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On special projections of varieties: *epitome* to a theorem of Beniamino Segre

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Abstract. In this paper we make some comments and improvements on a theorem of Beniamino Segre, concerning the locus of points from which an algebraic variety is not projected generically one-to-one.

1 Introduction

A well-known and useful technique in algebraic geometry is the linear projection of a given projective variety $X \subset \mathbb{P}^r$, which we will usually assume to be irreducible, reduced and non-degenerate, i.e. not contained in any proper subspace of $\mathbb{P}^r$. It is clear that the projection π from a general point of the ambient space of a variety X, which is not a hypersurface, is such that $\pi_{|X}$ is generically one-to-one, i.e. it is birational to its image. For example, if $n := \dim X < r - 1$, by applying $r - n - 1$ such projections, one may consider X as birationally equivalent to a hypersurface in $\mathbb{P}^{n+1}$.

However, it may be interesting to know also what is the locus $\mathfrak{S}(X)$ of points from which X is *projected multiply*, i.e. the locus of points from which the projection of X is not generically one-to-one. Since Beniamino Segre already studied in [4] the properties of $\mathfrak{S}(X)$, we will call it the *Segre locus* of X. More precisely, we say that the projection $\pi_z : \mathbb{P}^r \to \mathbb{P}^{r-1}$ from a point $z \notin X$ onto a hyperplane $\mathbb{P}^{r-1}$ not passing through z is a *special projection* of X if $\pi_{z|X}$ is not generically one-to-one. We define the Segre locus of an irreducible, reduced, algebraic variety $X \subset \mathbb{P}^r$ as:

$$\mathfrak{S}(X) := \overline{\{z \in \mathbb{P}^r - X : \pi_z \text{ is a special projection of } X\}}.$$

where $\overline{Y}$ is the Zariski closure of the subset Y in $\mathbb{P}^r$. For example, if $X \subset \mathbb{P}^{n+1}$ is a hypersurface (of degree > 1), then $\mathfrak{S}(X) = \mathbb{P}^{n+1}$. In [4] Segre proved the following:

Theorem 1. *Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, algebraic variety of dimension $n < r - 1$. Then the Segre locus $\mathfrak{S}(X)$ is the union of finitely many linear subspaces of $\mathbb{P}^r$ and all of its irreducible components have dimension strictly less than n. Furthermore, a linear k-space $\Pi \subset \mathbb{P}^r$, with $0 < k < n$, is contained in $\mathfrak{S}(X)$ if and only if either one of the following equivalent properties holds*:

(i) *X lies on an $(n+1)$-dimensional cone with vertex at* Π;

(ii) *the tangent space to X at a general point cuts Π in a subspace of dimension $k-1$.*

Finally Π is an irreducible component of $\mathfrak{S}(X)$ if and only if Π enjoys either (i) *or* (ii) *and it is maximal under this condition. This in turn happens if and only if the maximal vertex of the cone in* (i) *coincides with* Π.

Recall that a variety $X \subset \mathbb{P}^r$ is a *cone* if there is a point $z \in X$ such that for every other point $x \in X$ the line joining x and z lies in X. In this case z is called a *0-dimensional vertex* of X. The set of 0-dimensional vertices of X is a subspace of $\mathbb{P}^r$ called the *maximal vertex* of X. Any subspace of the maximal vertex is called a *vertex* of X.

Theorem 1 implies the following corollary, which has somewhat unexpected applications to problems in numerical algebraic geometry, as shown by Sommese, Verschelde and Wampler in [7]. Let $c = r - n - 1$ and let $\underline{x} = (x_1, \dots, x_c)$ be a point of X^c. Let us denote by $\mathrm{Cone}(X; \underline{x})$ the cone over X with vertex at the linear subspace of $\mathbb{P}^r$ spanned by $x_1, \dots, x_c$.

Corollary 2. *Let X be as in Theorem* 1. *Then*:

$$\bigcap_{\underline{x} \in X^c \setminus \Delta} \mathrm{Cone}(X; \underline{x}) = X \cup \mathfrak{S}(X)$$

where Δ is the set of $\underline{x} = (x_1, \dots, x_c) \in X^c$ such that the linear space spanned by $x_1, \dots, x_c$ has dimension strictly less than $c-1$ (if $c = 1$ then $\Delta = \emptyset$).

Note that Δ in the above statement is a proper subset of X^c by the General Position Theorem (see [1], pg. 109).

Section 2 will be devoted to explain some general properties of the Segre locus, to revise Segre's proof of Theorem 1 and to prove Corollary 2.

Following Segre, we define a zero- (resp. positive) dimensional component of $\mathfrak{S}(X)$ to be a *centre* (resp. an *axis*) of X. In [5] Segre shows that for every l, n, r such that $0 < l < n \leqslant r - 2$ and for every $m > 0$, there exists an irreducible algebraic variety $X \subset \mathbb{P}^r$ of dimension n such that X has an axis of dimension l and moreover m centers. He also studies the possible configurations of the axes of a surface. By extending some of these results, in section 3 we will show the following improvement of Theorem 1:

Theorem 3. *Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, algebraic variety of dimension $n < r-1$ which is not a cone. Then the Segre locus $\mathfrak{S}(X)$ is the disjoint union of finitely many linear subspaces of $\mathbb{P}^r$. If Π_1 and Π_2 are two distinct axes of X, then* $\dim(\Pi_1) + \dim(\Pi_2) \leqslant n + 1$. *Moreover*

(i) *If $\dim(\Pi_1) + \dim(\Pi_2) \geqslant n$, then Π_1 and Π_2 are the only axes of X, $n = r - 2$ and X is the complete intersection of two cones of dimension $n+1$ with vertices at Π_1 and Π_2.*

(ii) *If* $\dim(\Pi_1) + \dim(\Pi_2) = n + 1$ *then* $\mathfrak{S}(X) = \Pi_1 \cup \Pi_2$.

In the last section 4 we discuss some open problems.

2 Properties of the Segre locus

In this paper, $X \subset \mathbb{P}^r$ will always be an irreducible, reduced, algebraic variety of dimension n, where $\mathbb{P}^r = \mathbb{P}^r(\mathbb{C})$ is the projective r-dimensional space over the complex numbers. If x is a smooth point of X, we will denote by $T_{X,x}$ the projective tangent space to X at x. If Z is any subset of $\mathbb{P}^r$, we denote, as usual, by $\mathrm{Span}(Z)$ the smallest linear subspace of $\mathbb{P}^r$ containing Z.

If Y is a variety in $\mathbb{P}^r$ and Π is a subspace of $\mathbb{P}^r$ of dimension l, we denote by $\mathrm{Cone}(Y, \Pi)$ the cone over Y with vertex at Π, that is the Zariski closure of the union of all the $\mathbb{P}^{l+1}$'s joining Π with a point in $Y - \Pi \cap Y$.

Let X be a cone. If Π is a vertex of dimension l of X and Y is the intersection of X with a $\mathbb{P}^m$ independent of Π, for $r > m \geqslant r - l - 1$, then $X = \mathrm{Cone}(Y, \Pi)$. If $m = r - l - 1$, then Π is the maximal vertex of X if and only if Y is not a cone.

We can now list some basic properties of the Segre locus:

Lemma 4. *Let X be an irreducible, reduced, projective variety of dimension n in $\mathbb{P}^r$. Then*:

(i) *for every* $x \in X$, $\mathfrak{S}(X) \subseteq \mathrm{Cone}(X, x) \subseteq \mathrm{Span}(X)$;

(ii) $\dim \mathfrak{S}(X) \leqslant n + 1$;

(iii) *if* $X = \mathrm{Span}(X)$, *i.e. if X is a linear subspace of* $\mathbb{P}^r$, *then* $\mathfrak{S}(X) = \emptyset$;

(iv) $\dim(\mathfrak{S}(X)) = n + 1$ *if and only if* $\dim(\mathrm{Span}(X)) = n + 1$, *i.e. if and only if* $\mathfrak{S}(X) = \mathrm{Span}(X)$;

(v) *if* Π *is a subspace of* $\mathbb{P}^r$ *such that* $\mathrm{Span}(X) \cap \Pi = \emptyset$, *then* $\mathfrak{S}(\mathrm{Cone}(X, \Pi)) = \mathrm{Cone}(\mathfrak{S}(X), \Pi)$;

(vi) *if* Π *is a hyperplane of* $\mathbb{P}^r$ *such that* $X \cap \Pi$ *is irreducible and reduced, then* $\mathfrak{S}(X) \cap \Pi \subseteq \mathfrak{S}(X \cap \Pi)$;

(vii) *if* $z \in \mathbb{P}^r - (X \cup \mathfrak{S}(X))$, *set* $\pi = \pi_z$, *and suppose that for an irreducible component Z of $\mathfrak{S}(X)$, $\pi(Z)$ is not contained in $\pi(X)$. Then $\pi(Z) \subseteq \mathfrak{S}(\pi(X))$. In particular, if $z \in \mathbb{P}^r$ is a general point, then $\pi(\mathfrak{S}(X)) \subseteq \mathfrak{S}(\pi(X))$.*

Proof. (i) and (ii): By definition of the Segre locus, if $z \in \mathfrak{S}(X)$, the line Λ joining z with a general point $x \in X$ intersects X also at another point y, hence $z \in \Lambda \subseteq \mathrm{Cone}(X, x) \subseteq \mathrm{Span}(X)$. Moreover $\dim(\mathrm{Cone}(X, x)) \leqslant n + 1$.

(iii): For every $z \in \mathbb{P}^r - X$, any line through z intersects X in almost one point, thus π_z is not a special projection of X.

(iv): By (i) and (iii) we may assume that $\mathrm{Span}(X) = \mathbb{P}^r$ and $r > n$. If $\dim(\mathfrak{S}(X)) = n + 1$, then $\mathfrak{S}(X) = \mathrm{Cone}(X, x)$ for every $x \in X$ by (i). Hence the maximal vertex of the cone $\mathfrak{S}(X)$ coincides with $\mathrm{Span}(X)$, thus $\mathfrak{S}(X) = \mathbb{P}^r$. If $r = n + 1$, then a general line through any point $z \notin X$ intersects X in d points, where $d = \deg(X) > 1$, thus $z \in \mathfrak{S}(X)$, hence $\mathfrak{S}(X) = \mathbb{P}^r$. Finally, if $\mathfrak{S}(X) = \mathrm{Span}(X)$, then $\dim(\mathrm{Span}(X)) = \dim(\mathfrak{S}(X)) = n + 1$ by (ii) and (iii).

(v): We may assume that $\mathrm{Span}(X)$ and Π span the whole of $\mathbb{P}^r$.

Let $z \in \mathfrak{S}(X)$ and let w be a general point on the line joining z and a point $v \in \Pi$. Since π_z is a special projection of X, the line Λ joining z with a general point $x \in X$ intersects X at another point $y \in X$. Notice that Λ lies in $\mathrm{Span}(X)$, hence $\Lambda \cap \Pi = \emptyset$. Thus the lines joining v with x and y are distinct. Therefore the line joining w with a point of a general line $\overline{vx}$ of $\mathrm{Cone}(X, \Pi)$ intersects this cone at another point lying on the line $\overline{vy}$. This means that $w \in \mathfrak{S}(\mathrm{Cone}(X, \Pi))$.

Conversely, let $z \in \mathfrak{S}(\mathrm{Cone}(X, \Pi)) - \mathrm{Cone}(X, \Pi)$. The line Λ joining z with a general point $x \in \mathrm{Cone}(X, \Pi)$ intersects X at another point $y \in \mathrm{Cone}(X, \Pi)$. Notice that $\Lambda \cap \Pi = \emptyset$ otherwise Λ would be contained in $\mathrm{Cone}(X, \Pi)$, whereas $z \notin \mathrm{Cone}(X, \Pi)$. Let π be the projection of $\mathbb{P}^r$ from Π to $\mathrm{Span}(X)$. Then the line $\pi(\Lambda)$ contains $z' = \pi(z) \notin X$ and $\pi(x)$, which is a general point of X, and it also contains $\pi(y) \in X$; that implies that $z' \in \mathfrak{S}(X)$.

(vi): Let $z \in \Pi \cap \mathfrak{S}(X) - \Pi \cap X$. By the assumption that $X \cap \Pi$ is reduced, the line joining z with a general point $x \in X \cap \Pi$ intersects X, hence $X \cap \Pi$, at another point y, therefore $x \in \mathfrak{S}(X \cap \Pi)$.

(vii): Let $w \in Z$ be a general point. We may assume that $\pi(w) = w$ by choosing the hyperplane which π projects onto. Notice that $\pi(w) = w \notin \pi(X)$ by our assumption. Let $x \in X$ be a general point and set $y = \pi(x)$. Then the line joining w with x intersects X at another point x' (not lying on the line $\overline{zx}$), hence the line joining w with y intersects $\pi(X)$ also in $y' = \pi(x') \neq y$. The final assertion easily follows.

Notice that, in view of Lemma 4, we may and will assume, without loss of generality, that X is non-degenerate. We may also ignore, from now on, the trivial cases in which X is either a projective space or a hypersurface or a cone.

For the proof of Segre's theorem 1 we need to recall the following result from [2], Proposition 4.1:

Proposition 5. *Let X be an irreducible, reduced, projective variety of dimension n in $\mathbb{P}^r$ and let Π be a subspace of $\mathbb{P}^r$ of dimension l. The tangent space $T_{X,x}$ to X at a general point $x \in X$ intersects Π in a subspace of dimension h if and only if the projection Y of X from Π to a $\mathbb{P}^{r-l-1}$ has dimension $n - h - 1$. This in turn happens if and only if X sits on an $(n + l - h)$-dimensional cone with vertex at Π, namely on the* $\mathrm{Cone}(Y, \Pi)$. *In particular*:

- (i) *in case $h = l$: X is a cone with vertex at Π if and only if the tangent space $T_{X,x}$ to X at a general point $x \in X$ contains Π;*
- (ii) *in case $h = n$: X is contained in Π if and only if the tangent space $T_{X,x}$ to X at a general point $x \in X$ is contained in Π;*
- (iii) *in case $h = n - 1$: X is contained in a $\mathbb{P}^{l+1}$ containing Π if and only if the tangent space $T_{X,x}$ to X at a general point $x \in X$ intersects Π in a subspace of dimension $n - 1$.*

Now we are ready for the:

Proof of Theorem 1. Notice that properties (i) and (ii) in the statement of Theorem 1 are equivalent by Proposition 5.

Let Z be an irreducible component of $\mathfrak{S}(X)$. Fix a general point $x \in X$. For a general point $z \in Z$, the line Λ_z joining z with x intersects X at another point y. Notice that both $T_{X,x}$ and $T_{X,y}$ lie in a subspace of dimension $n+1$ which is tangent to $\mathrm{Cone}(X,z)$ along the line Λ_z. Let Y be an irreducible component of the Zariski closure of the locus of such y's as z varies in Z. Then $h := \dim Y \leqslant n$ and $T_{Y,y} \subseteq T_{X,y}$, hence $T_{Y,y}$ intersects $T_{X,x}$ in a subspace of dimension $h-1$, therefore Y is contained in a subspace Π_x of dimension $n+1$ containing $T_{X,x}$ by Proposition 5, (iii). Now Π_x contains both x and y hence it contains the line Λ_z, thus Π_x contains the general point $z \in Z$, i.e. $Z \subseteq \Pi_x$. Notice that Π_x depends on x, otherwise X would be contained in $\Pi_x = \mathbb{P}^{n+1}$, against the assumptions. Therefore $\Pi = \bigcap_{x \in X} \Pi_x$ is a linear subspace of dimension $l \leqslant n$ which contains Z. For every $x \in X$, the tangent space $T_{X,x}$ intersects Π in a subspace of dimension $l-1$, because both $T_{X,x}$ and Π lie in Π_x. The case $l = n$ is ruled out by Proposition 5 and the assumptions. Hence $l < n$ and X is contained in a cone with vertex at Π by Proposition 5, (i). Finally π_z is a special projection of X for every $z \in \Pi$, thus $Z = \Pi$.

At this point the statement clearly follows.

We finish this section with the

Proof of Corollary 2. Let $\Sigma = \bigcap_{\underline{x} \in X^c \setminus \Delta} \mathrm{Cone}(X; \underline{x})$. Clearly $X \subseteq \Sigma$. Lemma 4, (i) implies that $\mathfrak{S}(X) \subseteq \bigcap_{x \in X} \mathrm{Cone}(X, x) \subseteq \Sigma$. It remains to prove that $\Sigma \subseteq X \cup \mathfrak{S}(X)$. Assume first that $c = 1$. If $z \in \Sigma - X$, then the line Λ joining z to a general point $x \in X$ sits in $\mathrm{Cone}(X, x)$. Therefore Λ intersects X at some other point $y \in X$ and this means that $\pi_{z|X}$ is not birational, i.e. $z \in \mathfrak{S}(X)$.

Assume now that $c > 1$ and suppose that there is a point $z \in \Sigma - (X \cup \mathfrak{S}(X))$. Then the projection of X from z to a $\mathbb{P}^{r-1}$ would be an n-dimensional variety Y which would enjoy the following property: every $\mathbb{P}^{c-1}$, which cuts Y at c independent points, contains a further point of Y. This property contradicts the General Position Theorem (see [1], pg. 109).

3 On the axes of a variety

This section is devoted to the proof of Theorem 3. This will be done in a few different steps. The first one is the following:

Proposition 6. *Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, algebraic variety of dimension $n < r-1$ which is not a cone. Suppose that X has two distinct axes Π_1 and Π_2. Then $\Pi_1 \cap \Pi_2 = \emptyset$.*

Proof. Let $T := T_{X,x}$ be the tangent space to X at its general point x. We will use the following notation: for $h = 1, 2$, we set $l_h = \dim(\Pi_h)$, $T_h = T \cap \Pi_h$, $t_h = \dim(T_h)$,

$i = \dim(\Pi_1 \cap \Pi_2)$, $j = \dim(T_1 \cap T_2)$, $\Pi = \mathrm{Span}(\Pi_1 \cup \Pi_2)$, $l = \dim(\Pi) = l_1 + l_2 - i$, $\Theta = \mathrm{Span}(T_1 \cup T_2)$, $\theta = \dim(\Theta)$. Theorem 1 implies that $t_h = l_h - 1$, for $h = 1, 2$.

We argue by contradiction and we assume $i \geqslant 0$. Since X is not a cone, Proposition 5, (i) forces $j < i$, hence:

$$\begin{aligned} r - 2 \geqslant n \geqslant \theta &= t_1 + t_2 - j \\ &\geqslant (l_1 - 1) + (l_2 - 1) - (i - 1) = l_1 + l_2 - i - 1 = l - 1 \end{aligned}$$

thus Π is a proper subspace of $\mathbb{P}^r$. If $\theta = l$, then $\Theta = \Pi$ hence $\Pi \subseteq T$, but this is not possible because X should be a cone by Proposition 5, (i). It follows that $\theta = l - 1$, i.e. $j = i - 1$.

If $l < n$, then Π would enjoy property (ii) of the statement of Theorem 1, contradicting the fact that Π_1 and Π_2 are irreducible components of $\mathfrak{S}(X)$. The case $l = n$ is also excluded by Proposition 5, (iii), since $l = n \leqslant r - 2$. Therefore we may assume $l \geqslant n + 1$. On the other hand $l - 1 = \theta \leqslant n$, hence $l = n + 1$ and $\Theta = T$. Since $\Theta \subseteq \Pi$, we find a contradiction by Proposition 5, (ii).

The next step is as follows:

Proposition 7. *Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, algebraic variety of dimension $n < r - 1$ which is not a cone. Suppose that X has two distinct axes Π_1 and Π_2. Then* $\dim(\Pi_1) + \dim(\Pi_2) \leqslant n + 1$. *Moreover if* $\dim(\Pi_1) + \dim(\Pi_2) \geqslant n$ *then $n = r - 2$ and X is the complete intersection of two cones of dimension $n + 1$ with vertices respectively at Π_1 and Π_2.*

Proof. We keep the above notation. Since Π_1 and Π_2 are disjoint, the same is true for T_1 and T_2. Thus $n \geqslant \theta = t_1 + t_2 + 1 = l_1 + l_2 - 1$.

If $l_1 + l_2 = n + 1$, then $T = \Theta \subseteq \Pi$. Proposition 5, (ii) implies that $\Pi = \mathbb{P}^r$, thus $r = l = l_1 + l_2 + 1 = n + 2$, namely X has codimension 2 and X is contained in $\mathrm{Cone}(X, \Pi_1) \cap \mathrm{Cone}(X, \Pi_2)$. Actually X is equal to the complete intersection of the two cones. Indeed, for $i = 1, 2$, any $\mathbb{P}^{l_i + 1}$ generator of $\mathrm{Cone}(X, \Pi_i)$ cuts $\mathrm{Cone}(X, \Pi_{3-i})$ along a variety Y and $\mathrm{Cone}(X, \Pi_{3-i}) = \mathrm{Cone}(Y, \Pi_{3-i})$. This implies that Y is irreducible and reduced. One moment of reflection shows then that the complete intersection of $\mathrm{Cone}(X, \Pi_1)$ and $\mathrm{Cone}(X, \Pi_2)$ itself is irreducible and reduced, i.e. it coincides with X.

If $l_1 + l_2 = n$, then $\theta = n - 1$. Proposition 5, (iii) forces $n + 1 = l_1 + l_2 + 1 = l \geqslant r - 1$, hence again $r = n + 2$ and, as above, X is the complete intersection of $\mathrm{Cone}(X, \Pi_1)$ and $\mathrm{Cone}(X, \Pi_2)$.

The final step consists in proving the following result, which, together with Propositions 6 and 7, implies Theorem 3:

Proposition 8. *Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, algebraic variety of di-*

mension $n = r - 2$ which is not a cone. Suppose that Π_1 and Π_2 are two distinct irreducible components of $\mathfrak{S}(X)$. Then $\mathfrak{S}(X) \cap \mathrm{Span}(\Pi_1 \cup \Pi_2) = \Pi_1 \cup \Pi_2$.

Proof. Again we keep the above notation. We first treat the case $l_1 + l_2 \leqslant n$.

Let $[x_0, \dots, x_r]$ be projective coordinates in $\mathbb{P}^r$. We fix the hyperplane at infinity Π_∞ to be the one with equation $x_0 = 0$ and we consider $(x_1, \dots, x_r)$ as affine coordinates on $\mathbb{A}^r = \mathbb{P}^r - \Pi_\infty$. We denote by P_i the point at infinity of the x_i-axis, i.e. the point $[0, \dots, 1, \dots, 0]$ where 1 is at the $(i+1)$-th place. We may assume Π_1 to be generated by the points $P_1, \dots, P_{l_1+1}$ and Π_2 by the points $P_{r-l_2}, \dots, P_r$, thus $\Pi_1 \cup \Pi_2 \subset \Pi_\infty$. Hence local affine equations of X in a suitable open subset U of $\mathbb{A}^r$ may be written in the form:

$$x_1 = f(x_2, \dots, x_{r-l_2-1}), \quad x_r = g(x_{l_1+2}, \dots, x_{r-1})$$

where f and g are analytic functions of their variables.

Suppose that the assertion is false. Then we may assume that $\mathfrak{S}(X) \cap \Pi_\infty$ contains the point $C = [0, 1, \dots, 1, 0, \dots, 0, 1, \dots, 1]$, where the 0's appear at places 1 and $l_1 + 2, \dots, r - l_2 - 1$. Let

$$P = (f(u_2, \dots, u_{r-l_2-1}), u_2, \dots, u_{r-1}, g(u_{l_1+2}, \dots, u_{r-1}))$$

be a general point of X. Then there is a point

$$P' = (f(u'_2, \dots, u'_{r-l_2-1}), u'_2, \dots, u'_{r-1}, g(u'_{l_1+2}, \dots, u'_{r-1}))$$

on X such that the line joining P and P' has C as its point at infinity, i.e.:

$$\begin{aligned} f(u_2, \dots, u_{r-l_2-1}) - f(u'_2, \dots, u'_{r-l_2-1}) &= u_2 - u'_2 = \dots = u_{l_1+1} - u'_{l_1+1} \\ = u_{r-l_2} - u'_{r-l_2} = \dots = u_{r-1} - u'_{r-1} &= g(u_{l_1+2}, \dots, u_{r-1}) - g(u'_{l_1+2}, \dots, u'_{r-1}) \end{aligned} \tag{$*$}$$

and $u_i = u'_i$ for $i = l_1 + 2, \dots, r - l_2 - 1$. Notice that P' depends on P.

By shrinking the open subset U of $\mathbb{A}^r$ in which we are working, we may assume that $u'_2, \dots, u'_{l_1+1}, u'_{r-l_2}, \dots, u'_{r-1}$ are analytic functions of $u_2, \dots, u_{r-1}$. We claim that $u'_2, \dots, u'_{l_1+1}$ do not depend on $u_{r-l_2}, \dots, u_{r-1}$. We argue by contradiction and we assume this is not the case. By (*) we have:

$$f(u_2, \dots, u_{r-l_2-1}) - u_i = f(u'_2, \dots, u'_{r-l_2-1}) - u'_i$$

for every $i = 2, \dots, l_1 + 1$. Hence we deduce that:

$$\frac{\partial}{\partial u_j}(f(u'_2, \dots, u'_{r-l_2-1}) - u'_i) = 0$$

for every $j = r - l_2, \dots, r - 1$. This reads:

$$\sum_{h=2}^{l_1+1} \frac{\partial f}{\partial x_h} \frac{\partial u'_h}{\partial u_j} - \frac{\partial u'_i}{\partial u_j} = 0$$

which holds for every $i = 2, \dots, l_1 + 1$ and every $j = r - l_2, \dots, r - 1$. By linear algebra, this yields:

$$\sum_{h=2}^{l_1+1} \frac{\partial f}{\partial x_h} = 1.$$

Notice that the tangent space T to X at P has affine equations:

$$\sum_{h=2}^{r-l_2-1} \frac{\partial f}{\partial x_h}(x - u_h) = x_1 - u_1, \qquad \sum_{k=l_1+2}^{r-1} \frac{\partial g}{\partial x_k}(x - u_k) = x_r - u_r$$

and the projective hyperplane defined by the former equation contains Π_2 and C. This implies that the tangent space to X at P intersects the span M of Π_2 and C in a subspace of dimension l_2. Thus M is contained in an axis and we get a contradiction, because M strictly contains Π_2. This proves our claim that $u'_2, \dots, u'_{l_1+1}$ do not depend on $u_{r-l_2}, \dots, u_{r-1}$. Similarly one shows that $u'_{r-l_2}, \dots, u'_{r-1}$ do not depend on $u_2, \dots, u_{l_1+1}$.

Suppose that $l_1 = 0$. Since $l_2 \leqslant n - 1$ one has $r - l_2 > l_1 + 2 = 2$, and recall that in this case $u_i = u'_i$ for $i = 2, \dots, r - l_2 - 1$. Hence $f(u_2, \dots, u_{r-l_2-1}) = f(u'_2, \dots, u'_{r-l_2-1})$ and therefore all the differences in (*) are 0, i.e. $P = P'$, a contradiction. Similarly if $l_2 = 0$. Thus we may assume $l_1 \geqslant l_2 > 0$. In this case, as a consequence of the above analysis, we have that the differences in (*) are equal to a constant c, so the translation by c in the direction of the point at infinity C fixes X. Since X is algebraic, this yields that X is a cone with vertex at C, a contradiction. This ends the proof in the case $l_1 + l_2 \leqslant n$.

Suppose now that $l_1 + l_2 = n + 1$, thus in particular $l_1, l_2 > 1$. We may assume that Π_2 is generated by $P_{r-l_2}, \dots, P_r$, so $\Pi_2 \subset \Pi_\infty$. We may assume also that the subspace $\Pi_\infty \cap \Pi_1$ is generated by $P_1, \dots, P_{l_1}$. Notice that $l_1 + 1 = r - l_2$. Then we argue by contradiction supposing that the point $C = [0, 1, \dots, 1]$ sits in $\mathfrak{S}(X)$. By the same computations as before (by replacing only l_1 with $l_1 - 1$ in all the above formulae), it follows that X is a cone, a contradiction.

Remark 9. (i) We stress that the existence of varieties of dimension $n = r - 2$, with irreducible components Π_1 and Π_2 of $\mathfrak{S}(X)$ of dimension respectively l_1 and l_2, are possible for all values of l_1 and l_2 such that $l_1 + l_2 \leqslant n + 1$. It is sufficient to take for X a complete intersection of cones of dimension $n + 1$ with maximal vertices at two skew subspaces Π_1, Π_2 of the prescribed dimensions.

(ii) We want to point out the following interesting phenomenon. Let $X \subset \mathbb{P}^r$ be an irreducible, non-degenerate, algebraic variety of dimension $n < r - 1$ which is not a cone and let Π_1, Π_2 be two distinct, and therefore disjoint, axes of X. Consider the projection π from a point of $\Pi = \mathrm{Span}(\Pi_1 \cup \Pi_2)$ not on $X \cup \Pi_1 \cup \Pi_2$, and set $Y = \pi(X)$, which is a variety Y birational to X. Using Lemma 4, (vii), one sees that, in general, $\pi(\Pi) \subseteq \mathfrak{S}(Y)$.

4 Some open problems

In the present section we want to propose some open problems. First of all, the configuration and the number of the irreducible components of the Segre locus of a variety $X \subset \mathbb{P}^r$ of dimension $n \leqslant r-2$ is still pretty much a mystery. In the case $n = r-2$ some not exhaustive information is provided by Proposition 8. Is it possible to extend this result to the case $n < r-2$?

In general any information more detailed than the one we have given here would be welcome. In particular, Segre's theorem from [5], mentioned in the introduction, about the existence of varieties with as many centers as one wants, should be complemented with answers to questions like:

(i) are there bounds on the number of centers, or axes, depending on any invariant of the variety, like the (co)dimension, the degree, etc.?

(ii) is the configuration and the number of components of the Segre locus influenced by the smoothness of the variety?

A generalization of the Segre locus, that we call the *Grassmann–Segre locus*, can be defined as follows. Let $X \subset \mathbb{P}^r$ be an irreducible, projective variety of dimension n and let m be a non-negative integer such that $m \leqslant r-n-2$. If Π is a general linear subspace of $\mathbb{P}^r$ of dimension m, then the projection $\pi := \pi_\Pi$ of $\mathbb{P}^r$ to $\mathbb{P}^{r-m-1}$ from Π restricts to X to a birational morphism of X onto its image. If Π is still such that $\pi_{|X}$ is a morphism, i.e. $\Pi \cap X = \emptyset$, but $\pi_{|X}$ is no longer birational to its image, then we say that π is a *special projection* of X. We define the m-th *Grassmann–Segre locus* of X as:

$$\mathfrak{S}_m(X) := \overline{\{\Pi \in \mathbb{G}(m,r) : \pi_\Pi \text{ is a special projection of } X\}}.$$

Of course $\mathfrak{S}_0(X) = \mathfrak{S}(X)$. It would be nice to have extensions of Theorems 1 and 3 to these *Grassmann–Segre loci*. For instance, we have an argument, which we do not reproduce here, based on the theory of foci of planes in $\mathbb{P}^4$ (see [3]), to the effect that $\mathfrak{S}_1(X)$ is a finite set if X is a curve. Is it always the case that $\mathfrak{S}_m(X)$ is a finite set if X is a curve?

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