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Near-homogeneous 16-dimensional planes

Helmut Salzmann

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A projective plane $\mathscr{P} = (P, \mathfrak{L})$ with point set P and line set $\mathfrak{L}$ is a (compact) *topological* plane if P and $\mathfrak{L}$ are (compact) topological spaces and the geometric operations of joining two distinct points and intersecting two distinct lines are continuous. The classical examples are the Desarguesian planes over the real or complex numbers or the quaternions and the Moufang plane over the octonions, each taken with its natural topology on P and on $\mathfrak{L}$. These planes are also connected. Within the vast class of compact connected topological projective planes they are distinguished by their high degree of homogeneity: their automorphism groups are transitive on quadrangles. A detailed discussion of the classical planes can be found in Chapter I of the book *Compact Projective Planes* [15].

Each compact projective plane $\mathscr{P}$ with a point space P of positive (covering) dimension $\dim P < \infty$ has lines which are homotopy equivalent to an ℓ-sphere $\mathbb{S}_\ell$, where $\ell \mid 8$ and $\dim P = 2\ell$, see [15, 54.11]. In all known examples, the lines are actually *homeomorphic* to $\mathbb{S}_\ell$. The automorphism group $\Sigma = \operatorname{Aut}\mathscr{P}$ consists of all continuous collineations of $(P, \mathfrak{L})$. Taken with the compact-open topology (the topology of uniform convergence), Σ is a locally compact transformation group of P of finite dimension ([15, 83.2]). Let Δ denote a connected closed subgroup of Σ. If $\dim \Delta > 5\ell$, then $\mathscr{P}$ is a classical plane and Σ is a simple Lie group, see Salzmann [8], Theorem II, and [15, 87.7] for the cases $\ell \geqslant 4$. Suppose now that $3\ell \leqslant \dim \Delta \leqslant 5\ell$. Under suitable additional assumptions on the structure of Δ and its action on $\mathscr{P}$, *the classification problem requires to determine all possible pairs* $(\mathscr{P}, \Delta)$. The cases $\ell \mid 4$ are understood fairly well. In particular, the classification has been completed in the following cases:

(a) $\ell \leqslant 2$ and $\dim \Delta \geqslant 4\ell - 1$, see [15, 38.1, 74.27],

(b) $\ell = 4$ and $\dim \Delta \geqslant 17$, cf. [15, 84.28] or Salzmann [12],

(c) $\ell = 4$, Δ is almost simple, and $\dim \Delta > 10$, cf. Stroppel [16].

Less is known for $\ell = 8$, and it is this case we will be concerned with. Among other things, the following has been proved:

(1) *If* Δ *is transitive on* P, *then* $\mathscr{P}$ *is the classical Moufang plane* $\mathcal{O}$ *over the octonion algebra* $\mathbb{O}$ *and* Δ *contains the elliptic motion group*, [15, 63.8].

(2) *If* $\dim \Delta \geqslant 27$, *then* Δ *is a Lie group*, Priwitzer–Salzmann [7].

(3) *Assume that* $\dim \Delta \geqslant 29$. *If* Δ *fixes no point and no line, then* $\mathscr{P}$ *is classical, or* $\Delta' \cong \mathrm{SL}_3 \mathbb{H}$ *and* $\mathscr{P}$ *is a Hughes plane* (as described in [15, §86]). *If* Δ *fixes exactly one element, then* Δ *has a normal vector subgroup.*

In fact, either Δ is semi-simple, or Δ has a minimal commutative connected normal subgroup Θ, and Θ is compact or a vector group, cf. [15, 94.26]. For semi-simple groups the claim is an immediate consequence of Priwitzer's classification [5, 6]. If Θ is compact, the assertion follows from Salzmann [13]. In both of these cases, Δ fixes either no element or an anti-flag.

If Δ has a normal vector subgroup and $\dim \Delta \geqslant 24$, then there is always a fixed element, see Salzmann [10] or Grundhöfer–Salzmann [2], Proposition XI. 10. 19.

The purpose of this paper is to prove the following

Theorem. *If* $\dim \Delta \geqslant 35$ *and if* Δ *fixes exactly one line* W *and no point, then* $\mathscr{P}$ *is a translation plane.*

The *proof* depends on a recent improvement of [15, 87.4], cf. Salzmann [14]:

Proposition. *If* $\dim \Delta \geqslant 33$ *and if* Δ *has a normal vector subgroup, then there is also a minimal normal subgroup* $\Theta \cong \mathbb{R}^t$ *consisting of axial collineations.*

Under the hypotheses of the Theorem, Θ is contained in the translation group $\mathsf{T} = \Delta_{[W,W]}$. A detailed analysis of the irreducible representation induced by Δ on Θ will show finally that $\dim \mathsf{T} = 16$. Recently, Hähl and Löwe have determined explicitly all translation planes having a group Δ as in the Theorem, cf. Hähl [3]. In particular, their work implies the following:

Corollary. *Under the assumptions of the Theorem, either* $\mathscr{P} \cong \mathcal{O}$, *or* $\dim \Delta = 35$ *and the stabilizer of an affine point has an* 18*-dimensional semi-simple commutator group* Υ *isomorphic to one of the groups* $\mathrm{SL}_2 \mathbb{H} \cdot \mathrm{SU}_2 \mathbb{C}$ *or* $\mathrm{SU}_4 \mathbb{C} \cdot \mathrm{D}$ *with* $\mathrm{D} = \mathrm{SU}_2 \mathbb{C}$ *or* $\mathrm{D} = \mathrm{SL}_2 \mathbb{R}$.

Proof of the Theorem. Let $\Theta \cong \mathbb{R}^t$ be a minimal normal subgroup of Δ. First it will be shown that $t > 8$. Then each point $z \in W$ is the center of some one-parameter subgroup of Θ and the centralizer $\mathrm{Cs}\,\Theta$ coincides with T. Assuming that $\dim \mathsf{T} < 16$, it follows that the solvable radical $\Omega = \sqrt{\Delta}$ has dimension at most 16, and a Levi complement Ψ of Ω (a maximal semi-simple subgroup of Δ) satisfies $\dim \Psi \geqslant 19$. By minimality of Θ, the group Δ/T is an irreducible subgroup of $\mathrm{GL}_t \mathbb{R}$. Either Ψ acts irreducibly on Θ, or Θ splits into a direct product of two isomorphic minimal Ψ-invariant subgroups, and the actions of Ψ on the two factors are equivalent, compare [15, 95.6 (b)]. If $\dim \Delta \geqslant 40$, the Theorem is true by [15, (87.5)]. Therefore, only the cases $\dim \Psi \leqslant 39 - t$ have to be considered. The number of possibilities is further reduced by the fact that the torus rank $\mathrm{rk}\,\Delta$ is at most 4, see [15, 55.37 (a)]. For each admissible representation of Ψ on Θ, maximal sets of pairwise commuting involutions and their centralizers can be determined. The strategy is to find a suitable Baer involution β and to study the action of $\mathrm{Cs}\,\beta$ on the Baer plane $\mathscr{F}_\beta$. Known results for 8-dimensional planes (e.g. Stroppel [16]) will then lead to a contradiction in each case. Interestingly, the arguments for different groups turn out to be rather different.

For the first steps, the following *stiffness* theorem of Bödi [1] is needed:

($\square$) *If the fixed elements of the connected Lie group* Λ *form a connected subplane, then* Λ *is isomorphic to a compact group* $\mathrm{G}_2 = \mathrm{Aut}\,\mathbb{O}$ *or* $\mathrm{SU}_3\,\mathbb{C}$, *or* $\dim \Lambda < 8$.

The *notation* that has been introduced so far will be used throughout. $\langle S \rangle$ will denote the smallest closed subplane containing the set S. If S is not totally disconnected, then $\langle S \rangle$ is a connected plane, and $\dim\langle S \rangle$ divides 16, see [15, 54.11]. The connected component of the topological group A will be denoted by A^1, it should be distinguished from the commutator group A'. More generally, $[\mathsf{A}, \mathsf{B}]$ is the group generated by all commutators $\alpha^{-1}\beta^{-1}\alpha\beta$ with $\alpha \in \mathsf{A}$ and $\beta \in \mathsf{B}$. As customary, $\Gamma_{[z]}$ is the group of all collineations in Γ with center z. Without further mention, frequent use will be made of the dimension formula [15, 96.10] and of the List [15, 95.10] of all irreducible representations of almost simple Lie groups in dimension at most 16. In order to obtain information on representations of properly semi-simple groups, Clifford's Lemma [15, 95.5] is helpful:

Suppose that $\Gamma = \mathsf{AB}$ *is an irreducible subgroup of* $\mathrm{GL}_n\,\mathbb{R}$ *and that* A *and* B *centralize each other. If* $U \cong \mathbb{R}^t$ *is a minimal* A*-invariant subgroup of* $\mathbb{R}^n$, *then* $t \mid n$ *and* A *acts effectively* (*and irreducibly*) *on* U.

Note that the group $\mathrm{SO}_5\,\mathbb{R}$ cannot act on any compact plane [15, 55.40].

(1) The elements of any one-parameter subgroup of T have a common center [15, 61.8]. Since Δ fixes no point of W, it follows that $t > 1$ and that Θ contains translations in different directions. More generally, $t \geqslant 2\dim\Theta_{[z]}$. Note that the stabilizer Δ_ϱ in the action of Δ on Θ centralizes the one-parameter group Π containing ϱ. Choose any point $a \notin W$ and put $\Gamma = (\Delta_a)^1$. Then Γ_ϱ fixes each point of the orbit a^Π. Write $\Lambda = (\Gamma_{\varrho,\varrho'})^1$, where $\varrho, \varrho' \in \Theta$ are translations in different directions. The dimension formula gives $\dim\Delta = \dim a^\Delta + \dim\Delta_a \leqslant 16 + \dim\Gamma$. Now the stiffness theorem ($\square$) implies

$$19 \leqslant \dim\Gamma \leqslant 2t + \dim\Lambda \quad \text{and} \quad t \geqslant 6. \tag{$*$}$$

In fact, either $\dim\Lambda \leqslant 8$ or $\Lambda \cong \mathrm{G}_2$ and $t > 2$. In the second case, put $\Upsilon = \Theta \cap \mathrm{Cs}\,\Lambda$. By [15, 83.24], the fixed elements of $\Lambda \cong \mathrm{G}_2$ form a *flat* (i.e. 2-dimensional) subplane $\mathscr{E}$ containing a^Υ, and $\dim\Upsilon \leqslant 2$. On the other hand, $\varrho, \varrho' \in \Upsilon$ and $\Upsilon \cong \mathbb{R}^2$. Under the action of G_2, the vector group Θ splits into a product of Υ and a subgroup of dimension divisible by 7. Consequently, $\Lambda \cong \mathrm{G}_2$ implies $t = 9$. In any case, $t \geqslant 6$.

(2) If $t = 6$, then $\dim\Lambda < 8$ by ($\square$), and it follows from ($*$) that $\dim\varrho^\Gamma = 6$ for each $\varrho \neq \mathbb{1}$, and ϱ^Γ is open in Θ, compare [15, 96.11 (a)]. Hence Γ is transitive on $\Theta\backslash\{\mathbb{1}\}$ and Γ has a subgroup $\Phi \cong \mathrm{SU}_3\,\mathbb{C}$, see Völklein [17] or [15, 96.16, 96.19–96.22]. If $\mathsf{K} = \Gamma \cap \mathrm{Cs}\,\Theta \neq \mathbb{1}$, then a^Θ is contained in the fixed plane $\mathscr{F}_{\mathsf{K}}$, and $\mathscr{B} =$

$\langle a^\Theta \rangle$ is a Baer subplane. By Richardson's Theorem [15, 96.34], each action of $\mathrm{SU}_3\,\mathbb{C}$ on the 4-sphere $W \cap \mathscr{B}$ is trivial, and Φ would induce on $\mathscr{B}$ a group of homologies. This is impossible. Hence Γ acts effectively on Θ. By [15, 95.5, 95.6], the commutator group Γ' is almost simple and irreducible on Θ. Moreover, $(*)$ implies $17 \leqslant \dim \Gamma' \leqslant 20$, and then Γ' is locally isomorphic to $\mathrm{SO}_5\,\mathbb{C}$, but such a group has no subgroup $\mathrm{SU}_3\,\mathbb{C}$. This contradiction shows that $t > 6$.

(3) In the case $t = 7$, steps (1) and (2) show that Θ has no Γ-invariant proper subgroup: Γ acts irreducibly on Θ. Suppose that $\langle a^\Theta \rangle = \mathscr{B}$ is a Baer subplane. Then it follows from the theorems on large elation groups [15, 61.11–61.13] that $\dim \Theta_{[z]} \geqslant 3$ for $z \in W \cap \mathscr{B}$ and that $\Theta_{[z]} \cong \mathbb{R}^4$ for exactly one of these groups. Thus, Γ fixes in $\mathscr{B}$ some point $z \in W$, and $\Theta_{[z]}$ would be Γ-invariant. Therefore, $\langle a^\Theta \rangle = \mathscr{P}$, and Γ acts effectively on Θ. Again Γ' is almost simple and irreducible, see [15, 95.5, 95.6], moreover, $17 \leqslant \dim \Gamma' \leqslant 22$ by $(*)$. The list shows $\Gamma' \cong \mathrm{O}_7'(\mathbb{R}, r)$. In particular, $\dim \Gamma' = 21$ and $(*)$ implies as in (2) that Γ' is transitive on $\Theta \setminus \{\mathbb{1}\}$, but this is impossible for $r = 0$ as well as for $r > 0$.

(4) Next, let $t = 8$ and assume first that $\langle a^\Theta \rangle = \mathscr{B}$ is a Baer subplane. Then Θ is a transitive translation group of $\mathscr{B}$. Put $\overline{\Gamma} = \Gamma|_{\mathscr{B}} \cong \Gamma/\mathsf{K}$, where $\mathsf{K} = \Gamma \cap \mathrm{Cs}\,\Theta$. By [15, 83.22], the group K^1 is a subgroup of $\mathrm{SU}_2\,\mathbb{C}$, in particular, $\dim \mathsf{K} \leqslant 3$. Therefore, $16 \leqslant \dim \overline{\Gamma} \leqslant 19$ and $\mathscr{B} \cong \mathscr{P}_2(\mathbb{H})$, see [15, 83.26, 84.27]. The 19-dimensional stabilizer of a and W in $\mathrm{Aut}\,\mathscr{B}$ has a subgroup $\Psi \cong \mathrm{SL}_2\,\mathbb{H}$. Since a maximal compact subgroup $\Phi \cong \mathrm{U}_2\,\mathbb{H}$ of Ψ has no proper subgroup of dimension $\geqslant 7$, it follows that $\Phi \cap \overline{\Gamma} = \Phi$, and then $\Psi < \overline{\Gamma}$ by [15, 94.34). According to [15, 94.27], the group Ψ is covered by a subgroup Υ of Γ, and $\Upsilon \cong \Psi$ because Ψ is simply connected. The central involution σ of Υ cannot be planar, or else Υ would induce on the Baer subplane of fixed elements of σ a group containig $\mathrm{PU}_2\,\mathbb{H} \cong \mathrm{SO}_5\,\mathbb{R}$. Hence σ is a reflection of $\mathscr{P}$, it inverts each element of T. From [15, 61.20 (b)] it follows that $\dim \mathsf{T} = \dim a^\Delta = \dim \Delta - \dim \Gamma \geqslant 13$. By complete reducibility, Θ has a Υ-invariant complement Ξ in T. Because $\sigma|_\Xi \neq \mathbb{1}$, the representation of Υ on Ξ has trivial kernel, and $\Xi \cong \mathbb{R}^8$, but we have assumed that $\dim \mathsf{T} < 16$. Consequently, $\langle a^\Theta \rangle = \mathscr{P}$, and Γ acts effectively on Θ.

(5) Considering still the case $t = 8$, assume that Γ' is not almost simple, and put $\Gamma' = \mathsf{AB}$, where B is a factor of minimal dimension and $\mathsf{A} = \mathrm{Cs}\,\mathsf{B}$. Remember from $(*)$ that $\dim \Gamma' \geqslant 17$. Hence $\dim \mathsf{A} \geqslant 9$. If the action of A on Θ is irreducible, then $\mathsf{B} \leqslant \mathbb{H}^\times$ by Schur's Lemma [15, 95.4], and $\mathsf{A} \cong \mathrm{SL}_2\,\mathbb{H}$. If Θ contains a proper A-invariant subgroup, however, it follows, using Clifford's Lemma, that A is an irreducible subgroup of $\mathrm{SL}_4\,\mathbb{R}$ and then that A is almost simple. Hence $\mathrm{Sp}_4\,\mathbb{R} \hookrightarrow \mathsf{A}$, and for some $\gamma \in \mathsf{A}$ the fixed elements of γ in Θ form a 2-dimensional B-invariant subspace. Again by Clifford's Lemma, $\mathsf{B} \cong \mathrm{SL}_2\,\mathbb{R}$ and then $\mathsf{A} \cong \mathrm{SL}_4\,\mathbb{R}$. In both cases, the central involution $\alpha \in \mathsf{A}$ cannot be planar: for $\mathsf{A} \cong \mathrm{SL}_2\,\mathbb{H}$ this is true for the same reason as in step (4), for $\mathsf{A} \cong \mathrm{SL}_4\,\mathbb{R}$, a maximal compact subgroup of A would act as $(\mathrm{SO}_3\,\mathbb{R})^2$ on the 4-sphere consisting of the fixed elements of α on W, but this contradicts Richardson's Theorem [15, 96.34]. Consequently, α is a reflection. Because α fixes the center of each translation in Θ, the axis of α is W. Therefore, $\alpha^\Delta\alpha$ is contained in the translation group T, and $\dim \mathsf{T} = \dim a^\Delta \geqslant 15$, see [15, 61.19 (b)] and use the fact that $\dim \Gamma' = 18$ and $\dim \Gamma \leqslant 20$. Moreover, α inverts each transla-

tion. On the other hand, α induces on T a linear map of determinant 1 because α belongs to the connected group A. Hence $\dim \mathsf{T}$ is even, and T would be transitive contrary to the assumption.

(6) By the last step, Γ' is almost simple. Now $20 \leqslant \dim \Gamma' \leqslant 24$, and Γ' acts irreducibly on Θ. Inspection of the List leaves only the possibilities $\Gamma' \cong \mathrm{Sp}_4\,\mathbb{C} \cong \mathrm{Spin}_5\,\mathbb{C}$ and $\Gamma' \cong \mathrm{Spin}_7(\mathbb{R}, r)$ with $r \in \{0, 3\}$. Hence $\dim \Gamma \leqslant 22$. In each case, Γ' has a unique central involution σ. According to Stroppel [16] or [15, 84.19], the group $\Gamma'/\langle\sigma\rangle$ cannot act on an 8-dimensional plane, and σ is not planar. Because σ inverts the elements of Θ, it follows that σ is a reflection with axis W and center a. Exactly as at the end of step (4), this would imply $\dim \mathsf{T} = 16$. Together with the previous steps, this proves:

(7) If T is not transitive, then $t > 8$ as claimed at the very beginning of the proof. For $z \in W$, the action of Θ on the line pencil $\mathfrak{L}_z$ shows that $\dim \Theta_{[z]} \geqslant t - 8 > 0$, compare [15, 61.11 (a), (b)]. This has the consequence that $\mathrm{Cs}\,\Theta$ fixes each point of W, hence it consists of collineations with axis W. Because $a^\Theta \neq a$ for each $a \notin W$, no homology can belong to $\mathrm{Cs}\,\Theta$, and $\mathrm{Cs}\,\Theta \leqslant \mathsf{T}$. On the other hand, T is commutative since there are translations in different directions, see [15, 23.13]. Therefore, $\mathrm{Cs}_\Delta\,\Theta = \mathsf{T}$ as asserted.

(8) If $\dim \mathsf{T} = 15$, then [15, 61.11, 61.12] would imply $\dim \mathsf{T}_{[z]} = 8$ for some point $z \in W$, and T would be transitive since $z^\Delta \neq z$ by assumption. Hence $\dim \mathsf{T} \leqslant 14$. Minimality of Θ signifies that Δ induces on Θ an irreducible subgroup Δ/T of $\mathrm{GL}_t\,\mathbb{R}$. By the structure theorem [15, 95.6] for irreducible groups, Δ/T is a product of its center Ω/T and a semi-simple group, and Ω/T is isomorphic to a subgroup of $\mathbb{C}^\times$. Consequently, $\Omega = \sqrt{\Delta}$ and $\dim \Omega \leqslant 16$, moreover, $[\Delta, \Omega] \subseteq \mathsf{T}$.

(9) Any Levi complement Ψ of Ω acts effectively on Θ: otherwise, there is an element $\tau \neq \mathbb{1}$ in $\Psi \cap \mathsf{T}$, and τ is in the center of Ψ since Ψ is connected and semi-simple. Because Ω induces on T a group of complex dilatations, the center z of τ has a 1-dimensional orbit $z^\Delta = z^\Omega$, and Δ_z fixes each point of this orbit (since $[\Delta, \Omega] \subseteq \mathsf{T}$). Choose $c \in a^{\mathsf{T}_{[z]}}$ and note that the connected component Λ of $\Delta_{a,z,c}$ is not isomorphic to G_2 by the remarks following $(*)$. With $(\square)$ one would obtain $18 \leqslant \dim \Delta_{a,z} \leqslant \dim \mathsf{T}_{[z]} + \dim \Lambda \leqslant 7 + 8$, a contradiction.

(10) If t is odd, then $\dim \Omega/\mathsf{T} \leqslant 1$ and $20 \leqslant \dim \Psi \leqslant 39 - t \leqslant 30$. By Clifford's Lemma, Ψ is almost simple and irreducible on Θ. For prime numbers this is obvious, for $t = 9$ any proper factor of Ψ would act effectively on $\mathbb{R}^3$, and then $\dim \Psi \leqslant 16$. Inspection of the List shows that no almost simple group Ψ in the given dimension range has an irreducible representation in dimension 9, 11, or 13. Therefore, only the possibilities $t \in \{10, 12, 14\}$ remain. As in the proof of the Proposition, the case $t = 12$ turns out to be the most complicated one.

(11) In the other two cases, t is a product of two primes, and Clifford's Lemma implies that Ψ has at most two almost simple factors, compare Salzmann [14], Lemma 4 for details. Moreover, $19 \leqslant \dim \Psi \leqslant 39 - t$.

(12) The case $t = 10$ leads to a contradiction in the following way: an almost simple group Ψ with an effective irreducible representation on $\mathbb{R}^{10}$ is isomorphic to one of the groups $\mathrm{SL}_5\,\mathbb{R}$, $\mathrm{SO}_5\,\mathbb{C}$, or $\mathrm{SU}_5(\mathbb{C}, r)$. The first two of these groups and the compact unitary group have a subgroup $\mathrm{SO}_5\,\mathbb{R}$ which cannot act effectively on any

plane. If $\Psi \cong \mathrm{SU}_5(\mathbb{C}, r)$, then Ψ has compact subgroups $\Phi \cong \mathrm{SU}_3\,\mathbb{C}$ and $\mathsf{X} \cong \mathbb{T}^2$ such that $[\Phi, \mathsf{X}] = \mathbb{1}$. Each of the three involutions in X is a reflection (or else Φ would act on the 4-sphere consisting of the fixed elements of a Baer involution on W; by [15, 61.26], the action would be non-trivial contrary to Richardson's Theorem). Hence there is a reflection $\sigma \in \mathsf{X}$ with axis W, and σ inverts each translation. This implies $\sigma|_\Theta = -\mathbb{1}$, and σ would be in the center Z of Ψ, but this contradicts the fact that $\mathsf{Z} \cong \mathbb{Z}_5$. The possibilities $\Psi \cong \mathrm{SU}_5(\mathbb{C}, r)$ can also be excluded by the lemma in step (14).

(13) Assume still that $t = 10$. The arguments in step (12) show that $\Psi = \mathsf{AB}$ is a product of two almost simple factors with $\dim \mathsf{A} \geqslant 10$. By Clifford's Lemma, $\Theta = \Theta_1 \times \Theta_2$ and A acts equivalently on the two factors. For a suitable element $\alpha \in \mathsf{A}$ the fixed space $\mathsf{H} = \{\tau \in \Theta \mid \tau^\alpha = \tau\}$ is non-trivial and $\mathsf{H} = \mathsf{H}^\mathsf{B}$ has even dimension. Again by Clifford's Lemma, there is a 2-dimensional B-invariant subgroup of H, and $\mathsf{B} \cong \mathrm{SL}_2\,\mathbb{R}$. Consequently, $\dim \mathsf{A} \geqslant 16$, and the List shows that $\mathsf{A} \cong \mathrm{SL}_5\,\mathbb{R}$, but as before this is impossible.

(14) **Lemma.** *If* Ψ *has a subgroup* $\mathsf{X} \cong \mathbb{T}^4$ (i.e. if $\mathrm{rk}\,\Psi$ is as large as it can be), *then* X *contains* 3 *reflections and* $4 \mid t$, *hence* $t = 12$.

In fact, it follows from [15, 55.34] that X fixes a triangle a, u, v with $uv = W$, and that there is some Baer involution $\beta \in \mathsf{X}$. Each other involution in X acts non-trivially on the Baer plane $\mathscr{B} = \mathscr{F}_\beta$ of the fixed elements of β, and X induces on $\mathscr{B}$ a 3-dimensional torus group, see [15, 55.32 (ii), 55.37]. By Richardson's Theorem [15, 96.34], the group $\mathbb{T}^3$ cannot act effectively on $\mathbb{S}_4$. Consequently, there are 3 involutions $\sigma_\nu \in \mathsf{X}$ which induce reflections on $\mathscr{B}$ (with centers a, u, v). Now [15, 55.27] implies that for each ν either σ_ν or $\beta\sigma_\nu$ is a reflection of $\mathscr{P}$. This proves the first claim. Denote now the reflections by σ_ν and consider the action of σ_ν on Θ and the eigenspaces $\Theta_\nu^\pm = \{\tau \in \Theta \mid \tau^{\sigma_\nu} = \tau^{\pm 1}\}$. Put $q_\nu^\pm = \dim \Theta_\nu^\pm$, and note that q_ν^- is even because σ_ν belongs to the connected group X. If σ_0 has axis W and $\sigma_0\sigma_1 = \sigma_2$, then $q_0^- = t$ and $q_2^\pm = q_1^\mp$, moreover, $2q_\nu^+ \leqslant t = q_\nu^+ + q_\nu^-$ for $\nu \neq 0$ (remember from (1) that $2\dim \Theta_{[z]} \leqslant t$). Therefore, $q_\nu^+ \leqslant q_\nu^-$, hence $q_1^+ = q_1^-$ and $t \equiv 0 \bmod 4$.

(15) Next, let $t = 14$. Then $19 \leqslant \dim \Psi \leqslant 25$. As mentioned at the beginning of the proof, the semi-simple group $\Psi \cong (\Delta/\mathsf{T})'$ acts irreducibly and effectively on $\mathbb{R}^7$ or on $\mathbb{R}^{14}$. In the first case, Ψ is almost simple, and the List shows $\Psi \cong \mathrm{O}_7'(\mathbb{R}, r)$. Since Ψ has no subgroup $\mathrm{SO}_5\,\mathbb{R}$, the Witt index is $r = 3$. Hence a maximal compact subgroup Φ of Ψ is a product $\mathsf{A} \times \mathsf{B}$ with $\mathsf{A} \cong \mathrm{SO}_4\,\mathbb{R}$ and $\mathsf{B} \cong \mathrm{SO}_3\,\mathbb{R}$. The first factor contains 6 conjugate, pairwise commuting involutions, and these are planar by [15, 55.35]. If α, α', and $\alpha\alpha'$ are Baer involutions in A, then the common fixed elements of α and α' form a 4-dimensional subplane $\mathscr{C}$, see [15, 55.39]. Each involution in B induces on $\mathscr{C}$ a reflection [15, 55.21 (c)], and by [15, 55.35] one of these reflections would have the axis $W \cap \mathscr{C}$. Since B is a simple group, B would consist entirely of axial collineations of $\mathscr{C}$. This contradicts [15, 71.3], cf. also [15, 71.10].

(16) Assume that Ψ is almost simple and irreducible on $\mathbb{R}^t$, where still $t = 14$. According to the List, Ψ is a group of type C_3, in fact, Ψ is isomorphic to a motion group $\mathrm{PU}_3(\mathbb{H}, r)$ of the quaternion plane, or Ψ is covered by the symplectic group $\mathrm{Sp}_6\,\mathbb{R}$.

(17) Consider first the case $\kappa : \mathrm{U}_3(\mathbb{H}, r) \to \Psi$, where the unitary group preserves

the form $x_1\bar{y}_1 + x_2\bar{y}_2 + (-1)^r x_3\bar{y}_3$, and put $\mathrm{diag}(-1,1,1)^\kappa = \alpha$, $\mathrm{diag}(1,1,-1)^\kappa = \gamma$, $i\mathbb{1}^\kappa = \pi$, $j\mathbb{1}^\kappa = \varrho$. Note that $\Upsilon = \langle \alpha, \gamma, \pi, \varrho \rangle \cong \mathbb{Z}_2^4$ fixes a triangle a, u, v with $uv = W$, see [15, 55.34 (a)]. The involution α is conjugate ($\sim$) to $\beta = \alpha\gamma$ within Ψ, but $\alpha \not\sim \gamma$ if $r = 1$. Because any two pure units in $\mathbb{H}^\times$ are conjugate, we have also $\pi \sim \varrho \sim \pi\varrho = \varrho\pi$ and $\pi \sim \alpha\pi \sim \gamma\pi \sim \cdots$. Altogether, there are 12 conjugates of π in Υ, and these are planar by [15, 55.35]. On the other hand, $\Upsilon \setminus \{\mathbb{1}\}$ cannot entirely consist of Baer involutions, see [15, 55.39 (b)]. Since $\alpha \sim \beta$, it follows in any case that γ is a reflection. If γ has axis W, then γ inverts each translation in Θ, and $\gamma = (-\mathbb{1})^\kappa$, a contradiction. If γ has center v and axis au, however, and if M and N denote the positive and the negative eigenspace of γ on Θ respectively, then, for geometrical reasons, $a^{\mathsf{M}} \subseteq au$ and $a^{\mathsf{N}} \subseteq av$. This means that $\mathsf{M} \leqslant \Theta_{[u]}$ and $\mathsf{N} \leqslant \Theta_{[v]}$. Because γ belongs to a connected group, $\det \gamma = 1$ and $\dim \mathsf{N}$ is even. Therefore, one of the eigenspaces is 8-dimensional, and $\Theta_{[z]} \cong \mathbb{R}^8$ for one and then for several centers, and T would be transitive.

(18) Next, let $\kappa : \mathrm{Sp}_6\,\mathbb{R} \to \Psi$ be an isomorphism or a double covering. Write the symplectic form as $\Sigma_\nu(x_\nu y_{\nu+1} - x_{\nu+1} y_\nu)$, and define involutions

$$\mathrm{diag}(1,1,-1,-1,-1,-1)^\kappa = \alpha,\ \mathrm{diag}(-1,-1,1,1,-1,-1)^\kappa = \beta, \text{ and } \alpha\beta = \gamma.$$

Because a maximal compact subgroup of $\mathrm{Sp}_6\,\mathbb{R}$ is isomorphic to $\mathrm{U}_3\,\mathbb{C}$, an elementary abelian subgroup of Ψ has order at most 8, and one cannot argue as in step (17). If the conjugate involutions α, β, and γ would be reflections, then one of these would have axis W and could not be conjugate to the others (since $W^\Psi = W$). Therefore, α, β, and γ are planar. By [15, 55.39 (a)] their common fixed elements form a 4-dimensional plane $\mathscr{C} = \mathscr{F}_{\alpha,\beta} < \mathscr{F}_\beta < \mathscr{P}$. Because $\mathrm{Sp}_2\,\mathbb{R} = \mathrm{SL}_2\,\mathbb{R}$, there is a covering

$$\kappa : (\mathrm{SL}_2\,\mathbb{R})^3 \to \Omega \leqslant \mathrm{Cs}\{\alpha, \beta\}.$$

The group Ω induces on $\mathscr{C}$ a semi-simple group $\Omega|_{\mathscr{C}} = \Omega/\mathsf{K}$, and K is a *compact* normal subgroup of Ω by [15, 83.9]. This implies that K is discrete, and $\Omega|_{\mathscr{C}}$ is locally isomorphic to $(\mathrm{SL}_2\,\mathbb{R})^3$, but a semi-simple group of automorphisms of $\mathscr{C}$ is actually almost simple, see [15, 71.8].

(19) Thus, in the case $t = 14$, the group Ψ cannot be almost simple. From Clifford's Lemma it follows that Ψ is a product of two almost simple factors A and B, where $\dim \mathsf{B} \geqslant 10$ and B acts irreducibly on $\mathbb{R}^7$, cf. also Salzmann [14], Lemma 4. The List shows that B is of type G_2 or B_3. As noted above, $\dim \Psi \leqslant 25$, and hence $\dim \mathsf{A} \leqslant 10$. Consequently, A acts irreducibly on $\mathbb{R}^2$ and $\mathsf{A} \cong \mathrm{SL}_2\,\mathbb{R}$. Therefore, $\dim \mathsf{B} > 14$ and B is of type B_3. In particular, $\Psi = \mathsf{AB}$ has torus rank $\mathrm{rk}\,\Psi = 4$, and then lemma (14) implies $t = 12$, a contradiction.

(20) Only the possibility $t = 12$ remains. Several arguments of steps (14)–(19) fail in this case, and the proof will become more cumbersome. An improved lower bound for $\dim \Psi$ will somewhat reduce the number of cases to be considered. In the next step, it will be shown that Θ is the connected component of T. This implies that $\dim \Omega \leqslant 14$ and $21 \leqslant \dim \Psi \leqslant 39 - t = 27$.

(21) If $\dim \mathsf{T} > 12$, then, by complete reducibility, Θ has a Ψ-invariant complement Π in T^1, and $\dim \Pi \leqslant 2$. Because Ψ acts irreducibly on Θ or each Ψ-invariant

subspace of Θ is 6-dimensional [15, 95.6], the group Π is unique. From $[\Omega, \Psi] \subseteq \mathsf{T}$ it follows that $\Pi^{\omega\psi} = \Pi^{\omega}$ for each $\omega \in \Omega$ and $\psi \in \Psi$. Hence $\Pi^{\omega} = \Pi$ and $\Pi^{\Delta} = \Pi^{\Psi\Omega} = \Pi$, but this is impossible by step (1).

(22) Suppose first that Ψ is almost simple. According to the List, $\Psi \cong \mathrm{Sp}_6\,\mathbb{R}$ or $\Psi \cong \mathrm{U}_3(\mathbb{H}, r)$, and Ψ acts on Θ in the natural way. Exactly as in step (18), the symplectic case leads to a contradiction (κ being an isomorphism). Other than the projective forms, the simply connected groups $\mathrm{U}_3(\mathbb{H}, r)$ do not contain an elementary abelian subgroup of order 16, and one cannot reason as in step (17). The central involution $\varepsilon = -\mathbb{1}$ of $\Psi = \mathrm{U}_3(\mathbb{H}, r)$ inverts each translation in Θ and hence is a reflection with axis W. Use the same unitary form as in (17) and assume that $\alpha = \mathrm{diag}(-1, -1, 1)$ is planar. The involution α is contained in the center of a compact subgroup $\mathsf{A} \cong \mathrm{U}_2\,\mathbb{H}$ of Ψ, and A would induce on the fixed plane $\mathscr{F}_\alpha$ a group $\mathrm{SO}_5\,\mathbb{R}$. Therefore, $\sigma = \varepsilon\alpha = \mathrm{diag}(1, 1, -1)$ is a reflection with an axis au. Put $\mathsf{M} = \Theta \cap \mathrm{Cs}\,\sigma$, and note that $\mathsf{M} \cong \mathbb{R}^8$ (by the construction of σ). Now $a^{\mathsf{M}} \subseteq au$ and $\mathsf{M} \leqslant \Theta_{[u]}$. Because $u^{\Delta} \neq u$, the translation group T would be transitive, contrary to what has been assumed.

(23) Finally, still for $t = 12$, let $\Psi = \mathsf{AB}$ be a product of semi-simple factors A and B, where $[\mathsf{A}, \mathsf{B}] = \mathbb{1}$ and $0 < \dim \mathsf{A} \leqslant \dim \mathsf{B}$. Remember from (9) that A is faithfully represented on Θ. Hence A contains at least one involution, and even two commuting involutions if $\dim \mathsf{A} > 6$, see [15, 94.37] and note that the simply connected covering group of $\mathrm{SL}_3\,\mathbb{R}$ has no faithful linear representation. In order to determine an upper bound for $\dim \mathsf{B}$, a few results on orbits of a point $z \in W$ are needed.

(24) In the situation of (23), each point $z \in W$ has an orbit z^{Δ} of dimension $\dim z^{\Delta} = k > 2$. Indeed, choose points $u, v \in z^{\Delta}$ such that u, v, z are distinct. Let ∇ denote the stabilizer of the triangle a, u, v. Consider the equivalent actions of ∇_z on $\mathsf{H} = \Theta_{[z]} \cong \mathbb{R}^s$ and on a^{H}, and note that $4 \leqslant s \leqslant 6$. Whenever $a \neq c \in a^{\mathsf{H}}$, then $\nabla_c \not\cong \mathrm{SU}_3\,\mathbb{C}$, and ($\square$) implies $\dim\nabla_c \leqslant 7$. The dimension formula gives $19 \leqslant \dim \Delta_a \leqslant 3k + \dim c^{\nabla_z} + \dim\nabla_c$. Hence, if $k \leqslant 2$, then $\dim c^{\nabla_z} = 6$ and $\dim\nabla_z = 13$, moreover, ∇_z is transitive and effective on $\mathsf{H} \cong \mathbb{R}^6$, and then $\nabla_z \hookrightarrow \mathbb{C}^{\times}\mathrm{SU}_3\,\mathbb{C}$ and $\dim\nabla_z \leqslant 10$, see Völklein [17] or [15, 96.19–96.22, 94.34]. Therefore, $3k \geqslant 7$.

(25) More can be said if $z^{\Omega} \neq \{z\}$. Since $[\Delta, \Omega] \subseteq \mathsf{T}$, the orbit z^{Ω} is fixed pointwise by Δ_z, and since Ω induces on $\Theta = \mathsf{T}^1$ a group of complex dilatations, $\dim z^{\Omega} = 1$. Hence the arguments of the last step give $k \geqslant 7$ instead of $3k \geqslant 7$. Consequently, $\dim z^{\Psi} \geqslant 6$. (Note that in general $\dim z^{\Delta} \leqslant \dim z^{\Psi} + \dim z^{\Omega} \leqslant \dim z^{\Psi} + 1$.)

(26) The factor A does not contain any reflection with an axis $L \neq W$. Assume, in fact, that α is a reflection in A with center $u \in W$ and axis $L \neq W$, and put $L \cap W = v$. Then $u^{\mathsf{B}} = u$ and $u^{\Psi} = u^{\mathsf{A}}$, and, if $u^{\Omega} = u$, then even $u^{\Delta} = u^{\mathsf{A}}$. In any case, $\dim u^{\mathsf{A}} \geqslant 3$ by the last steps, and B acts trivially on u^{A}. Similarly, $\dim v^{\mathsf{A}} \geqslant 3$ and B fixes each line in the orbit L^{A} and hence also some intersection point $a \notin W$ of such lines. If $a \neq c \in a^{\Theta_{[v]}}$, then the fixed elements of $\Lambda = \mathsf{B}_c$ form a subplane $\mathscr{F}_\Lambda$ with lines of dimension at least 3. Therefore, $\mathscr{F}_\Lambda$ is a Baer plane, or $\mathscr{F}_\Lambda = \mathscr{P}$. From [15, 83.22] it follows that $\dim \Lambda \leqslant 3$. Hence $\dim \mathsf{B} \leqslant \dim \Theta_{[v]} + 3 \leqslant 9$ and $\dim \Psi < 19$, a contradiction.

(27) If $\mathscr{B} = \mathscr{F}_\alpha$ is the fixed plane of a Baer involution $\alpha \in \mathsf{A}$, then B acts effec-

tively on $\mathscr{B}$. Assume on the contrary that B contains an element $\beta \neq \mathbb{1}$ which induces the identity on $\mathscr{B}$. Then $\mathscr{B} = \mathscr{F}_\beta$ is even Ψ-invariant. Put $\overline{\Psi} = \Psi|_{\mathscr{B}} = \Psi/\Phi$. From [15, 84.16] it follows that $\dim \overline{\Psi} \leqslant 19$ and $\dim \Phi \geqslant 3$. By [15, 83.22], the connected component of the kernel Φ is isomorphic to $\mathrm{SU}_2\,\mathbb{C}$, and $\dim \overline{\Psi} \geqslant 18$. One may now choose $\mathsf{A} = \Phi^1$. The group B is then a covering group of $\overline{\Psi}$. Because $\mathsf{M} = \Theta \cap \mathrm{Cs}\,\alpha$ is B-invariant, Clifford's Lemma implies that $\mathsf{M} \cong \mathbb{R}^6$. Moreover, $\mathscr{B}^{\mathsf{MB}} = \mathscr{B}$, and $\mathscr{B}$ is isomorphic to the quaternion plane $\mathscr{P}_2\mathbb{H}$, see [15, 84.27] or Salzmann [9]. The large semi-simple groups of the affine quaternion plane can easily be determined, they are described, e.g., in Salzmann [9], §3. In particular, such a group has dimension at most 13, or it contains $\mathrm{SL}_2\,\mathbb{H}$, but the latter group does not have a faithful representation on M.

(28) Assume again that $\mathscr{B}$ is the fixed point plane of a Baer involution $\alpha \in \mathsf{A}$. Then $\mathscr{B} \cong \mathscr{P}_2\mathbb{H}$ is true in any case. In fact, $\dim \mathsf{B} \geqslant 11$ (since $\dim \mathsf{A} + \dim \mathsf{B} \geqslant 21$). By Clifford's Lemma, either $\mathsf{M} = \Theta \cap \mathrm{Cs}\,\alpha \cong \mathbb{R}^4$ and $\mathsf{B} \cong \mathrm{SL}_4\,\mathbb{R}$, or $\mathsf{M} \cong \mathbb{R}^6$. Therefore, $\dim \mathsf{BM} \geqslant 17$. The last step implies $\mathsf{BM} \hookrightarrow \mathrm{Aut}\,\mathscr{B}$, and $\mathscr{B}$ is not a proper Hughes plane, see [15, 86.35]. The theorem in Salzmann [12] shows that $\mathscr{B}$ is a translation plane, and hence the dimension of $\mathrm{Aut}\,\mathscr{B}$ is at least 19. The claim is now a consequence of [15, 82.25].

(29) From (28) and the last remark in (27) it follows that $\dim \mathsf{B} \leqslant 13$ (whenever there is a Baer involution $\alpha \in \mathsf{A}$). All remaining cases lead to a contradiction: if $\dim \mathsf{B} = 13$, then $\dim \mathsf{A} \geqslant 8$, the torus rank $\mathrm{rk}\,\mathsf{B} = 3$, and $\mathrm{rk}\,\mathsf{A} = 1$, cf. [15, 55.37]. Consequently, $\mathsf{A} \cong \mathrm{SL}_3\,\mathbb{R}$ contains even 3 pairwise commuting planar involutions. Their common fixed elements form a 4-dimensional B-invariant subplane $\mathscr{C} < \mathscr{F}_\alpha$, see [15, 55.39]. Because of [15, 83.11], the semi-simple group B would act with a discrete kernel on $\mathscr{C}$, but this contradicts [15, 71.8]. Therefore, $\dim \mathsf{B} < 13$, $\dim \mathsf{A} > 8$, and $\mathrm{rk}\,\mathsf{A} = \mathrm{rk}\,\mathsf{B} = 2$. If $\dim \mathsf{B} = 11$, then B has a factor $\Gamma \cong \mathrm{SL}_3\,\mathbb{R}$. This case can be ruled out, applying the previous arguments to Γ instead of A. Clifford's Lemma shows that a group B which is locally isomorphic to $(\mathrm{SL}_2\,\mathbb{C})^2$ cannot act effectively on $\mathbb{R}^6$. For this reason, the last possibility $\dim \mathsf{B} = 12$ is also excluded.

(30) Whenever $\Psi = \mathsf{AB}$ as in (23), one can conclude from (23)–(29) that each involution $\alpha \in \mathsf{A}$ is a reflection with axis W. It follows that $\alpha|_\Theta = -\mathbb{1}$. Hence α is unique, and α is contained in the center of Ψ. In particular, $\mathrm{rk}\,\mathsf{A} = 1$ and A is a subgroup of $\mathrm{SL}_2\,\mathbb{C}$. Consequently, $\dim \mathsf{A} \in \{3, 6\}$ and $15 \leqslant \dim \mathsf{B} \leqslant 24$. Suppose that B is an almost direct product of proper normal subgroups Γ and P with $\dim \Gamma \leqslant \dim \mathsf{P}$. It has just been proved that in any factorization of Ψ one of the factors is a subgroup of $\mathrm{SL}_2\,\mathbb{C}$. This is true, in particular, for $\Psi = (\Gamma\mathsf{A})\mathsf{P}$. Because $\mathrm{rk}\,\Gamma\mathsf{A} > 1$, one has necessarily $\mathsf{P} \hookrightarrow \mathrm{SL}_2\,\mathbb{C}$, and then $\dim \mathsf{B} \leqslant 12$, a contradiction. Therefore, B is almost simple.

(31) Since almost simple groups of type A_4 or B_3 and the group $\mathrm{Spin}_5\,\mathbb{C}$ do not admit an irreducible representation in a dimension dividing 12, Clifford's Lemma (together with the List) shows that $\dim \mathsf{B} \leqslant 21$ and that B is not an orthogonal group. The arguments of step (22) exclude the possibilities $\mathsf{B} \cong \mathrm{Sp}_6\,\mathbb{R}$ and $\mathsf{B} \cong \mathrm{U}_3(\mathbb{H}, r)$. Hence only the cases $\dim \mathsf{B} \leqslant 16$ and $\mathsf{A} \cong \mathrm{SL}_2\,\mathbb{C}$ remain to be discussed.

(32) If $\dim \mathsf{B} = 16$, then $\mathsf{B} \cong \mathrm{SL}_3\,\mathbb{C}$, and B contains 3 pairwise commuting involutions conjugate to $\beta = \mathrm{diag}(-1, -1, 1)$. These cannot be reflections (because $W^\Delta =$

W). The planar involution β is in the center of a subgroup $\Gamma \cong \mathrm{SL}_2\,\mathbb{C}$ of B, and Γ induces on the Baer plane $\mathscr{B} = \mathscr{F}_\beta$ a group $\bar{\Gamma}$ isomorphic to the Möbius group $\mathrm{PSL}_2\,\mathbb{C}$ (note that the kernel of the action of Γ on $\mathscr{B}$ is compact). The central involution $\alpha \in \mathsf{A}$ is a reflection with axis W, and $\mathscr{B}^{\mathsf{A}} = \mathscr{B}$. Consequently, A induces on $S = W \cap \mathscr{B} \approx \mathbb{S}_4$ also a Möbius group $\bar{\mathsf{A}}$. The direct product $\bar{\mathsf{A}} \times \bar{\Gamma}$ acts effectively on S (since $\bar{\mathsf{A}}$ and $\bar{\Gamma}$ are the only proper normal subgroups), but a maximal compact subgroup of $\bar{\mathsf{A}} \times \bar{\Gamma}$ is isomorphic to $(\mathrm{SO}_3\,\mathbb{R})^2$, and this group cannot act effectively on $\mathbb{S}_4$ by Richardson's Theorem. Thus $\dim \mathsf{B} \neq 16$.

(33) Finally, if $\dim \mathsf{B} = 15$, then $\dim \Psi = 21$ and $\dim \Delta = 35$. Let a be the center of the reflection $\alpha \in \mathsf{A}$ and put $\Gamma = (\Delta_a)^1$. Since $\Gamma \cap \mathrm{Cs}\,\Theta = \mathbb{1}$, the group Γ acts effectively on Θ. From (21) and the well-known fact that the product of two reflections with the same axis and different centers is an elation [15, 23.20], it follows that $\alpha\alpha^\Delta \subseteq \Theta$, and this implies successively $\alpha^\Delta = \alpha^\Theta$, $\dim a^\Delta = 12$, and $\dim \Gamma = 23$. Obviously, $\Psi < \Gamma$, the radical $\mathsf{P} = \sqrt{\Gamma}$ is 2-dimensional, and $[\Psi, \mathsf{P}] = \mathbb{1}$. With Clifford's Lemma, one can conclude that Ψ acts irreducibly on Θ, and [15, 95.6 (b)] shows that $\mathsf{P} \cong \mathbb{C}^\times$. Consequently, $\Psi \hookrightarrow \mathrm{SL}_6\,\mathbb{C}$. By Schur's Lemma, B is not irreducible on $\mathbb{C}^6$, and the complex version of Clifford's Lemma implies $\mathsf{B} \hookrightarrow \mathrm{SL}_3\,\mathbb{C}$, but the latter group does not have a subgroup of codimension one. This contradiction completes the proof of the theorem.

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H. Salzmann, Mathematisches Institut der Universität Tübingen, Auf der Morgenstelle 10, 72076 Tübingen, Germany
E-mail: helmut.salzmann@uni-tuebingen.de