehat{Z},\hat\tau)$ is in the support of ω_∞. We want to prove that there exists $(Z_j,\tau_j) \in \partial\Omega$ such that

$$(\widetilde{Z}_j,\tilde\tau_j) := \big((Z_j - X_j)/r_j, (\tau_j - t_j)/r_j^2\big) \to (\widehat{Z},\hat\tau).$$

If this is not the case then there exists $\varepsilon > 0$ and j_0 such that for any sequence $(Z_j,\tau_j) \in \partial\Omega$ as above $d\big((\widetilde{Z}_j,\tilde\tau_j),(\widehat{Z},\hat\tau)\big) \ge \varepsilon$ if $j \ge j_0$. In particular in this case $C_{\varepsilon/2}(\widehat{Z},\hat\tau) \cap \partial\Omega_j = \emptyset$. If we take $\phi \in C_0^\infty\big(C_{\varepsilon/2}(\widehat{Z},\hat\tau)\big)$ we then get as $\omega_j \to \omega_\infty$ that

$$0 = \int \phi(Y,s)\, d\omega_j(Y,s) \to \int \phi(Y,s)\, d\omega_\infty(Y,s).$$

I.e.,

$$\int \phi(Y,s)\, d\omega_\infty(Y,s) = 0$$

for all $\phi \in C_0^\infty\big(C_{\varepsilon/2}(\widehat{Z},\hat{\tau})\big)$. This contradicts the assumption that $(\widehat{Z},\hat{\tau})$ is in the support of ω_∞. I.e., $(\widehat{Z},\hat{\tau}) \in \Sigma_\infty$ and we can conclude that the support of ω_∞ coincides with Σ_∞.

We now prove that $\partial\Omega_\infty \subset \Sigma_\infty$. To do this we let $(Z,\tau) \in \partial\Omega_\infty = \overline{\Omega_\infty} \cap \overline{\mathbf{R}^{n+1} \setminus \Omega_\infty}$ and note that given $\varepsilon > 0$, there exist $(Y,s) \in \Omega_\infty \cap C_\varepsilon(Z,\tau)$ and $(\widehat{Y},\hat{s}) \in [\mathbf{R}^{n+1} \setminus \overline{\Omega_\infty}] \cap C_\varepsilon(Z,\tau)$. There also exist sequences of points $(Y_j,s_j) \in \Omega$ and $(\widehat{Y}_j,\hat{s}_j) \in [\mathbf{R}^{n+1} \setminus \overline{\Omega}]$ such that

$$(Y,s) = \big((Y_j - X_j)/r_j, (s_j - t_j)/r_j^2\big), \quad (\widehat{Y},\hat{s}) = \big((\widehat{Y}_j - X_j)/r_j, (\hat{s}_j - t_j)/r_j^2\big).$$

Let l_j be the parabolic line connecting (Y_j,s_j) and $(\widehat{Y}_j,\hat{s}_j)$ and pick $(Z_j,\tau_j) \in l_j \cap \partial\Omega$. As $\partial\Omega$ separates $\mathbf{R}^{n+1}$ at least one such point exists. As $\{(Z_j - X_j)/r_j\}_j$ as well as $\{(\tau_j - t_j)/r_j^2\}_j$ are bounded sequences there exists a subsequence (which we relabel) such that

$$\big((Z_j - X_j)/r_j, (\tau_j - t_j)/r_j^2\big) \to (\widehat{Z},\hat{\tau}) \in \Sigma_\infty.$$

Furthermore as

$$d\,\Big(\big((Z_j - X_j)/r_j, (\tau_j - t_j)/r_j^2\big), \big((Y_j - X_j)/r_j, (s_j - t_j)/r_j^2\big)\Big)$$

can be bounded by $r_j^{-1} d\big((Y_j,s_j), (\widehat{Y}_j,\hat{s}_j)\big)$, we can conclude, by letting $j \to \infty$, that

$$d\big((Y,s),(\widehat{Z},\hat{\tau})\big) \le d\big((Y,s),(\widehat{Y},\hat{s})\big) \le C\varepsilon$$

for a universal constant C. By the same line of thought

$$\begin{aligned} d\big((Z,\tau),(\widehat{Z},\hat{\tau})\big) &\le d\big((Z,\tau),(\widehat{Y},\hat{s})\big) + d\big((\widehat{Z},\hat{\tau}),(\widehat{Y},\hat{s})\big) \\ &\le d\big((Z,\tau),(\widehat{Y},\hat{s})\big) + d\big((Y,s),(\widehat{Y},\hat{s})\big) \le \widetilde{C}\varepsilon. \end{aligned}$$

In total we have proved that for any $(Z,\tau) \in \partial\Omega_\infty$ and for any $\varepsilon > 0$ there exists $(\widehat{Z},\hat{\tau}) \in \Sigma_\infty$ such that $d\big((Z,\tau),(\widehat{Z},\hat{\tau})\big) \le \varepsilon$. This argument proves that (Z,τ) is in the closure of the set Σ_∞. But the closure of the set Σ_∞ equals, as we have proven above, the closure of the support of ω_∞. The latter equals the support of ω_∞ as the support is closed. Based on this we can conclude that $(Z,\tau) \in \Sigma_\infty$ and that $\partial\Omega_\infty \subset \Sigma_\infty$.

Left is to prove that $\Sigma_\infty \subset \partial\Omega_\infty$. We let $(\widehat{Z},\hat{\tau}) \in \Sigma_\infty$. By construction there exists a sequence (Z_j,τ_j) such that $(Z_j,\tau_j) \in \partial\Omega$ and

$$(\widetilde{Z}_j,\tilde{\tau}_j) := \big((Z_j - X_j)/r_j, (\tau_j - t_j)/r_j^2\big) \to (\widehat{Z},\hat{\tau}).$$

In order to argue as in the proof of Theorem 4.1 in [KT2] we start by fixing M and by considering arbitrary $\varrho > 0$. Let $\hat{n}(Z_j, \tau_j, \varrho r_j)$ be the normal of the plane through (Z_j, τ_j), associated with the scale ϱr_j, which appears in the definition of Reifenberg flatness in Definition 1. We define

$$\begin{aligned} A_{\varrho r_j}(Z_j, \tau_j) &= \big(Z_j + \varrho r_j \hat{n}(Z_j, \tau_j, \varrho r_j), \tau_j\big), \\ \tilde{A}_{\varrho r_j}(Z_j, \tau_j) &= \big(Z_j - \varrho r_j \hat{n}(Z_j, \tau_j, \varrho r_j), \tau_j\big) \end{aligned}$$

where we assume that $A_{\varrho r_j}(Z_j, \tau_j) \in \Omega$ and $\tilde{A}_{\varrho r_j}(Z_j, \tau_j) \in \mathbf{R}^{n+1} \setminus \overline{\Omega}$. Choosing M large enough we can conclude that

$$\begin{aligned} &C_{\varrho r_j/M}\big(A_{\varrho r_j}(Z_j, \tau_j)\big) \subset \Omega, &&d\big(A_{\varrho r_j}(Z_j, \tau_j), (Z_j, \tau_j)\big) \le \varrho r_j, \\ &C_{\varrho r_j/M}\big(\tilde{A}_{\varrho r_j}(Z_j, \tau_j)\big) \subset \mathbf{R}^{n+1} \setminus \overline{\Omega}, &&d\big(\tilde{A}_{\varrho r_j}(Z_j, \tau_j), (Z_j, \tau_j)\big) \le \varrho r_j. \end{aligned}$$

We also define

$$\begin{aligned} A_j(\varrho) &= \big((Z_j + \varrho r_j \hat{n}(Z_j, \tau_j, \varrho r_j) - X_j) r_j^{-1}, (\tau_j - t_j) r_j^{-2}\big), \\ \tilde{A}_j(\varrho) &= \big((Z_j - \varrho r_j \hat{n}(Z_j, \tau_j, \varrho r_j) - X_j) r_j^{-1}, (\tau_j - t_j) r_j^{-2}\big). \end{aligned}$$

Then

$$\begin{aligned} &C_{\varrho/M}\big(A_j(\varrho)\big) \subset \Omega_j, &&d\big(A_j(\varrho), (\widetilde{Z}_j, \tilde{\tau}_j)\big) \le \varrho, \\ &C_{\varrho/M}\big(\tilde{A}_j(\varrho)\big) \subset \mathbf{R}^{n+1} \setminus \overline{\Omega}_j, &&d\big(\tilde{A}_j(\varrho), (\widetilde{Z}_j, \tilde{\tau}_j)\big) \le \varrho. \end{aligned}$$

Going to the limit we can conclude that, for every $\varrho > 0$, there exist points $A_\infty(\varrho) \in \Omega_\infty$ and $\tilde{A}_\infty(\varrho) \in \mathbf{R}^{n+1} \setminus \overline{\Omega}_\infty$ such that

$$\begin{aligned} &C_{\varrho/M}\big(A_\infty(\varrho)\big) \subset \Omega_\infty, &&d\big(A_\infty(\varrho), (\widehat{Z}, \hat{\tau})\big) \le \varrho, \\ &C_{\varrho/M}\big(\tilde{A}_\infty(\varrho)\big) \subset \mathbf{R}^{n+1} \setminus \overline{\Omega}_\infty, &&d\big(\tilde{A}_\infty(\varrho), (\widehat{Z}, \hat{\tau})\big) \le \varrho. \end{aligned}$$

If we let $\varrho \to 0$ we can therefore conclude that $(\widehat{Z}, \hat{\tau}) \in \partial\Omega_\infty$ and hence that $\Sigma_\infty \subset \partial\Omega_\infty$. In total we have proven that $\Sigma_\infty = \partial\Omega_\infty$.

Left is to estimate the Reifenberg constant of $\partial\Omega_\infty$. To do this we again let $(\widehat{Z}, \hat{\tau}) \in \partial\Omega_\infty$ and consider $r > 0$. By construction there exists a sequence (Z_j, τ_j) such that $(Z_j, \tau_j) \in \partial\Omega$ and

$$(\widetilde{Z}_j, \tilde{\tau}_j) := \big((Z_j - X_j)/r_j, (\tau_j - t_j)/r_j^2\big) \to (\widehat{Z}, \hat{\tau}).$$

Let $\varepsilon > 0$. As $\partial\Omega$ is δ_0-Reifenberg flat there exists, for each j, an n-dimensional plane $\widehat{P}_j = \widehat{P}(Z_j, \tau_j, r_j r)$, containing (Z_j, τ_j) and a line parallel to the t axis, having unit normal $\hat{n}_j = \hat{n}(Z_j, \tau_j, r_j r)$ such that

$$D\big(\partial\Omega \cap C_{r_j r}(Z_j, \tau_j), \widehat{P}_j \cap C_{r_j r}(Z_j, \tau_j)\big) \le r_j r(\delta_0 + \varepsilon).$$

Also as $\partial\Omega_j \to \partial\Omega_\infty$ in the parabolic Hausdorff distance sense there exists $j_0 \geq 1$ such that if $j \geq j_0$ then

$$D\big(\partial\Omega_j \cap C_r(\widehat{Z},\hat{\tau}), \partial\Omega_\infty \cap C_r(\widehat{Z},\hat{\tau})\big) < \varepsilon r, \quad d\big((\widetilde{Z}_j,\tilde{\tau}_j),(\widehat{Z},\hat{\tau})\big) < \varepsilon r.$$

We define a new plane $\Lambda_j := \widehat{P}(Z_j,\tau_j,r_j r) - (Z_j,\tau_j) + (\widehat{Z},\hat{\tau})$. Note that this is a plane containing $(\widehat{Z},\hat{\tau})$. Then,

$$\begin{aligned} D\big(\partial\Omega_\infty \cap C_r(\widehat{Z},\hat{\tau}), \Lambda_j \cap C_r(\widehat{Z},\hat{\tau})\big) &\leq D\big(\partial\Omega_\infty \cap C_r(\widehat{Z},\hat{\tau}), \partial\Omega_j \cap C_r(\widehat{Z},\hat{\tau})\big) \\ &\quad + D\big(\partial\Omega_j \cap C_r(\widehat{Z},\hat{\tau}), \Lambda_j \cap C_r(\widehat{Z},\hat{\tau})\big). \end{aligned}$$

Left is therefore to estimate $D\big(\partial\Omega_j \cap C_r(\widehat{Z},\hat{\tau}), \Lambda_j \cap C_r(\widehat{Z},\hat{\tau})\big)$. As

$$d\big((\widetilde{Z}_j,\tilde{\tau}_j),(\widehat{Z},\hat{\tau})\big) < \varepsilon r,$$

we have

$$\partial\Omega_j \cap C_r(\widehat{Z},\hat{\tau}) \subset \partial\Omega_j \cap C_{r(1+\varepsilon)}(\widetilde{Z}_j,\tilde{\tau}_j).$$

But by construction of the plane Λ_j we get

$$\begin{aligned} D\big(\partial\Omega_j \cap C_{r(1+\varepsilon)}(\widetilde{Z}_j,\tilde{\tau}_j), \widehat{P}(\widetilde{Z}_j,\tilde{\tau}_j,r) \cap C_{r(1+\varepsilon)}(\widetilde{Z}_j,\tilde{\tau}_j)\big) &< r(1+\varepsilon)(2\delta_0+\varepsilon) \\ &< 4\delta_0 r + 2\varepsilon r. \end{aligned}$$

Summing up we can conclude that

$$D\big(\partial\Omega_\infty \cap C_r(\widehat{Z},\hat{\tau}), \Lambda_j \cap C_r(\widehat{Z},\hat{\tau})\big) \leq 4\delta_0 r + 2\varepsilon r.$$

As ε is arbitrary this completes the proof. ▫

To continue we proceed with the proof of Lemma 17.

Proof. Recall that from the argument of the proof of Lemma 16 it follows that $\omega_j \to \omega_\infty$ and that the support of ω_∞ coincides with $\partial\Omega_\infty$. Let $\phi \in C_0^\infty\big(\mathbf{R}^{n+1} \setminus (\widehat{X},\hat{t})\big)$ and define $\phi_j(Y,s) = \phi\big(r_j^{-1}(Y-X_j), r_j^{-2}(s-t_j)\big)$. By the Riesz representation formula we have

$$\int_{\partial\Omega} \phi_j(Y,s)\, d\omega(\widehat{X},\hat{t},Y,s) = \int_\Omega G(\widehat{X},\hat{t},Z,\tau)(\Delta\phi_j - \partial_\tau\phi_j)\, dZ\, d\tau.$$

If we let $(\widehat{X}_j,\hat{t}_j) = \big(r_j^{-1}(\widehat{X}-X_j), r_j^{-2}(\hat{t}-t_j)\big)$, then by a change of variables,

$$\int_{\partial\Omega_j} \phi(Y,s)\, d\omega_j(\widehat{X}_j,\hat{t}_j,Y,s) = \int_{\Omega_j} u_j(Z,\tau)(\Delta\phi - \partial_\tau\phi)\, dZ\, d\tau$$

where ω_j and u_j were introduced above the statement of the lemma. Defining $u_j \equiv 0$ on the complement of Ω_j we can conclude, using the same argument as in the proof of Lemma 14, that $\{u_j\}$ is a uniformly bounded sequence on compacts. By the Arzela–Ascoli theorem, $u_j \to u_\infty$ uniformly on compact subsets, and u_∞ is a positive adjoint caloric function in Ω_∞ such that $u_\infty = 0$ on $\partial\Omega_\infty$. By weak convergence we can therefore conclude that

$$\int_{\partial\Omega_\infty} \phi(Y,s)\, d\omega_\infty(Y,s) = \int_{\Omega_\infty} u_\infty(Y,s)(\Delta - \partial_s)\phi(Y,s)\, dY\, ds$$

for all $\phi \in C_0^\infty(\mathbf{R}^{n+1})$. □

Finally we will now explore the information contained in the condition

$$\log k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot) \in \mathrm{VMO}\big(d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot)\big).$$

Lemma 18. *Let Ω be a δ_0-Reifenberg flat domain with $\delta_0 > 0$ small and define $\Omega^1 = \Omega$ and $\Omega^2 = \mathbf{R}^{n+1} \setminus \overline{\Omega}$. We also let $(\widehat{X}^i, \hat{t}^i) \in \Omega^i$, for $i \in \{1,2\}$, and define $\omega^i(\widehat{X}^i, \hat{t}^i, \cdot)$ to be the caloric measure defined with respect to Ω^i. Assume that $\omega^2(\widehat{X}^2, \hat{t}^2, \cdot)$ is absolutely continuous with respect to $\omega^1(\widehat{X}^1, \hat{t}^1, \cdot)$ on $\partial\Omega$ and that the Radon–Nikodym derivative $k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot) = d\omega^2(\widehat{X}^2, \hat{t}^2, \cdot)/d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot)$ is such that $\log k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, \cdot) \in \mathrm{VMO}\big(d\omega^1(\widehat{X}^1, \hat{t}^1, \cdot)\big)$. If, using the notation of Lemma 17, $\omega_j^i \to \omega_\infty^i$ for $i \in \{1,2\}$ then*

$$\omega_\infty^1 \equiv \omega_\infty^2.$$

Proof. To prove the lemma we prove that for any $\phi \in C_0^\infty(\mathbf{R}^{n+1})$, $\phi \ge 0$, we have

$$\int_{\partial\Omega_\infty} \phi\, d\omega_\infty^1 = \int_{\partial\Omega_\infty} \phi\, d\omega_\infty^2.$$

In the following we write $k(Y,s) = k(\widehat{X}^1, \hat{t}^1, \widehat{X}^2, \hat{t}^2, Y, s)$, $\omega^1(E) = \omega^1(\widehat{X}^1, \hat{t}^1, E)$ and $\omega^2(E) = \omega^2(\widehat{X}^2, \hat{t}^2, E)$. Applying Lemma 12 there exists $\alpha \in (0,1)$ and a constant $C = C(n, \delta_0, A)$ such that for all $(X,t) \in \partial\Omega$, $r < r_0$, $|X - \widehat{X}^i|^2 \le A(\hat{t}^i - t)$ for some $A \ge 2$ and for $i \in \{1,2\}$, $\min\{\hat{t}^1, \hat{t}^2\} - t \ge 8r^2$ and $E \subset \Delta(X,t,r)$,

$$\frac{\omega^2(E)}{\omega^2\big(\Delta(X,t,r)\big)} \le C\left(\frac{\omega^1(E)}{\omega^1\big(\Delta(X,t,r)\big)}\right)^\alpha.$$

Let $\phi \in C_0^\infty(\mathbf{R}^{n+1})$ and recall that in our blow-up argument we considered a sequence of points $(X_j, t_j) \in \partial\Omega \to (\widetilde{X}, \tilde{t}) \in \partial\Omega$ and a sequence of scales $\{r_j\}$, $r_j \to 0$, and we assumed for simplicity that $(\widetilde{X}, \tilde{t}) = (0,0)$. Let $\Delta_j = \Delta(X_j, t_j, r_j)$. In the following we will assume that supp $\phi \subset C_M(0,0)$ for some

$M > 1$ and that $\phi \geq 0$. Let $\varepsilon > 0$ be given. As $\log k(\,\cdot\,,\cdot\,) \in \mathrm{VMO}(d\omega^1)$ there exists, by the John–Nirenberg inequality, j_0 such that for $j \geq j_0$, there exists $G_j \subset \Delta(X_j, t_j, Mr_j) := \tilde{\Delta}_j$, $\omega^1(\tilde{\Delta}_j) \leq (1+\varepsilon)\omega^1(G_j)$ and such that for every $(Y,s) \in G_j$

$$(1-\varepsilon)\frac{1}{\omega^1(\tilde{\Delta}_j)}\int_{\tilde{\Delta}_j} k\, d\omega^1 \leq k(Y,s) \leq (1+\varepsilon)\frac{1}{\omega^1(\tilde{\Delta}_j)}\int_{\tilde{\Delta}_j} k\, d\omega^1.$$

Using this inequality we have,

$$(1-\varepsilon)\omega^1(\Delta_j \cap G_j) \leq \frac{\omega^2(\Delta_j \cap G_j)\omega^1(\tilde{\Delta}_j)}{\omega^2(\tilde{\Delta}_j)} \leq (1+\varepsilon)\omega^1(\Delta_j \cap G_j).$$

In the following C_M will denote constants which depend on M and other parameters but are independent of j. Using these inequalities and the consequence of the VMO condition stated above, the constant A can be chosen uniformly and independent of j as $r_j \to 0$ and as the sequence (X_j, t_j) converges to a point located below $(\widehat{X}^1, \hat{t}^1)$ as well as $(\widehat{X}^2, \hat{t}^2)$, we have

$$\begin{aligned}
\frac{\omega^2(\Delta_j)}{\omega^1(\Delta_j)} &= \frac{\omega^2(\Delta_j \cap G_j)}{\omega^1(\Delta_j)} + \frac{\omega^2(\Delta_j \setminus G_j)}{\omega^1(\Delta_j)} \\
&\leq (1+\varepsilon)\frac{\omega^1(\Delta_j \cap G_j)}{\omega^1(\Delta_j)}\frac{\omega^2(\tilde{\Delta}_j)}{\omega^1(\tilde{\Delta}_j)} + \frac{\omega^2(\Delta_j \setminus G_j)}{\omega^1(\Delta_j)} \\
&\leq (1+\varepsilon)\frac{\omega^2(\tilde{\Delta}_j)}{\omega^1(\tilde{\Delta}_j)} + \frac{\omega^2(\Delta_j)}{\omega^1(\Delta_j)}\left(\frac{\omega^1(\Delta_j \setminus G_j)}{\omega^1(\Delta_j)}\right)^{\alpha} \\
&\leq (1+\varepsilon)\frac{\omega^2(\tilde{\Delta}_j)}{\omega^1(\tilde{\Delta}_j)} + C_M\varepsilon^{\alpha}\frac{\omega^2(\Delta_j)}{\omega^1(\Delta_j)}.
\end{aligned}$$

By similar deductions

$$\begin{aligned}
\frac{\omega^2(\Delta_j)}{\omega^1(\Delta_j)} &\geq \frac{\omega^2(\Delta_j \cap G_j)}{\omega^1(\Delta_j)} \geq (1-\varepsilon)\frac{\omega^1(\Delta_j \cap G_j)}{\omega^1(\Delta_j)}\frac{\omega^2(\tilde{\Delta}_j)}{\omega^1(\tilde{\Delta}_j)} \\
&= (1-\varepsilon)\left(1 - \frac{\omega^1(\Delta_j \setminus G_j)}{\omega^1(\Delta_j)}\right)\frac{\omega^2(\tilde{\Delta}_j)}{\omega^1(\tilde{\Delta}_j)} \\
&\geq (1-\varepsilon)(1 - C_M\varepsilon)\frac{\omega^2(\tilde{\Delta}_j)}{\omega^1(\tilde{\Delta}_j)}.
\end{aligned}$$

To continue we define $\widehat{G}_j = \big\{\big(r_j^{-1}(Y - X_j), r_j^{-2}(s - t_j)\big) : (Y,s) \in G_j\big\}$, $\widehat{E}_j = \Delta(0,0,M) \setminus \widehat{G}_j$. Using this notation we can conclude that for $(Y,s) \in \widehat{G}_j$,

$$(1-\varepsilon)I_j \leq k_j(Y,s) \leq (1+\varepsilon)I_j, \quad I_j := \frac{\omega^1(\Delta_j)}{\omega^1(\tilde{\Delta}_j)}\frac{\omega^2(\tilde{\Delta}_j)}{\omega^2(\Delta_j)}.$$

Based on the deductions above we can conclude that

$$I_j^{-1} \leq \frac{\omega^1(\tilde{\Delta}_j)}{\omega^2(\tilde{\Delta}_j)}\left[(1+\varepsilon)\frac{\omega^2(\tilde{\Delta}_j)}{\omega^1(\tilde{\Delta}_j)} + C_M\varepsilon^\alpha \frac{\omega^2(\Delta_j)}{\omega^1(\Delta_j)}\right] \leq (1+C_M\varepsilon^\alpha)$$

and

$$I_j \leq (1-\varepsilon)^{-1}(1-C_M\varepsilon)^{-1}.$$

In total it follows there exist two functions $A(\varepsilon)$ and $B(\varepsilon)$ such that if $j \geq j_0$ then

$$A(\varepsilon) \leq I_j \leq B(\varepsilon).$$

Furthermore, $A(\varepsilon) \to 1$ and $B(\varepsilon) \to 1$ as $\varepsilon \to 0$. Continuing we have that

$$(1-\varepsilon)I_j \int_{\widehat{G}_j} \phi\, d\omega_j^1 \leq \int_{\widehat{G}_j} \phi k_j\, d\omega_j^1 \leq (1+\varepsilon)I_j \int_{\widehat{G}_j} \phi\, d\omega_j^1.$$

Note that

$$\begin{aligned}
\int_{\partial\Omega_j} \phi\, d\omega_j^2 = \int_{\partial\Omega_j} \phi k_j\, d\omega_j^1 &= \int_{\widehat{G}_j} \phi k_j\, d\omega_j^1 + \int_{\widehat{E}_j} \phi k_j\, d\omega_j^1 \\
&\leq (1+\varepsilon)I_j \int_{\widehat{G}_j} \phi\, d\omega_j^1 + \|\phi\|_\infty \omega_j^2(\widehat{E}_j) \\
&\leq (1+\varepsilon)I_j \int_{\partial\Omega_j} \phi\, d\omega_j^1 + C_M\varepsilon^\alpha \|\phi\|_\infty.
\end{aligned}$$

Similarly

$$\begin{aligned}
\int_{\partial\Omega_j} \phi\, d\omega_j^2 \geq \int_{\widehat{G}_j} \phi k_j\, d\omega_j^1 &\geq (1-\varepsilon)I_j \int_{\widehat{G}_j} \phi\, d\omega_j^1 \\
&= (1-\varepsilon)I_j \int_{\partial\Omega_j} \phi\, d\omega_j^1 - (1-\varepsilon)I_j \int_{\widehat{E}_j} \phi\, d\omega_j^1 \\
&\geq (1-\varepsilon)I_j \int_{\partial\Omega_j} \phi\, d\omega_j^1 - C_M(1-\varepsilon)I_j\varepsilon\|\phi\|_\infty.
\end{aligned}$$

Based on this we can conclude that

$$\begin{aligned}
\int_{\partial\Omega_j} \phi\, d\omega_j^2 &\leq (1+\varepsilon)B(\varepsilon) \int_{\partial\Omega_j} \phi\, d\omega_j^1 + C_M\varepsilon^\alpha \|\phi\|_\infty, \\
\int_{\partial\Omega_j} \phi\, d\omega_j^2 &\geq (1-\varepsilon)A(\varepsilon) \int_{\partial\Omega_j} \phi\, d\omega_j^1 - C_M(1-\varepsilon)B(\varepsilon)\varepsilon\|\phi\|_\infty.
\end{aligned}$$

Hence

$$\int_{\partial\Omega_\infty} \phi\, d\omega_\infty^2 = \lim_{j\to\infty} \int_{\partial\Omega_j} \phi\, d\omega_j^2 = \lim_{j\to\infty} \int_{\partial\Omega_j} \phi\, d\omega_j^1 = \int_{\partial\Omega_\infty} \phi\, d\omega_\infty^2.$$

This completes the proof of the lemma. □

4.2. Classification of global solutions and the proof of Theorem 1

Lemma 19. *Assume that the original domain Ω is δ_0-Reifenberg flat with δ_0 small enough. Assume furthermore that the assumptions in Lemma 18 are fulfilled and, using the notation of Lemma 17, $u_j^i \to u_\infty^i$ for $i \in \{1,2\}$. Define for $(Y,s) \in \mathbf{R}^{n+1}$, $u_\infty(Y,s) = u_\infty^1(Y,s) - u_\infty^2(Y,s)$, where $u_\infty^1(Y,s) \equiv 0$ in Ω_∞^2, $u_\infty^2(Y,s) \equiv 0$ in Ω_∞^1. Then u_∞ is a linear function in the space variables and Ω_∞ is a half space containing a line parallel to the time-axis.*

Proof. Applying Lemma 18 we can conclude that $\omega_\infty^1 \equiv \omega_\infty^2$ and that for all $\phi \in C_0^\infty(\mathbf{R}^{n+1})$

$$\int_{\partial\Omega_\infty^i} \phi(Y,s)\, d\omega_\infty^i(Y,s) = \int_{\Omega_\infty^i} u_\infty^i(Y,s)(\Delta - \partial_s)\phi(Y,s)\, dY\, ds.$$

Hence

$$\begin{aligned}\int_{\mathbf{R}^{n+1}} u_\infty(Y,s)(\Delta-\partial_s)\phi(Y,s)\,dY\,ds &= \int_{\Omega_\infty^1} u_\infty^1(Y,s)(\Delta-\partial_s)\phi(Y,s)\,dY\,ds \\ &\quad - \int_{\Omega_\infty^2} u_\infty^2(Y,s)(\Delta-\partial_s)\phi(Y,s)\,dY\,ds \\ &= 0.\end{aligned}$$

As u_∞ is continous, it is weakly adjoint caloric in $\mathbf{R}^{n+1}$ and therefore adjoint caloric in $\mathbf{R}^{n+1}$. By a change of the time direction we can assume that u_∞ is caloric in $\mathbf{R}^{n+1}$. We also note that $u_\infty(0,0) = 0$. By standard estimates for the heat equation we have that

$$\max_{C_{r/2}(0,0)} |D_Z^k D_\tau^l u_\infty(Z,\tau)| \le \frac{C_{kl}}{r^{k+2l}} \max_{C_r(0,0)} |u_\infty(Z,\tau)|.$$

As, according to Lemma 16, $\partial\Omega_\infty$ is Reifenberg flat we define for $(X,t) \in \partial\Omega$, $r > 0$

$$A_r^1(X,t) = (X + r\hat{n}, t) \in \Omega_\infty, \quad A_r^2(X,t) = (X - r\hat{n}, t) \in \mathbf{R}^{n+1} \setminus \overline{\Omega}_\infty.$$

Here $\hat{n} = \hat{n}(X,t,r)$. Using this notation we have by the backward in time Harnack principle in Lemma 8 that

$$\max_{C_r(0,0)} |u_\infty(Z,\tau)| \le C \max\bigl\{u_\infty^1\bigl(A_r^1(0,0)\bigr), u_\infty^2\bigl(A_r^2(0,0)\bigr)\bigr\}.$$

Using Lemmas 7 and 9 we have

$$\frac{r^n u_\infty^1\bigl(A_r^1(0,0)\bigr)}{\omega_\infty^1\bigl(\Delta(0,0,r)\bigr)} \sim \frac{r^n u_\infty^2\bigl(A_r^2(0,0)\bigr)}{\omega_\infty^2\bigl(\Delta(0,0,r)\bigr)} \sim 1.$$

Hence as $\omega^1_\infty \equiv \omega^2_\infty$ we can conclude that $u^1_\infty\big(A^1_r(0,0)\big) \sim u^2_\infty\big(A^2_r(0,0)\big)$. As, according to Lemma 16, Ω_∞ is $4\delta_0$-Reifenberg flat we can apply Lemma 13 to appropriate functions defined in Ω_∞. In particular according to Lemma 13, given $\varepsilon > 0$, there exists $\hat\delta_0 = \hat\delta_0(n,\varepsilon) > 0$ and a constant $C = C(n,\varepsilon)$ such that if $\delta_0 \le \hat\delta_0$ and $\hat r \le r$ then

$$C^{-1}\Big(\frac{\hat r}{r}\Big)^{1+\varepsilon} u^1_\infty\big(A^1_r(0,0)\big) \le u^1_\infty\big(A^1_{\hat r}(0,0)\big) \le C\Big(\frac{\hat r}{r}\Big)^{1-\varepsilon} u^1_\infty\big(A^1_r(0,0)\big).$$

If we choose $\hat r = 1$ we can conclude that

$$u^1_\infty\big(A^1_r(0,0)\big) \le C u^1_\infty\big(A^1_1(0,0)\big) r^{1+\varepsilon}.$$

In total we can conclude that

$$\max_{C_{r/2}(0,0)} |D^k_Z D^l_\tau u_\infty(Z,\tau)| \le \frac{C_{kl}}{r^{k+2l}} u^1_\infty\big(A^1_r(0,0)\big) \le \frac{C_{kl}}{r^{k+2l-1-\varepsilon}} u^1_\infty\big(A^1_1(0,0)\big).$$

By letting $r \to \infty$ we get $D^k_Z D^l_\tau u_\infty(Z,\tau) = 0$ for all $(Z,\tau) \in \mathbf{R}^{n+1}$ and all (k,l) such that $k+2l-1-\varepsilon > 0$. We can therefore conclude that u_∞ is in fact a linear function in the space variables and Ω_∞ is a half space containing a line parallel to the time-axis. □

We can now prove Theorem 1 using Lemma 19. According to Definition 1, if $\partial\Omega$ is δ_0 Reifenberg flat, then given any $(\widetilde X, \tilde t) \in \partial\Omega$, $R > 0$, there exists an n-dimensional plane $\widehat P = \widehat P(\widetilde X, \tilde t, R)$, containing $(\widetilde X, \tilde t)$ and a line parallel to the t-axis, having unit normal $\hat n = \hat n(\widetilde X, \tilde t, R)$ such that

$$\begin{aligned}&\big\{(Y,s) + r\hat n \in C_R(\widetilde X, \tilde t) : (Y,s) \in \widehat P,\ r > \delta_0 R\big\} \subset \Omega,\\ &\big\{(Y,s) - r\hat n \in C_R(\widetilde X, \tilde t) : (Y,s) \in \widehat P,\ r > \delta_0 R\big\} \subset \mathbf{R}^{n+1} \setminus \Omega.\end{aligned}$$

We therefore introduce the quantity

$$\Theta(\widetilde X, \tilde t, R) := \frac{1}{R} \inf_{\widehat P} D\big[\partial\Omega \cap C_R(\widetilde X, \tilde t), \widehat P \cap C_R(\widetilde X, \tilde t)\big]$$

where the infimum is taken over all n-dimensional planes $\widehat P = \widehat P(\widetilde X, \tilde t, R)$, containing $(\widetilde X, \tilde t)$ and a line parallel to the t-axis. For any compact set $K \subset \mathbf{R}^{n+1}$ we also introduce

$$\Theta_K(R) := \sup_{(\widetilde X, \tilde t) \in K} \Theta(\widetilde X, \tilde t, R).$$

If $(X,t)\in\partial\Omega$, $r>0$, then the statement that $C_r(X,t)\cap\partial\Omega$ is Reifenberg flat with vanishing constant in the parabolic sense is equivalent to the statement that

$$\lim_{\hat r\to 0}\Theta_{C_r(X,t)\cap\partial\Omega}(\hat r)=0.$$

To prove Theorem 1 we assume, using the notation of the theorem, that $(X,t)\in\partial\Omega$, $\hat t^2>t+4r^2$ and that

$$\lim_{\hat r\to 0}\Theta_{C_r(X,t)\cap\partial\Omega}(\hat r)=\beta$$

for some $\beta>0$. We intend to prove that this is impossible and that $\beta=0$. Let $(X_j,t_j)\in C_r(X,t)\cap\partial\Omega$, $(X_j,t_j)\to(\widehat X,\hat t)\in C_r(X,t)\cap\partial\Omega$ and r_j be a sequence of real numbers tending to zero such that

$$\lim_{j\to\infty}\Theta(X_j,t_j,r_j)=\beta.$$

By a translation argument we can without loss of generality assume that $(\widehat X,\hat t)=(0,0)$ and that $(0,0)\in C_r(X,t)\cap\partial\Omega$. Define the domains,

$$\Omega_j^i=\big\{\big(r_j^{-1}(X-X_j),r_j^{-2}(t-t_j)\big):(X,t)\in\Omega^i\big\}.$$

Then according to Lemma 16 we can assume that $\Omega_j^i\to\Omega_\infty^i$, $\partial\Omega_j^i\to\partial\Omega_\infty^i$ in the parabolic Hausdorff distance sense uniformly on compact subsets of $\mathbf{R}^{n+1}$. Furthermore, $\Omega_\infty=\Omega_\infty^1$ and $\Omega_\infty^2=\mathbf{R}^{n+1}\setminus\overline{\Omega}_\infty$ are Reifenberg flat domains. We can furthermore apply Lemmas 18 and 19 in order to conclude that Ω_∞ is a half space containing a line parallel to the time-axis. Still our assumption above gives at hand that

$$\Theta_\infty(0,0,1):=\beta>0$$

where $\Theta_\infty(0,0,1)$ is defined with respect to $\partial\Omega_\infty$. Clearly this is a contradiction and we can conclude that $C_r(X,t)\cap\partial\Omega$ is Reifenberg flat with vanishing constant.

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