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A rough classification of symmetric planes

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Abstract. Symmetric planes are stable planes carrying an additional structure of a symmetric space such that the symmetries are also automorphisms of the geometry. The hyperbolic, affine, and projective planes over the real alternative division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ are examples of symmetric planes.

Lie triple planes arise naturally as local linear approximations of symmetric planes. The aim of the paper is to give a "rough classification": Every Lie triple plane is abelian, or semisimple, or splits. The proof of this fact is the first step towards a complete classification of symmetric planes, which will be carried out in a series of subsequent papers.

1 Introduction and statement of results

The investigation of the hyperbolic planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ (quaternions) and $\mathbb{O}$ (octonions) was the starting point of several branches of geometry, among them the theory of symmetric spaces and the theory of stable planes[1]. The notion of a symmetric plane (introduced in Löwen's fundamental paper [12]) links the latter two branches; a *symmetric plane* is a stable plane whose point space is, in addition, a symmetric space[2] such that the symmetries are reflections in the geometric sense. Besides the hyperbolic planes, the classical affine and projective planes are examples of symmetric planes. Another class is derived from Frobenius partitions of sharply 2-transitive Lie groups, see Löwe [8]. Some of these do not embed in classical projective planes, see Löwe [7]. We point out that the dimension of (the point space of) a symmetric plane has to be one of the numbers 2, 4, 8, and 16.

Symmetric planes in dimension 2 and 4 were completely classified by Löwen, see [12], [13]. These planes are divided into three major classes:

(1) Simple symmetric planes, i.e. symmetric planes whose motion group is an almost simple Lie group. Examples are the classical projective and the hyperbolic planes.

[1] See below for a definition. Prime examples are the open subgeometries of classical projective planes.

[2] We shall always use the term symmetric space in the sense of Loos [15]. In particular, we do not require a Riemannian connection on these spaces.

(2) So-called split symmetric planes; see (2.13) for the definition. Examples are the symmetric planes derived from sharply 2-transitive Lie groups mentionend above.

(3) Abelian symmetric planes, i.e. topological affine translation planes (which are always symmetric planes in a trivial way).

Löwen's key to the classification result is a linearization of the problem: The local structure of a symmetric plane $\mathbb{E} = (P, \mathscr{L})$ at a point o is approximated by its so-called tangent translation plane $T_o\mathbb{E}$ (cp. (2.4), (2.5) for details). This fact prompted us to introduce a "local counterpart" of the notion of a symmetric plane: A *Lie triple plane* is a topological affine translation plane $(M, \mathscr{M})$ whose point space M is, in addition, a Lie triple system such that (1) every line through the origin is a subsystem of M, and (2) every inner automorphisms of the Lie triple system M is also an automorphism of the geometry $(M, \mathscr{M})$.

While every tangent translation plane of a symmetric plane is a Lie triple plane, there exist Lie triple planes which do not arise in this way (see [5]). Nevertheless, these "non-integrable" examples are related to interesting stable planes which one may describe as locally symmetric planes (cp. the discussion in [5, 5.2(B)]). For this reason, the intention of this paper and a series of subsequent ones (which cover the material of the author's thesis [5] and Habilitationsschrift [10]) is to solve the following two problems.

(1.1) Local classification problem. Classify all Lie triple planes whose underlying Lie triple system is nonabelian, i.e. give a complete list of examples which contains all Lie triple planes of the specified kind up to isomorphism.

Remark. Every topological affine translation plane can be considered as a Lie triple plane with vanishing triple bracket. Of course, the additional structure does not give further information on the geometry; there are a vast number of non-isomorphic translation planes. This fact causes us to restrict ourselves to the nonabelian case.

(1.2) Global classification problem. Classify all non-abelian symmetric planes.

As a first step towards the solution of these problems, the following "rough classification" of Lie triple planes is the aim of the present paper:

(1.3) Main Result. *Every Lie triple plane is abelian, or simple, or splits.*

Remark. For the definition of the term "split" see (2.14) and (2.13) (which motivates the definition). Notice that the main result (1.3) is analogous to the case of low-dimensional symmetric planes, see above.

(1.4) Organization of the paper. The second section contains an extended version of the preceding discussion: We collect basic definitions and facts concerning symmetric planes and Lie triple planes.

Section 3 is entirely devoted to the proof of (1.3), which will be completed in (3.10). Important steps are to show that a semisimple Lie triple plane is simple (cf. (3.4)), and that the center of a nonabelian Lie triple plane vanishes (this will be done in (3.6) to (3.9)).

The appendix (Section 4) gives a comprehensive introduction to the theory of Lie triple systems (most of the material is taken from Lister [4]). We remark that Theorem 4.9.c (which states that Levi complements of a Lie triple system are conjugate by particular inner automorphisms) seems to be new.

2 Definitions and prerequisites

(2.1) Stable planes. A *stable plane* is a pair $(P, \mathscr{L})$ consisting of a locally compact Hausdorff space P (whose elements are called *points*) together with a system $\mathscr{L}$ of subsets of P, called *lines*, such that the following properties hold:

(1) Every two distinct points $p, q \in P$ are contained in precisely one line $p \vee q \in \mathscr{L}$. Every line contains at least two points. Moreover, there exists a quadrangle, i.e. four points no three of which are on the same line. (Notice that two lines $K, L \in \mathscr{L}$ may not meet; but if they do, then their intersecting point $K \wedge L$ is unique.)

(2) The covering dimension of the topological space P is positive and finite.

(3) There exists a topology on $\mathscr{L}$ such that the operations $\vee$ and $\wedge$ are continuous, where defined, and such that the domain of definition of $\wedge$ is an open subset of $\mathscr{L} \times \mathscr{L}$.

An *isomorphism* of stable planes is a homeomorphism between the point spaces which maps lines onto lines.

We refer to Grundhöfer, Löwen [2] for a detailed survey on stable planes. For our purposes, it suffices to know the following results, for which Löwen [11], [14] contains the details: The covering dimension $\dim P$ equals one of the numbers 2, 4, 8, and 16. The line space $\mathscr{L}$ is locally homeomorphic to P. Every line $L \in \mathscr{L}$ is a closed subspace of P of dimension $l := (\dim P)/2$. Every line pencil $\mathscr{L}_p$ (i.e. the set of all lines through $p \in P$) is a compact connected homotopy l-sphere.

Prime examples of stable planes are the open subgeometries of the classical projective planes over $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$. Another class of examples are the topological (= stable) affine translation planes, which we define next.

(2.2) Topological translation planes. Instead of an axiomatic definition we give the following description which covers all topological affine translation planes, see [19, 64.4ff]: The point space of such a topological translation plane is the real affine space $\mathbb{R}^{2l}$ (where $l \in \{1, 2, 4, 8\}$ holds by (2.1)); the line set $\mathscr{L}$ is a translation invariant system of l-dimensional affine subspaces of $\mathbb{R}^{2l}$. Clearly, $\mathscr{L}$ equals the set of all affine cosets of all elements of the line pencil $\mathscr{L}_0$ (the set of lines through the origin 0). This line pencil has the following properties [19, 64.4]:

1. $\mathscr{L}_0$ is a *spread*, i.e. every point $p \in \mathbb{R}^{2l} \setminus \{0\}$ is contained in a unique element of $\mathscr{L}_0$, and
2. $\mathscr{L}_0$ is a *compact* subset of the Grassmannian manifold of all l-dimensional vector subspaces of $\mathbb{R}^{2l}$.

We shall call such a family of l-dimensional vector subspaces of $\mathbb{R}^{2l}$ a *compact spread*.

Conversely, if $\mathscr{L}_0$ is a compact spread on $\mathbb{R}^{2l}$, define $\mathscr{L} = \mathscr{L}_0$ as the set of all cosets of all elements of $\mathscr{L}_0$. Then $(\mathbb{R}^{2l}, \mathscr{L})$ is a topological translation plane [19, 64.4]. The topology on $\mathbb{R}^{2l}$ is the usual one; the topology on $\mathscr{L}$ is derived from the Grassmannian topology on $\mathscr{L}_0$; see the proof of [19, 64.4]. Clearly, the group Σ of vector translations of $\mathbb{R}^{2l}$ consists of automorphisms of the plane $(\mathbb{R}^{2l}, \mathscr{L})$.

While the affine real plane is the only 2-dimensional topological translation plane (there is only one spread on $\mathbb{R}^2$), there are a vast number of examples in the higher dimensions, see e.g. [19, Sections 73, 82].

(2.3) Symmetric spaces. A *symmetric space* consists of a smooth manifold (of finite dimension) together with a family $\{s_x \mid x \in P\}$ of involutory diffeomorphisms (the *symmetries*), such that (1) the "multiplication map" $P \times P \to P$; $(x, y) \mapsto x \cdot y := s_x(y)$ is smooth, (2) the symmetries s_x are morphisms of $(P, \cdot)$, and (3) x is an isolated fixed point of s_x.

The group Σ generated by the set $\{s_x s_y \mid x, y \in P\}$ is called the *motion group* of $(P, \{s_x\})$. If P is connected, then Σ is a transitive connected Lie transformation group of P, see [15, p. 91].

Concerning symmetric spaces, we use the terminology of Loos [15], [16].

(2.4) Symmetric planes. A *symmetric plane* is a triple $(P, \mathscr{L}, \{s_x \mid x \in P\})$ such that (1) $(P, \mathscr{L})$ is a stable plane, (2) $(P, \{s_x\})$ is a symmetric space, and (3) every symmetry s_x is an automorphism of the stable plane $(P, \mathscr{L})$.

An *isomorphism* between symmetric planes is an isomorphism of the underlying symmetric spaces which maps lines onto lines, i.e. which is also an isomorphism of the underlying stable planes. The group of automorphisms of a symmetric plane $\mathbb{E}$ will be denoted by $\mathrm{Aut}(\mathbb{E})$. Notice that the motion group of the symmetric space P is a normal subgroup of $\mathrm{Aut}(\mathbb{E})$.

For examples of symmetric planes, we refer to Löwen [12], Stroppel [21] and Löwe [5], [7], [8], [10].

Let $\mathbb{E} = (P, \mathscr{L}, \{s_p\})$ be a symmetric plane. By [12, 1.4], the symmetry s_p is a reflection at p in the geometric sense, i.e. s_p leaves every line through p invariant. This implies that a line L is invariant under the symmetries at points of L. Since L is a closed subset of P, [15, p. 125] shows that L is a symmetric subspace of P, cp. [12, 4.2]. Therefore, if $o \in L$, then the tangent space T_oL is a subsystem of the Lie triple system T_oP, cf. [15, p. 121]. For $o \in P$, let $T_o\mathscr{L}_o$ be the set of all subsystems T_oL, $L \in \mathscr{L}_o$. Moreover, let $T_o\mathscr{L}$ be the set of all affine cosets of all elements of $T_o\mathscr{L}_o$. We refer to $T_o\mathbb{E} := (T_oP, T_o\mathscr{L})$ as the *tangent translation plane* of $\mathbb{E}$ at o; this name is justified by the following result:

(2.5) Theorem (Löwen [12, 4.6]). *Let $(P, \mathscr{L}, \{s_p\})$ be a symmetric plane with base point $o \in P$ and motion group Σ.*

a. *The tangent translation plane $T_o\mathbb{E}$ is a topological affine translation plane of dimension* $\dim T_oP = \dim P$. *In particular, the set $T_o\mathscr{L}_o$ is a compact spread.*

b. *The isotropy representation $D : \Sigma_o \to \mathrm{GL}(T_oP)$; $\gamma \mapsto T_o\gamma$ of the stabilizer Σ_o on the Lie triple system T_oP is faithful.*

c. *$D(\Sigma_o)$ consists of automorphisms of the translation plane $T_o\mathbb{E}$; its identity component $D(\Sigma_o)^e$ coincides with the group* $\exp \operatorname{ad}[T_oP, T_oP]$ *of inner automorphisms of the Lie triple system T_oP.*

Remarks. The equation $\dim T_oP = \dim P$ follows from the fact that connected components of a symmetric plane are open and, hence, can be considered as symmetric planes of the same dimension, see the considerations in Section 4 of Löwen [12].

Locally at the point o, the tangent translation plane $T_o\mathbb{E}$ uniquely determines the geometry and the symmetric structure of $\mathbb{E}$. (In fact, if $\mathbb{E}$ is connected and satisfies rather mild conditions (see [12, 4.11] for details), then $T_o\mathbb{E}$ determines $\mathbb{E}$ globally.) Since we are interested in a local theory of symmetric planes, the statement of Theorem 2.5 motivates the following.

(2.6) Definition. Let M be a $2l$-dimensional Lie triple system and let $\mathscr{M}_0$ be a set of l-dimensional subsystems of M. Define $\mathscr{M}$ as the set of all affine cosets of all elements of $\mathscr{M}_0$. Then $(M, \mathscr{M})$ is called a *Lie triple plane*, if the following conditions are satisfied:

1. $\mathscr{M}_0$ is a compact spread, i.e. $(M, \mathscr{M})$ is a topological translation plane.
2. The group $\exp \operatorname{ad}[M, M]$ consists of automorphisms of the plane $(M, \mathscr{M})$.

If $(M, \mathscr{M})$ is a Lie triple plane, then the standard embedding $\operatorname{ad}[M, M] \oplus M$ of its underlying Lie triple system M is called the *motion algebra* of $(M, \mathscr{M})$.

An *isomorphism* of two Lie triple planes is an isomorphism between the underlying Lie triple systems which maps lines onto lines.

(2.7) Remarks. (a) A Lie triple plane $(M, \mathscr{M})$ is called abelian, semisimple etc., if its underlying Lie triple system M has the respective property.

(b) We agree on the following exception to the rule (a): A Lie triple plane $(M, \mathscr{M})$ is called *simple*, if the standard embedding $\operatorname{ad}[M, M] \oplus M$ of its underlying Lie triple system is a simple Lie algebra.

(c) Let $\mathbb{M} = (M, \mathscr{M})$ be a Lie triple plane. If we replace M by the dual Lie triple system M^* (i.e. the vector space M with the triple bracket $[\cdot,\cdot,\cdot]^* := -[\cdot,\cdot,\cdot]$), then $\mathbb{M}^* = (M^*, \mathscr{M})$ clearly is a Lie triple plane. We refer to $\mathbb{M}^*$ as the *antipodal Lie triple plane* of $\mathbb{M}$.

The following lemma is a direct consequence of (2.5) and provides examples of Lie triple planes:

(2.8) Lemma. *The tangent translation planes of symmetric planes are Lie triple planes.*

(2.9) Remark. The converse of the statement of (2.8) does not hold: There are Lie triple planes which are not isomorphic to a tangent translation plane of any symmetric plane; see [5, 4.4.1] for examples.

(2.10) Abelian Lie triple planes. Let $(M, \mathscr{M})$ be a topological affine translation plane. The point space $M = \mathbb{R}^{2l}$ is a symmetric space with multiplication $x \cdot y := 2x - y$

(the symmetries then are defined by $s_x(y) := x \cdot y$). It is clear that $\mathbb{E} := (M, \mathscr{M}, \{s_x\})$ is a symmetric plane whose motion group coincides with the group of vector translations of M. We refer to these planes as *abelian symmetric planes*, because their underlying symmetric space is abelian.

The tangent translation plane $T_0\mathbb{E}$ equals $(M, \mathscr{M})$; its underlying Lie triple system M is equipped with the trivial triple bracket $[\cdot, \cdot, \cdot] \equiv 0$; i.e. $T_0\mathbb{E}$ is an abelian Lie triple plane. The following result shows that abelian Lie triple planes occur precisely as the tangent translation planes of abelian symmetric planes:

(2.11) Theorem (Löwen [12, 4.14]). *For a symmetric plane* $\mathbb{E} = (P, \mathscr{L}, \{s_p\})$, *the following properties are equivalent*:

1. $\mathbb{E}$ *is an abelian symmetric plane* (*i.e. an affine translation plane with the symmetric structure defined above*).
2. $\mathbb{E}$ *is an affine plane.*
3. P *is an abelian symmetric space.*
4. *The motion group* Σ *is abelian.*
5. *The Lie triple system* T_oP *is abelian for some point* $o \in P$.

(2.12) Remark. The above theorem is stated in [12] for connected planes only. Using that

(i) an affine plane is a dense open subplane of its projective closure and, therefore, cannot be a connected component of a disconnected plane, and
(ii) the connected components of (abelian) symmetric planes are (abelian) symmetric planes again,

one easily sees that this assumption is unnecessary.

(2.13) Split symmetric planes. By the definition of Löwen, see [12, 3.1][3], a nonabelian symmetric plane $(P, \mathscr{L}, \{s_x\})$ is called *split*, if for some connected abelian subgroup Δ of its motion group Σ there exists a set $\mathscr{F}$ of lines such that the orbits of Δ are precisely the connected components of the elements of $\mathscr{F}$, and such that Δ is normalized by all symmetries. Examples of split symmetric planes are the punctured classical projective planes (cp. [12, 2.9]) and the symmetric planes derived from Frobenius partitions of sharply 2-transitive Lie groups, see [8].

From the point of view of differential geometry, the connected components of the elements of $\mathscr{F}$ are the leaves of an abelian congruence[4] on the symmetric space P. If $o \in P$, then the connected component of a line $L \in \mathscr{L}_o$ containing o is the leaf of an

[3] In addition, Löwen requires that Δ is a maximal abelian normal subgroup. We replace this part of the definition by the condition that the plane is nonabelian.

[4] A *congruence* of a symmetric space P is a equivalence relation $\mathfrak{C} \subseteq P \times P$ which is a symmetric subspace of $P \times P$; the equivalence classes (called the "leaves") of $\mathfrak{C}$ then are symmetric subspaces of P. An abelian congruence is a congruence $\mathfrak{C}$ which is an abelian symmetric space. We refer to Loos [15, p. 130ff] for further information.

abelian congruence if and only if T_oL is a totally abelian ideal of T_oP, cf. Loos [15, p. 131]. For this reason we introduce the term "split Lie triple plane" as follows:

(2.14) Definition. A Lie triple plane $\mathbb{M} = (M, \mathscr{M})$ is called *split*, if it is nonabelian and if one element $A \in \mathscr{M}_0$ is a totally abelian ideal of the Lie triple system M. In this case, we refer to A as the *splitting line* of $\mathbb{M}$.

Remark. It can be shown that a symmetric plane splits if and only if its tangent translation planes are split Lie triple planes.

We close this section with some frequently used results on topological translation planes:

(2.15) Theorem ([19, 44.4, 81.5], [20, 6.3, 6.8]). *Let $\mathbb{M} = (M, \mathscr{M})$ be a $2l$-dimensional topological translation plane and let $\Delta \neq \{\mathrm{id}\}$ be a subgroup of $\mathrm{GL}(M)$ consisting of automorphisms of $\mathbb{M}$. Let $F \subseteq M$ denote the set of fixed points of Δ.*

a. *Either F is contained in a line, or F is a subplane of $\mathbb{M}$ of dimension $2, 4, 8 \lneqq 2l$. In the latter case, Δ is relatively compact in $\mathrm{GL}(M)$.*
b. *In addition, assume that F is a Baer subplane of $\mathbb{M}$ (i.e. a subplane of dimension l) and that Δ is connected. Then one of the following possibilities occurs:*
   1. *Δ is isomorphic to $\mathrm{SO}_2\mathbb{R}$ and $\dim P \in \{8, 16\}$, or*
   2. *Δ is isomorphic to $\mathrm{Spin}_3\mathbb{R}$ and $\dim P = 16$.*

Let $\mathfrak{n} \leqslant \mathfrak{gl}(\mathbb{R}^n)$ be a Lie subalgebra whose elements are nilpotent endomorphisms of the vector space $\mathbb{R}^n$, and put $\Gamma := \exp \mathfrak{n}$. We collect some properties of $\mathfrak{n}$ and Γ (see [23, §§3.5, 3.6] for details and proofs): There exists a basis of $\mathbb{R}^n$ such that $\mathfrak{n}$ consists of upper right triangular matrices. In particular, $\mathfrak{n}$ is a nilpotent Lie algebra and annihilates some nonzero vector v. It is clear that such a vector v is a fixed point of Γ. Moreover, the exponential map $\exp : \mathfrak{n} \to \Gamma$ is a diffeomorphism; its inverse $\log : \Gamma \to \mathfrak{n}$ is given by

$$\log(\gamma) = \sum_{s=1}^{n-1} (-1)^{s-1} \frac{(\gamma - \mathrm{id})^s}{s}. \tag{1}$$

It follows that Γ is a closed connected simply connected subgroup of $\mathrm{SL}_n\mathbb{R}$. Furthermore, Γ is compact-free (i.e. only the trivial subgroup of Γ is relatively compact in $\mathrm{GL}_n\mathbb{R}$).

(2.16) Proposition. *Let $\mathbb{M} = (M, \mathscr{M})$ be a $2l$-dimensional translation plane. Let $\mathfrak{n}$ be a nontrivial Lie subalgebra of $\mathfrak{gl}(M)$ such that every element of $\mathfrak{n}$ is a nilpotent endomorphism of M.*

Moreover, assume that $\Gamma := \exp \mathfrak{n}$ consists of automorphisms of $(M, \mathscr{M})$. Then Γ fixes exactly one line $L \in \mathscr{M}_0$ and acts freely on $\mathscr{M}_0 \setminus \{L\}$.

Proof. Choose a vector $v \in M \backslash \{0\}$ which is annihilated by $\mathfrak{n}$. Then v is a fixed point of Γ, whence Γ leaves the line $L \in \mathscr{M}_0$ containing v invariant. Let $\gamma \in \Gamma \backslash \{e\}$. Aiming at a contradiction we assume that $K \in \mathscr{M}_0 \backslash \{L\}$ is γ-invariant. Since $\log \gamma$ is a nilpotent endomorphism, the group $\Theta := \exp(\mathbb{R} \cdot \log \gamma)$ fixes some nonzero vector $w \in K$. This implies that Θ is planar and, hence, is relatively compact in $\mathrm{GL}(M)$ (cf. 2.15.a). But this is impossible, because Γ is compact-free.

(2.17) Proposition. *Let $\mathbb{M} = (M, \mathscr{M})$ be a $2l$-dimensional translation plane and let $d : M \to M$ be a nilpotent linear map different from* 0. *In addition, suppose that $d(M)$ is contained in some line $S \in \mathscr{M}_0$. If* $\exp d$ *is an automorphism of the translation plane* $\mathbb{M}$, *then $d(L) = S$ holds for every line $L \in \mathscr{M}_0 \backslash \{S\}$.*

Proof. As a vector space, M can be written as a direct sum $M = L \oplus S$. Choose a basis $\mathscr{B} = \{b_1, \dots, b_{2l}\}$ of M such that $\{b_1, \dots, b_l\}$ and $\{b_{l+1}, \dots, b_{2l}\}$ are bases of L and S, respectively.

With respect to $\mathscr{B}$, the linear map d (whose image is contained in S) is represented by some matrix

$$d = \begin{pmatrix} 0 & 0 \\ X & Y \end{pmatrix},$$

consisting of four $(l \times l)$-blocks. From the nilpotency of d it follows that Y is nilpotent, too. Choose $m \in \mathbb{N}$ with $Y^m = 0$. A short computation shows that

$$\exp d = \begin{pmatrix} E & 0 \\ Z \cdot X & \exp Y \end{pmatrix}, \quad \text{where } Z = \sum_{k=0}^{m-1} \frac{1}{(k+1)!} Y^k.$$

Notice that Z is invertible, because $Z - E = \sum_{k=1}^{m-1} Y^k/(k+1)!$ is a nilpotent matrix.

Since $\exp d$ is a collineation, the image $(\exp d)(L) = \{(x, ZXx)^t \mid x \in L\}$ of the line L is a line through 0 again. Thus, the intersection $L \cap (\exp d)(L)$ either equals $\{0\}$ or coincides with L. It follows that ZX either is an invertible matrix or equals 0. Assume that $ZX = 0$. Being the exponential of a nilpotent matrix, $\exp Y$ fixes some nonzero vector $y \in S$. Thus, $\exp d$ fixes the $(l+1)$-dimensional subspace $L \oplus \langle y \rangle$ pointwise. According to (2.15), this implies that $\exp d$ is the identity map. In contradiction to our assumption $d \neq 0$, it follows that $d = 0$, because d is a nilpotent linear map.

We have proved that ZX is invertible. Therefore, X is invertible, too, and we obtain $d(L) = S$.

3 The proof of the main result

(3.1) Notation and conventions. Throughout this section, $\mathbb{M} = (M, \mathscr{M})$ denotes a $2l$-dimensional Lie triple plane. We let $\mathfrak{g} = \mathrm{ad}[M, M] \oplus M$ be the standard embedding of the Lie triple system M and $\sigma : \mathfrak{g} \to \mathfrak{g}$ the standard involution. We shall refer to $\mathfrak{g}$ as the *motion algebra* of $\mathbb{M}$.

We start our investigation with two useful lemmas:

(3.2) Lemma. *If two distinct lines $A_1, A_2 \in \mathscr{M}_0$ are ideals of M, then $\mathbb{M}$ is an abelian Lie triple plane.*

Proof. We may choose a coordinatization[5] of $\mathbb{M}$ by a locally compact, connected quasifield $(Q = \mathbb{R}^l, +, \circ)$ such that the following holds:

$$M = Q \times Q, \quad A_1 = Q \times \{0\}, \quad A_2 = \{0\} \times Q.$$
$$\mathscr{M} = \{L_a \mid a \in Q\} \cup \{A_2\}, \quad \text{where}$$
$$L_a = \left\{ \begin{pmatrix} x \\ a \circ x \end{pmatrix} \middle| x \in Q \right\} \quad \text{for } a \in Q;$$

Since "mixed products" between the elements of the ideals A_1 and A_2 vanish, there are triple products $[\cdot,\cdot,\cdot]_i$ on Q such that the equation

$$\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right] = \begin{pmatrix} [x_1, y_1, z_1]_1 \\ [x_2, y_2, z_2]_2 \end{pmatrix},$$

holds for all $x_1, x_2, y_1, y_2, z_1, z_2 \in Q$. The fact that L_a is a subsystem implies

$$[a \circ x, a \circ y, a \circ z]_2 = a \circ [x, y, z]_1 \quad \text{for all } a, x, y, z \in Q. \tag{2}$$

Setting $a = 1$ we infer that $[x, y, z]_2 = [x, y, z]_1 =: [x, y, z]$. It remains to show that the Lie triple system $(Q, [\cdot,\cdot,\cdot])$ is abelian. Of course it suffices to consider the case $\dim Q \geqslant 2$.

For $a \in Q$, let λ_a denote the left multiplication $\lambda_a : Q \to Q; x \mapsto a \circ x$. Then λ_a is an automorphism of the Lie triple system Q, thanks to equation (2). Moreover, the map $\lambda : Q \backslash \{0\} \to \mathrm{GL}(Q); a \mapsto \lambda_a$ is continuous, whence the image $\Lambda := \lambda(Q)$ is contained in the connected component Γ of the automorphism group of the Lie triple system Q. As a consequence, we obtain that Γ acts transitively on $Q \backslash \{0\}$. This implies that every characteristic ideal of Q equals Q or $\{0\}$. Applying this fact to the radical $\mathrm{Rad}(Q)$ and the commutator ideal $[Q, Q, Q]$, we derive that Q is abelian or semisimple.

Aiming at a contradiction, we assume that Q is semisimple. Then every derivation of Q is inner [4, 2.11], whence the connected group Γ is a subgroup of $\exp \mathrm{ad}[Q, Q]$. Moreover, since Q is semisimple, the Ricci form

$$\rho : (x, y) \mapsto \mathrm{trace}(z \mapsto [z, x, y])$$

[5] For details concerning the coordinatization procedure and locally compact, connected quasifields we refer to [19, Sections 22, 25, 42].

of Q is a symmetric bilinear form on Q, cf. Loos [15, 1.3 (p. 142)]. Applying the Jacobi identity, we conclude that $\operatorname{trace}(\operatorname{ad}[x,y]) = \rho(y,x) - \rho(x,y)$ vanishes for all $x, y \in Q$. Consequently, $\exp\operatorname{ad}[Q,Q]$ is a subgroup of $\mathrm{SL}(Q)$. In contrast, the subset $\Lambda \subseteq \Gamma$ is not a subset of $\mathrm{SL}(Q)$, since the module function $Q \to \mathbb{R}$; $a \mapsto |\det \lambda_a|$ of the quasifield Q is continuous and not constant; cf. [19, 81.3f].

(3.3) Corollary. *The center Z of a Lie triple plane $(M, \mathscr{M})$ is not a line.*

Proof. Assume that Z is an element of $\mathscr{M}_0$. Choose $L \in \mathscr{M}_0 \setminus \{Z\}$. Since L is a subsystem of M complementary to the center, we infer that L is an ideal of M, too, in contradiction to (3.2).

Next, we turn to semisimple Lie triple planes.

(3.4) Theorem. *The motion algebra of a semisimple Lie triple plane is simple. In other words, every semisimple Lie triple plane is simple.*

Proof. If the Lie triple system M is simple, then either its standard embedding $\mathfrak{g} := \operatorname{ad}[M,M] \oplus M$ is simple (and we are finished) or M is a simple Lie algebra (considered as a Lie triple system), cf. (4.10).

In the latter case the representation of $\operatorname{ad}[M,M] \cong M$ is the adjoint representation of the simple Lie algebra M. Hence, the almost simple Lie group $\Gamma = \exp\operatorname{ad}[M,M]$ acts on the translation plane $\mathbb{M}$ in its adjoint representation. By [9, Theorem A][6], Γ is a compact group. Its Lie algebra M has dimension $n = 2, 4, 8$ or 16. Checking the possibilities for such Lie algebras, see e.g. Tits, [22], we infer that M is isomorphic to the 8-dimensional Lie algebra $\mathfrak{su}_3(\mathbb{C})$. But also this case is impossible: M is 8-dimensional and the group Γ is locally isomorphic to $\mathrm{SU}_3(\mathbb{C})$. By [18], Γ cannot act almost effectively on the 4-sphere $\mathscr{M}_0$. It follows that every line through 0 is Γ-invariant, in contradiction to [19, 81.0].

Therefore, it sufficies to prove that M is simple. Assume that this is not true. According to [4, Thm. 2.9], M is a direct sum

$$M = U_1 \oplus \cdots \oplus U_k$$

of at least two simple ideals $U_j \trianglelefteq M$. We will show that this is impossible:

We prove first that $\dim U_j = l$ for $1 \leqslant j \leqslant k$; this will imply that $k = 2$. Set

$$V_j := U_1 \oplus \cdots \oplus U_{j-1} \oplus U_{j+1} \oplus \cdots \oplus U_k.$$

Then V_j is a semisimple ideal of M centralizing U_j. Thus, the nontrivial group

[6] Theorem A in [9] states that a noncompact, almost simple group Δ of automorphisms of a topological translation plane $\mathbb{M}$ is a 2-fold covering group of $\mathrm{PSO}_m(\mathbb{R},1)^e$ for some m, $3 \leqslant m \leqslant 10$. Consequently, the center of Δ is not trivial, whence the representation of Δ on $\mathbb{M}$ is not equivalent to the adjoint representation.

$\exp \operatorname{ad}[V_j, V_j]$ fixes U_j pointwise and we conclude $\dim U_j \leqslant l$ from (2.15). Analogously, we may prove that $\dim V_j \leqslant l$ and obtain $\dim U_j = \dim M - \dim V_j \geqslant l$ as a consequence.

We have shown that $M = U_1 \oplus U_2$ is a direct sum of two l-dimensional simple ideals $U_1, U_2 \trianglelefteq M$. According to (3.2), at least one of these ideals, say U_1, is not a line.

Since the nontrivial group $\exp \operatorname{ad}[U_2, U_2]$ fixes the l-dimensional subspace U_1 of M pointwise, (2.15) shows that U_1 is a Baer subplane of $\mathbb{M}$, and that there are only the following three possibilities:

(i) $\exp \operatorname{ad}[U_2, U_2] = \mathrm{SO}_2\,\mathbb{R}$ and $l = 4$;

(ii) $\exp \operatorname{ad}[U_2, U_2] = \mathrm{SO}_2\,\mathbb{R}$ and $l = 8$;

(iii) $\exp \operatorname{ad}[U_2, U_2] = \mathrm{Spin}_3\,\mathbb{R}$ and $l = 8$.

In any case, the dimension of the subalgebra $\mathfrak{u}_2 := \operatorname{ad}[U_2, U_2] \oplus U_2$ of $\mathfrak{g}$ satisfies

$$\dim \mathfrak{u}_2 = \dim U_2 + \dim \operatorname{ad}[U_2, U_2] = l + \dim \exp \operatorname{ad}[U_2, U_2] \in \{5, 9, 11\}.$$

Notice that $\mathfrak{u}_2$ is isomorphic to the standard embedding of the simple Lie triple system U_2, cp. (4.10). Moreover, the odd-dimensional Lie algebra $\mathfrak{u}_2$ is not a direct sum of two isomorphic ideals. By (4.10) again, we obtain that $\mathfrak{u}_2$ is a simple Lie algebra. But this is impossible, since no simple Lie algebra has dimension 5, 9 or 11, cf. [22]. This contradiction finishes the proof.

(3.5) Corollary. *If the motion algebra $\mathfrak{g}$ of a Lie triple plane $\mathbb{M} = (M, \mathscr{M})$ is a complex, semisimple Lie algebra, then $\mathfrak{g}$ is simple and $\mathbb{M}$ is a non-Riemannian Lie triple plane.*

Proof. By (3.4), $\mathfrak{g}$ is simple. Assume that M is Riemannian, i.e. that $\mathfrak{k} = \operatorname{ad}[M, M]$ is a maximal compact subalgebra of the complex, simple Lie algebra $\mathfrak{g}$. Looking at the second part of the proof of Theorem 1.9 in Loos [16, p. 151], we see that the motion algebra $\mathfrak{g}^*$ of the antipodal plane $\mathbb{M}^*$ is isomorphic to $\mathfrak{k} \times \mathfrak{k}$. In contrast to (3.4), $\mathfrak{g}^*$ is semisimple, but not simple.

Having treated the semisimple case, we return to general Lie triple planes. The following result excludes direct products of a semisimple and an abelian Lie triple system as the underlying Lie triple system of a Lie triple plane:

(3.6) Lemma. *Let $\mathbb{M} = (M, \mathscr{M})$ be a nonabelian Lie triple plane of dimension $n = 2l$. If the radical $R = \operatorname{Rad} M$ coincides with the center Z of M, then $\mathbb{M}$ is a simple Lie triple plane, i.e. $R = Z = \{0\}$.*

Proof. If we can prove $R = \{0\}$, then $\mathbb{M}$ will be a semisimple Lie triple plane; now (3.4) shows the assertion. Aiming for a contradiction we assume $R = Z \neq \{0\}$.

(1) *$M = H \oplus Z$ is a direct sum of Z and a simple ideal $H \trianglelefteq M$ with* $\dim H \geqslant l$:

By assumption, the radical $R \neq M$ coincides with the center Z. We obtain a Levi decomposition $M = H_1 \oplus \cdots \oplus H_k \oplus Z$ of M, where $H_1, \ldots, H_k$ are simple ideals of M.

For every i, $1 \leqslant i \leqslant k$, the nontrivial group $\exp \operatorname{ad}[H_i, H_i]$ fixes the subspace $V_i := H_1 \oplus \cdots \oplus H_{i-1} \oplus H_{i+1} \oplus \cdots \oplus H_k \oplus Z$ of M pointwise. According to (2.15), the dimension of V_i is at most l. This proves $\dim H_i = \dim M - \dim V_i \geqslant l$ for every i. Together with $\dim Z \geqslant 1$ this inequality shows $2l = \dim M \geqslant k \cdot l + 1$. We conclude that $k = 1$, i.e. M is a direct sum $M = H \oplus Z$ of a simple Lie triple system H of dimension $\dim H \geqslant l$ and the center Z.

(2) Clearly, the group $\Gamma := \exp \operatorname{ad}[M, M] = \exp \operatorname{ad}[H, H]$ fixes every point of Z and no point of $H \backslash \{0\}$. Therefore, Z is the set of fixed points of Γ and thus is contained in some line, or Z is a subplane.

(3) *Z is contained in some line*:

Assume that the assertion is false. Then Z is a subplane and thus Γ is a planar group. In particular, $\Gamma = \exp \operatorname{ad}[H, H]$ is relatively compact (2.15) and H is a Riemannian Lie triple system, see (4.12). According to [15, Cor. 2 on p. 147], the representation of Γ on H is irreducible.

On the other hand, choose some nonzero element $z \in Z$. Since Γ fixes z, the line $L \in \mathscr{M}_0$ containing z is Γ-invariant. Looking at the Γ-invariant subspace $L \cap H$ of H, the irreducibility of Γ on H shows that $H \cap L = \{0\}$ or $H \leqslant L$. In both cases, $\dim H \geqslant l$ (cp. (1)) implies that $\dim H = l$ and that $\dim Z = \dim M - \dim H = l$.

Therefore, Z is a Baer subplane on which $\Gamma = \exp \operatorname{ad}[H, H]$ acts trivially. Now, a contradiction may be obtained similar to part (2) of the proof of (3.4): The dimension of the standard embedding $\operatorname{ad}[H, H] \oplus H$ equals 5, 9, or 11, and none of these numbers occurs as the dimension of the standard embedding of a simple Lie triple system.

(4) *The contradiction*:

Combining (3) and (3.3), we infer that the center Z is a proper subspace of some line $L \in \mathscr{M}_0$. Consequently, we have $\dim H = \dim M - \dim Z \geqslant l + 1$. It follows that $U := H \cap L$ is a proper, $(\operatorname{ad}[H, H])$-invariant subspace of H different from $\{0\}$. Applying [4, Lemma 4.4], we infer $\dim H = 2 \cdot \dim U$. We obtain

$$2l = \dim M = \dim H + \dim Z = \underbrace{\dim U}_{\lneqq l} + \underbrace{\dim U + \dim Z}_{\leqslant \dim L = l} \lneqq 2l,$$

and this contradiction finishes the proof.

Let $\mathbb{M} = (M, \mathscr{L})$ be a Lie triple plane. Suppose that $\mathbb{M}$ is neither abelian nor (semi)simple. Then the radical R of M is different from the center Z of M by the preceding result. This fact ensures that $\operatorname{ad}[R, M]$ contains elements different from zero. Since every element of $\operatorname{ad}[R, M]$ is a nilpotent derivation of M (see (4.4)), we obtain the following by (2.16):

(3.7) Proposition. *Let $\mathbb{M} = (M, \mathscr{L})$ be a Lie triple plane which is neither abelian nor simple. Then the nontrivial group $\exp \operatorname{ad}[R, M]$ (where R denotes the radical of M) leaves precisely one line $L \in \mathscr{M}_0$ invariant and acts freely on $\mathscr{M}_0 \backslash \{L\}$.*

We continue with a more technical lemma concerning Lie triple planes whose radicals are lines.

(3.8) Lemma. *Let* $\mathbb{M} = (M, \mathscr{M})$ *be a Lie triple plane of dimension* $n = 2l$. *Suppose that the radical* R *of* M *is a line* (*i.e.* $R \in \mathscr{M}_0$). *Recall from* (4.4) *that* $\mathfrak{r} := \mathrm{ad}[R, M] \oplus R$ *is the radical of the motion algebra* $\mathfrak{g} = \mathrm{ad}[M, M] \oplus M$ *of* M. *The adjoint representation of the Lie algebra* $\mathfrak{g}$ *on its ideal* $\mathfrak{r}$ *will be denoted by* φ. *In this situation the following assertions hold*:

a. *Every line* $L \in \mathscr{M}_0 \backslash \{R\}$ *is a* (*semisimple*) *Levi complement of* M.

b. *The action of the group* $\Sigma := \exp \mathrm{ad}[R, M]$ *on the* (Σ-*invariant*) *set* $\mathscr{M}_0 \backslash \{R\}$ *is sharply transitive. In particular, the dimension of the Lie algebra* $\mathfrak{d} := \mathrm{ad}[R, M]$ *of* Σ *coincides with* $l = \dim \mathscr{M}_0 \backslash \{R\}$.

c. *For every* $x \in M \backslash R$ *the kernel* $\ker(\varphi(x))$ *is a subspace of* R.

d. *Let* $L \in \mathscr{M}_0 \backslash \{R\}$ *and let* $a \in L \backslash \{0\}$ *be an element of some Cartan complement of the semisimple Lie algebra* $\mathrm{ad}[L, L] \oplus L$. *Then* $\varphi(a) : \mathfrak{r} \to \mathfrak{r}$ *is an isomorphism which interchanges the subspaces* $\mathfrak{d}$ *and* R *of* $\mathfrak{r}$.

e. *The Lie triple system* M *is center-free*.

Proof. (a) Every line $L \in \mathscr{M}_0 \backslash \{R\}$ is a subsystem of M which is complementary to the radical R. According to (4.9), L is a Levi complement.

(b) From part a. it follows that every two elements of $\mathscr{M}_0 \backslash \{R\}$ are conjugate by an element of Σ, cf. (4.9). Moreover, the set $\mathscr{M}_0 \backslash \{R\}$ is Σ-invariant and hence Σ acts transitively on it. In view of (3.7), this proves part b.

(c) We emphasize that the following holds for all $x \in M$:

$$\varphi(x)(\mathfrak{d}) \leqslant R \quad \text{and} \quad \varphi(x)(R) \leqslant \mathfrak{d}. \tag{3}$$

By (4.4.c), every $d \in \mathfrak{d} \backslash \{0\}$ is a nilpotent derivation of M whose image $d(M)$ is a subspace of the line R. We apply (2.17) to infer $d(L) = R$ for $L \in \mathscr{M}_0 \backslash \{R\}$. Consequently, $\varphi(x)(d) = d(x) \neq 0$ holds for all $x \in L \backslash \{0\}$. Moreover, $\varphi(x)(d + r) = \varphi(x)(d) + \varphi(x)(r) \neq 0$ holds for every $r \in R$ (use formula (3) above) and we are finished.

(d) We infer from (b) that the dimension of $\mathfrak{r}$ equals $\dim \mathrm{ad}[R, M] + \dim R = 2l$.

The representation φ of $\mathfrak{g}$ induces a representation of the semisimple Lie algebra $\mathrm{ad}[L, L] \oplus L$ on $\mathfrak{r}$. Let $a \in L \backslash \{0\}$ be an element of some Cartan complement $\mathfrak{p}$ of $\mathrm{ad}[L, L] \oplus L$. Then a is contained in some maximal abelian subalgebra $\mathfrak{a} \subseteq \mathfrak{p}$, see [17, p. 153]. In fact, the proof of Proposition 4.3 in [17, p. 159] shows that $\varphi(a)$ is diagonalizable on $\mathfrak{r}$. Choose a basis $\{d_i + r_i \mid i = 1, \ldots, 2l\}$ (with $d_i \in \mathfrak{d}$ and $r_i \in R$) and real numbers λ_i, $1 \leqslant i \leqslant 2l$, such that $\varphi(a)(d_i + r_i) = \lambda_i \cdot (d_i + r_i)$ holds for all i, $1 \leqslant i \leqslant 2l$. From formula (3) one easily sees that

$$\varphi(a)(d_i) = \lambda_i r_i \quad \text{and} \quad \varphi(a)(r_i) = \lambda_i d_i \tag{4}$$

holds for all $i, 1 \leqslant i \leqslant 2l$.

Without loss of generality we assume that $\{d_1, \dots, d_l\}$ is a basis of the l-dimensional subspace $\mathfrak{d}$. By part c, the kernel of $\varphi(a)$ has trivial intersection with $\mathfrak{d}$. Equation (4) now yields that $\lambda_i \neq 0$ for all i, $1 \leqslant i \leqslant l$, and that

$$\mathscr{B} := \{r_i = \varphi(a)(\lambda_i^{-1} d_i) \mid i = 1, \dots, l\}$$

is a linearly independent subset of the l-dimensional vector space R and hence is a basis of R. Repeating this argument shows that $\mathscr{B}$ is mapped onto a basis of $\mathfrak{d}$ and part d is proved.

(e) Choose a line $L \in \mathscr{M}_0 \backslash \{R\}$. Notice that L is semisimple. By possibly passing to the antipodal plane $\mathbb{M}^*$ we may assume that L is not a compact Lie triple system.

Let σ denote the standard involution of $\mathfrak{l} = \operatorname{ad}[L, L] \oplus L$. Choose a Cartan involution ι of $\mathfrak{l}$ which commutes with σ. Notice that L is not contained in the eigenspace $\mathfrak{k}$ of ι with respect to the eigenvalue 1; otherwise we would obtain the contradiction that $\mathfrak{l} = [L, L] + L \leqslant \mathfrak{k} \lneqq \mathfrak{l}$. Consequently, the intersection of L with the Cartan complement of $\mathfrak{l}$ (with respect to ι) contains an element a with $a \neq 0$. By part d, the map $\varphi(a) : \mathfrak{r} \to \mathfrak{r}$ is an isomorphism.

Thus, if z is an element of the center of M, then we see from $z \in R$ and from $\varphi(a)(z) = \operatorname{ad}[z, a] = 0$ that $z = 0$.

(3.9) Lemma. *Every Lie triple plane is abelian or center-free.*

Proof. Let $\mathbb{M} = (M, \mathscr{M})$ be a Lie triple plane. Aiming at a contradiction we assume that the center Z of M satisfies $Z \neq \{0\}, M$.

Let $R = \operatorname{Rad} M$ denote the radical of M. The group $\Sigma := \exp \operatorname{ad}[R, M]$ fixes some line $L \in \mathscr{M}_0$ and acts freely on $\mathscr{M}_0 \backslash \{L\}$, cf. (3.7). It follows that every fixed point x of an element $\varphi \in \Sigma \backslash \{e\}$ is an element of L, because the line $x \vee 0$ is φ-invariant. In particular, the center Z of M is a subspace of L.

We claim that $[R \cap K, K, M] = \{0\}$ holds for every $K \in \mathscr{M}_0 \backslash \{L\}$: Let $x \in R \cap K$ and $y \in K$. Then $d := \operatorname{ad}[x, y] \in \operatorname{ad}[K, K]$ leaves K invariant. Being an element of $\operatorname{ad}[R, M]$, the map d is nilpotent (4.4.c) and hence there exists an element $u \in K \backslash \{0\}$ with $d(u) = 0$. Thus, the map $\exp(d) \in \Sigma$ fixes $u \in M \backslash L$ and we infer $\exp(d) = e$. Since d is a nilpotent map, this implies $d = 0$, i.e. $[x, y, M] = \{0\}$.

We assume $R \nleqslant L$ and obtain a contradiction as follows. Let $x \in R \backslash L$, choose some element $z \in Z \backslash \{0\}$ and define $y := x + z$. Then the lines $L = 0 \vee z$, $K_x := 0 \vee x$ and $K_y := 0 \vee y$ are pairwise distinct. Notice that $x + z \in R$ and use the result above to obtain $[y, K_y, M] = [x, K_x, M] = \{0\}$. The computation

$$\begin{aligned} [x, M, M] &= [x, K_x + K_y, M] && (\text{since } K_x \neq K_y) \\ &= [x, K_y, M] && (\text{since } [x, K_x, M] = \{0\}) \\ &= [y, K_y, M] && (\text{since } x - y \in Z) \\ &= \{0\} \end{aligned}$$

shows that x is an element of $Z \cap K_x$—in contradiction to $Z \leqslant L$.

We have proved that L contains the radical R. Again, let $d \in \mathrm{ad}[R, M]$ be a nonzero element. Observe that $d(M) \leqslant [R, M, M] \leqslant R \leqslant L$, then apply (2.17) to show $R \leqslant L = d(M) \leqslant R$. Contradicting (3.8.e), the radical $R = L$ is a line and M is not center-free.

We are now ready to prove the main result:

(3.10) Proof of (1.3). Let $\mathbb{M} = (M, \mathscr{M})$ be a nonabelian Lie triple plane of dimension $n = 2l$. If M is semisimple, then $\mathbb{M}$ is a simple Lie triple plane by (3.4). If M is not semisimple, then M contains some totally abelian ideal A with $A \neq \{0\}$, M, and we have to show that $\mathbb{M}$ is a split Lie triple plane.

Since M is center-free, $(\mathrm{ad}[A, M])\backslash\{0\}$ contains at least one element d. Notice that $d(M)$ is contained in A, because A is an ideal.

By (4.7), $\mathrm{ad}[M, A]$ consists of nilpotent elements. Thus, (2.16) shows that the corresponding group $\Gamma = \exp \mathrm{ad}[M, A]$ leaves precisely one line $L \in \mathscr{M}_0$ invariant. We conclude that the totally abelian ideal A (which is fixed pointwise by Γ, see (4.7)) is contained in L.

Consequently, the subspace $d(M)$ of A is contained in L, too, and (2.17) shows that $d(M) = L$. Now, $L = d(M) \subseteq A \subseteq L$ implies that $A = L$ is a line. Just by definition, $\mathbb{M}$ is a split Lie triple plane.

In fact, (3.10) shows the following:

(3.11) Proposition. *The splitting line of a split Lie triple plane* $\mathbb{M}$ *is the only nontrivial totally abelian ideal of* $\mathbb{M}$.

4 Appendix: Facts concerning Lie triple systems

(4.1) Notation and conventions. Throughout this section, let T be a Lie triple system, i.e. a finite-dimensional real vector space T equipped with a trilinear map $[\cdot,\cdot,\cdot] : T^3 \to T$, the *triple bracket*, which satisfies the following conditions:

(1) $[x, y, z] = -[y, x, z]$;

(2) $[x, y, z] + [y, z, x] + [z, x, y] = 0$ (Jacobi identity);

(3) The maps $\mathrm{ad}[x, y] : T \to T; z \mapsto [x, y, z]$ are derivations[7] of T.

We emphasize that the more complicated definition of a Lie triple system given in [4] is equivalent to the one above, see [24].

The set of all derivations is a subalgebra $\mathrm{Der}(T)$ of the Lie algebra $\mathfrak{gl}(T)$. Homomorphisms of Lie triple systems are defined in the obvious way; the group of automorphisms of T will be denoted by $\mathrm{Aut}(T)$. Notice that $\mathrm{Aut}(T)$ is a closed subgroup of $\mathrm{GL}(T)$ with Lie algebra $\mathrm{Der}(T)$, cp. [4].

[7] A (trilinear) derivation is a linear map $d : T \to T$ such that $d([x, y, z]) = [d(x), y, z] + [x, d(y), z] + [x, y, d(z)]$ holds for all $x, y, z \in T$.

If X, Y, Z are vector subspaces of T, then we shall write $[X, Y, Z]$ for the vector subspace of T generated by the set $\{[x, y, z] \mid x \in X, y \in Y, z \in Z\}$. The vector subspace of $\mathrm{Der}(T)$ generated by the set $\{\mathrm{ad}[x, y] \mid x \in X, y \in Y\}$ will be denoted by $\mathrm{ad}[X, Y]$.

One easily obtains that $\mathrm{ad}[T, T]$ is an ideal of the Lie algebra $\mathrm{Der}(T)$; the elements of this ideal are called *inner derivations* of T. We refer to the subgroup $\exp \mathrm{ad}[T, T]$ of $\mathrm{Aut}(T)$ as the *group of inner automorphisms* of T.

An *ideal* of T is a vector subspace $B \leqslant T$ satisfying $[B, T, T] \subseteq B$.

Endowing a Lie algebra $\mathfrak{g}$ with the bracket $[x, y, z] := [[x, y], z]$ yields a Lie triple system $\mathscr{T}(\mathfrak{g}) := (\mathfrak{g}, [\cdot, \cdot, \cdot])$. If σ is an involutive automorphism of $\mathfrak{g}$ and if $\mathfrak{g}_-$ is the eigenspace of σ with respect to the eigenvalue -1, then $\mathscr{T}(\mathfrak{g}, \sigma) := \mathfrak{g}_-$ is a subsystem of $\mathscr{T}(\mathfrak{g})$.

(4.2) Embeddings. An *embedding* of a Lie triple system T into the Lie algebra $\mathfrak{g}$ is an injective homomorphism $\iota : T \to \mathscr{T}(\mathfrak{g})$ such that $\iota(T)$ generates $\mathfrak{g}$ as a Lie algebra. We shall identify T and $\iota(T)$ whenever no confusion can occur. The *standard embedding* of T is the natural embedding $T \to \mathrm{ad}[T, T] \oplus T$, where the vector space $\mathrm{ad}[T, T] \oplus T$ is endowed with the Lie bracket defined by

$$[x, y] := \begin{cases} xy - yx & \text{if } x, y \in \mathrm{ad}[T, T], \\ x(y) & \text{if } x \in \mathrm{ad}[T, T] \text{ and } y \in T, \\ -y(x) & \text{if } x \in T \text{ and } y \in \mathrm{ad}[T, T], \\ \mathrm{ad}[x, y] & \text{if } x, y \in T, \end{cases}$$

see [4]. The *standard involution* $\sigma : \mathrm{ad}[T, T] \oplus T \to \mathrm{ad}[T, T] \oplus T;\ d + x \mapsto d - x$ is an involutive automorphism of the Lie algebra $\mathrm{ad}[T, T]$; notice that T equals $\mathscr{T}(\mathrm{ad}[T, T] \oplus T, \sigma)$.

Remark. The notion of embeddings requires neither that $T \cap [T, T] = \{0\}$ nor that T is the (-1)-eigenspace of an involution, nor that $x \in [T, T] \backslash \{0\}$ implies $\mathrm{ad}\, x \neq 0$ on T. Examples illustrating this are (1) the embedding $\mathrm{id} : \mathscr{T}(\mathfrak{g}) \to \mathfrak{g}$ for any Lie algebra $\mathfrak{g}$ and (2) the embedding of the trivial Lie triple system with basis x, y into the Lie algebra with basis x, y, z, where $z = [x, y]$ is central.

(4.3) The radical. Let T be embedded in a Lie algebra $\mathfrak{g} = [T, T] + T$. Following Lister [4], we introduce the *derived series* of an ideal $B \trianglelefteq T$ by putting recursively $B^{(0)} = B$ and $B^{(k+1)} := [T, B^{(k)}, B^{(k)}]$. According to [4, Lemma 2.1], every $B^{(k)}$ is an ideal of T. An ideal $B \trianglelefteq T$ is called *solvable in* T, if there exists an element $k \in \mathbb{N}$ with $B^{(k)} = 0$. Since the sum of two solvable ideals is solvable again ([4, Lemma 2.2]), we obtain a unique maximal ideal R of T which is solvable in T. We refer to R as the *radical* of T and shall write $\mathrm{Rad}(T) := R$. A Lie triple system T is called *solvable* if $\mathrm{Rad}(T) = T$ and it is called *semisimple* if $\mathrm{Rad}(T) = \{0\}$.

The radical may be derived from the radical of any embedding of T:

(4.4) Theorem. *Let T be a Lie triple system which is embedded in a Lie algebra $\mathfrak{g} = [T, T] + T$. Put $R := \mathrm{Rad}(T)$ and let $\mathfrak{r}$ denote the radical of $\mathfrak{g}$.*

a. *The intersection of* $\mathfrak{r}$ *and* T *coincides with* R. *Conversely*, *the ideal* $[R, T] + R \trianglelefteq \mathfrak{g}$ *generated by* R *coincides with* $\mathfrak{r}$.

b. T *is solvable* [*respectively*, *semisimple*] *if and only if* $\mathfrak{g}$ *is a solvable* [*respectively*, *semisimple*] *Lie algebra*.

c. *If* x *is an element of* $[R, T]$, *then* $d = (\operatorname{ad} x)|_T$ *is a nilpotent derivation of* T *satisfying* $d(T) \subseteq R$.

Proof. For (a) we refer to Lister, [4, Lemma 2.15, Theorem 2.16]. Part (b) is a consequence of (a). We proceed with part (c): By Bourbaki [1, Th. 1, p. 45; Cor. 7, p. 47], the map $\operatorname{ad} x$ is a nilpotent derivation of $\mathfrak{g}$ for every $x \in \mathfrak{u} := [\mathfrak{g}, \mathfrak{r}]$, because $\mathfrak{u}$ is contained in the nilradical of $\mathfrak{g}$. The assertion follows from the observation $[R, T] \subseteq [\mathfrak{r}, \mathfrak{g}]$.

(4.5) Totally abelian ideals. A *totally abelian ideal* of T is an ideal $A \trianglelefteq T$ satisfying $[T, A, A] = 0$. The *center* $Z := \{z \in T \mid [z, T, T] = 0\}$ of the Lie triple system T is an example. We emphasize that every totally abelian ideal is contained in the radical.

If A is a totally abelian ideal, then $\mathfrak{a} = \operatorname{ad}[T, A] \oplus A$ is an abelian ideal of the standard embedding of T. Conversely, the intersection of a σ-invariant abelian ideal of $\operatorname{ad}[T, T] \oplus T$ and T is a totally abelian ideal of T.

We include two results on totally abelian ideals:

(4.6) Lemma. *A Lie triple system* T *is semisimple if and only if every totally abelian ideal of* T *vanishes*.

Proof. If A is a totally abelian ideal of T, then $\operatorname{ad}[A, T] \oplus T$ is an abelian ideal of the standard embedding $\mathfrak{g}$ of T. If T is semisimple, then $\mathfrak{g}$ is a semisimple Lie algebra and we infer $A = \{0\}$. Conversely, suppose that T is not semisimple and let R be the radical of T. Then there exists a number k such that $R^{(k)} \neq \{0\}$ and $R^{(k+1)} = [T, R^{(k)}, R^{(k)}] = \{0\}$. Therefore, $R^{(k)}$ is a nonvanishing totally abelian ideal of T.

(4.7) Lemma. *Let* A *be a totally abelian ideal of the Lie triple system* T. *Then* $d^2 = 0$ *holds for every element* $d \in \operatorname{ad}[T, A]$. *In particular*, $\operatorname{ad}[T, A]$ *consists of nilpotent elements*. *Moreover*, *the closure of the group* $\exp \operatorname{ad}[T, A]$ *in* $\mathrm{GL}(T)$ *fixes* A *pointwise*.

Proof. If d is an element of $\operatorname{ad}[T, A]$, then $d|_A = 0$, because $d(A)$ lies in $[T, A, A] = \{0\}$. This implies that $\exp \operatorname{ad}[T, A]$ fixes A pointwise; and so does the closure of $\exp \operatorname{ad}[T, A]$ in $\mathrm{GL}(T)$.

Computing $[T, A, [T, A, T]] \leqslant [T, A, A] = \{0\}$ we derive that $d_1 d_2 = 0$ holds for all elements $d_1, d_2 \in \operatorname{ad}[T, A]$.

(4.8) The Levi decomposition. Let S be a semisimple subsystem of T which is complementary to the radical $R = \operatorname{Rad}(T)$. Then L is called a *Levi complement* of T. We refer to the vector space decomposition $T = S \oplus R$ as a *Levi decomposition* of T. If T is embedded in a Lie algebra $\mathfrak{g} = [T, T] + T$, then every Levi decomposition $T = S \oplus R$ extends to a Levi decomposition $\mathfrak{g} = ([S, S] + S) \oplus ([R, T] + R)$ of $\mathfrak{g}$.

(4.9) Theorem. *Let T be a Lie triple system with radical R.*

a. *There exists a Levi decomposition of T.*

b. *Every subsystem S of T which is complementary to R is a Levi complement of T.*

c. *If S_1 and S_2 are Levi complements of T, then there exists an element $d \in \operatorname{ad}[R, T]$ such that* $\exp(d)(S_1) = S_2$.

Remark. Part (c) of the theorem should be a well known result. Nevertheless, I did not find a proof for this in the literature.

Proof. See Lister [4, Thm. 2.21] for (a). For (b), choose a Levi complement L of M and observe that $L \cong T/R \cong S$ is semisimple. It remains to show (c):

(1) Let $\mathfrak{g}$ be the standard embedding of T and let σ be the standard involution. In order to avoid confusion, we will write $\mathfrak{g} = [T, T] \oplus T$ (instead of $\operatorname{ad}[T, T] \oplus T$).

Let $\mathfrak{r} = [R, T] \oplus R$ denote the radical of $\mathfrak{g}$. Then the ideal $[\mathfrak{g}, \mathfrak{r}]$ of $\mathfrak{g}$ is σ-invariant (because $\mathfrak{r}$ is). By [1, Thm. 1, p. 45], $\operatorname{ad}[\mathfrak{g}, \mathfrak{r}]$ consists of nilpotent derivations of $\mathfrak{g}$. In particular, the exponential function $\exp : \operatorname{ad}[\mathfrak{g}, \mathfrak{r}] \to \exp \operatorname{ad}[\mathfrak{g}, \mathfrak{r}] =: \Gamma$ is bijective ([23, 3.6.2]). We put $e^{\operatorname{ad} x} := \exp \operatorname{ad}(x)$ for $x \in [\mathfrak{g}, \mathfrak{r}]$.

Notice that the center $\mathfrak{z}$ of $\mathfrak{g}$ is contained in T, because $[T, T]$ contains no ideal of $\mathfrak{g}$ except $\{0\}$.

(2) For $j \in \{1, 2\}$, the subalgebra $\mathfrak{h}_j = [S_j, S_j] + S_j$ generated by S_j is a σ-invariant Levi complement of $\mathfrak{g}$. By the theorem of Levi–Malcev ([1, Thm. 5, p. 63]), there exists an element $x \in [\mathfrak{g}, \mathfrak{r}]$ such that $e^{\operatorname{ad} x}(\mathfrak{h}_1) = \mathfrak{h}_2$. Because $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are σ-invariant, we infer $e^{\operatorname{ad} \sigma(x)}(\mathfrak{h}_1) = \sigma e^{\operatorname{ad} x} \sigma(\mathfrak{h}_1) = \mathfrak{h}_2$. This implies that the map $\varphi := e^{-\operatorname{ad} \sigma(x)} e^{\operatorname{ad} x}$ leaves $\mathfrak{h}_1$ invariant.

(3) Choose an element $y \in [\mathfrak{g}, \mathfrak{r}]$ with $\varphi = e^{\operatorname{ad} y}$. Then

$$e^{\operatorname{ad} \sigma(y)} = \sigma e^{\operatorname{ad} y} \sigma = \sigma e^{-\operatorname{ad} \sigma(x)} e^{\operatorname{ad} x} \sigma = e^{-\operatorname{ad} x} e^{\operatorname{ad} \sigma(x)} = e^{-\operatorname{ad} y}$$

shows that $\operatorname{ad} \sigma(y) = \operatorname{ad}(-y)$, since $\exp : \operatorname{ad}[\mathfrak{g}, \mathfrak{r}] \to \Gamma$ is bijective. Consequently, $y + \sigma(y)$ is an element of $\mathfrak{z}$ which is fixed by σ. Since $\mathfrak{z}$ is a subspace of T, we derive $\sigma(y) = -y$ and, hence, $\sigma(y/2) = -y/2$. Now $(e^{\operatorname{ad} y/2})^2 = e^{\operatorname{ad} y} = e^{-\operatorname{ad} \sigma(x)} e^{\operatorname{ad} x}$ implies that

$$e^{\operatorname{ad} x} e^{-\operatorname{ad} y/2} = e^{\operatorname{ad} \sigma(x)} e^{\operatorname{ad} y/2} = \sigma e^{\operatorname{ad} x} e^{-\operatorname{ad} y/2} \sigma. \tag{5}$$

Moreover, $e^{\operatorname{ad} y}$ leaves $\mathfrak{h}_1$ invariant. Since the nilpotent map $\operatorname{ad} y$ can be expressed as a polynomial in $e^{\operatorname{ad} y}$ (see formula (1) on p. 7 for the logarithm), we conclude that $\operatorname{ad} y$ leaves $\mathfrak{h}_1$ invariant, too. It follows that also $e^{-\operatorname{ad} y/2}$ leaves $\mathfrak{h}_1$ invariant.

(4) Choose $z \in [\mathfrak{g}, \mathfrak{r}]$ with $e^{\operatorname{ad} z} = e^{\operatorname{ad} x} e^{-\operatorname{ad} y/2}$. From equation (5) above we infer that $e^{\operatorname{ad} \sigma(z)} = \sigma e^{\operatorname{ad} z} \sigma = e^{\operatorname{ad} z}$. Consequently, $z - \sigma(z)$ is an element of the center $\mathfrak{z}$. Setting $d := (z + \sigma(z))/2$ and $c := (z - \sigma(z))/2$ yields the decomposition $z = d + c$ with $d \in [T, R]$ (because $\sigma(d) = d$) and $c \in \mathfrak{z}$. We claim that $e^{\operatorname{ad} d}(\mathfrak{h}_1) = \mathfrak{h}_2$—then $e^{\operatorname{ad} d}(S_1) = e^{\operatorname{ad} d}(\mathfrak{h}_1 \cap T) = e^{\operatorname{ad} d}(\mathfrak{h}_1) \cap e^{\operatorname{ad} d}(T) = \mathfrak{h}_2 \cap T = S_2$ shows the assertion. First, observe

that $e^{\operatorname{ad} d} = e^{\operatorname{ad}(d+c)} = e^{\operatorname{ad} z}$, because c is an element of the center of $\mathfrak{g}$. Since $e^{-\operatorname{ad} y/2}$ leaves $\mathfrak{h}_1$ invariant (cf. (3)), we obtain $e^{\operatorname{ad} d}(\mathfrak{h}_1) = e^{\operatorname{ad} z}(\mathfrak{h}_1) = e^{\operatorname{ad} x}e^{-\operatorname{ad} y/2}(\mathfrak{h}_1) = e^{\operatorname{ad} x}(\mathfrak{h}_1) = \mathfrak{h}_2$.

(4.10) Semisimple Lie triple systems. Let T be embedded in the Lie algebra $\mathfrak{g} = [T, T] + T$ such that $[T, T] \cap T = \{0\}$. If T is semisimple, then $\mathfrak{g}$ is isomorphic to the standard embedding of T (this is a consequence of [4, Thm. 2.7], [3, Thm. 7.3]). If, in particular, T is a subsystem of a Lie triple system L, then the subalgebra $\operatorname{ad}_L[T, T] \oplus T$ of the standard embedding of L is isomorphic to the standard embedding of T. Moreover, every derivation of a semisimple Lie triple system is inner, see [4, Thm. 2.17].

An at least 2-dimensional Lie triple system T is called *simple*, if it contains no proper ideal. According to [4, Thm. 2.9], every semisimple Lie triple system T is the direct sum of simple ideals. Conversely, the direct sum of simple Lie triple systems is semisimple.

Let T be a simple Lie triple system with standard embedding $\mathfrak{g}$ and standard involution σ. Then $\mathfrak{g}$ is a semisimple Lie algebra. Let $\mathfrak{h}$ be a simple ideal of $\mathfrak{g}$. Observe that $\mathfrak{h} + \mathfrak{h}^\sigma$ is a σ-invariant ideal of $\mathfrak{g}$. Since $\operatorname{ad}[T, T]$ contains no proper ideal of $\mathfrak{g}$, we infer that $(\mathfrak{h} + \mathfrak{h}^\sigma) \cap T$ is a nonvanishing ideal of T. Thus, T is a subset of $\mathfrak{h} + \mathfrak{h}^\sigma$, whence $\mathfrak{h} + \mathfrak{h}^\sigma = \mathfrak{g}$. It may happen that $\mathfrak{h} = \mathfrak{h}^\sigma$, and we conclude: The standard embedding $\mathfrak{g}$ of a simple Lie triple system T either is a simple Lie algebra, or $\mathfrak{g}$ is isomorphic to a direct sum of two isomorphic Lie algebras. In the latter case, the standard involution interchanges the simple summands.

(4.11) Riemannian Lie triple systems. We call a Lie triple system *Riemannian*, if its group of inner automorphisms is compact. (The name indicates that Riemannian Lie triple systems are precisely the tangent objects of Riemannian symmetric spaces, cp. Loos [15, Chap. 4].)

By Loos [15, p. 145], every Riemannian Lie triple system T has a unique decomposition $T = T_+ \oplus T_- \oplus T_0$ into ideals T_0, T_+, T_-, where T_0 is the center of T, where T_+ is a of noncompact type (that means that T_+ is a Cartan complement of its standard embedding $\operatorname{ad}[T_+, T_+] \oplus T_+$), and where T_- is of compact type (i.e. the standard embedding of T_- is a compact semisimple Lie algebra).

(4.12) Lemma. *A Lie triple system T is Riemannian if and only if its group of inner automorphisms is relatively compact in* $\mathrm{GL}(T)$.

Proof. Let T be a Lie triple system. Assume that $\Delta = \exp \operatorname{ad}[T, T]$ is a relatively compact subgroup of $\mathrm{GL}(T)$. In order to prove that Δ is compact it sufficies to show that Δ is closed in $\mathrm{GL}(T)$.

(1) First, suppose that T is semisimple. Then $\operatorname{ad}[T, T]$ equals the Lie algebra of the group $\mathrm{Aut}(T)$. Therefore, Δ and the connected component of $\mathrm{Aut}(T)$ coincide. Since $\mathrm{Aut}(T)$ is a closed subgroup of $\mathrm{GL}(T)$, its connected component Δ is closed, too.

(2) Suppose that $T = L \oplus Z$ is a direct sum of a semisimple Lie triple system L and the center Z of T. Then $\operatorname{ad}[T, T]$ and $\operatorname{ad}[L, L]$ coincide. Moreover, $\operatorname{ad}[L, L]$ acts

trivially on Z. Therefore, Δ is isomorphic to $\exp \mathrm{ad}_L[L, L]$, whence Δ is a compact group.

(3) We turn to the general case: Let R denote the radical of T. Choose a Levi decomposition $T = L \oplus R$ of T. According to (4.4.c), every element $d \in \mathrm{ad}[R, T]$ is a nilpotent derivation of T. This implies that d vanishes, because the closure of $\exp(\mathbb{R} \cdot d)$ in $\mathrm{GL}(T)$ is compact. We conclude that the radical R and the center of T coincide. According to (2), Δ is a compact group and T is a Riemannian Lie triple system.

(4.13) Corollary. *Subsystems of Riemannian Lie triple systems are Riemannian.*

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