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Topological and smooth unitals

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Abstract. We first show that under natural topological assumptions in compact, connected projective planes there are no objects besides ovals which have incidence geometric properties analogous to those of unitals in finite projective planes. After giving a brief account of 'classical unitals', i.e. sets of absolute points of continuous polarities in the classical projective planes $\mathscr{P}_2\mathbb{F}$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, we define unitals by means of their intersection properties with respect to lines, in analogy to properties of classical unitals. Under suitable topological assumptions, unitals turn out to be homeomorphic to spheres. The existence of exterior lines is related to the codimension of the unital in the point space. Our main result is that unitals satisfying an additional regularity condition have the same dimensions as classical unitals. Finally, we consider smooth unitals in smooth projective planes. In this case some results can be improved by using transversality arguments.

This paper is inspired by the beautiful paper [3] by Buchanan, Hähl and Löwen on topological ovals. The methods which we will use here, however, are often quite different. Unitals were defined originally as sets of absolute points of unitary polarities of Desarguesian finite projective planes. Such unitals have the characteristic property that every line intersects in 0, 1 or n points ($n \geq 2$ fixed). This property can be used to define unitals in arbitrary finite projective planes (cf. [4], in particular 3.3.7). At the beginning of Section 2, we will give a short proof that under natural topological assumptions there are no such objects besides ovals ($n = 2$) in compact, connected projective planes. Hence, this is not the right way to generalize the concept of a unital to compact, connected projective planes.

This gives the motivation for introducing the notion of a *unital* in another way, as a generalization of topological ovals and of sets of absolute points of continuous polarities in the classical projective planes $\mathscr{P}_2\mathbb{F}$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, where $\mathbb{H}$ denotes the skew-field of quaternions, and $\mathbb{O}$ the alternative division algebra of octonions (or Cayley numbers). In these planes, there are two conjugacy classes of continuous polarities with non-empty sets of absolute points (called *classical unitals* and denoted by U in the following), the hyperbolic polarities and the planar polarities. In the hyperbolic case, U is a sphere of codimension 1. The planar case occurs only for $l > 1$ (where l denotes the dimension of $\mathbb{F}$ over $\mathbb{R}$), and then U is a sphere of dimension

$\frac{3}{2}l - 1$. In 2.4, we will discuss various properties of classical unitals and will sketch proofs for some of these properties. Unitals in compact, connected projective planes in general will be defined by their intersection properties with respect to lines, see Definition 2.5, analogously to the definition of ovals in [3]. We will show that under natural topological assumptions unitals are homeomorphic to spheres. For the remaining part of this paper we will then concentrate on spherical unitals (see Definition 2.7). At the end of Section 2, we will investigate the existence of exterior lines of unitals according to their codimension in the point space. The main result of this paper is the following

Theorem. *Let U be a spherical unital. Then, under natural assumptions, U has the same dimension as one of the classical unitals.*

In Section 3, we will make this statement precise and prove this theorem by means of a Gysin sequence. A partial result of this type will be obtained under weaker assumptions by means of the Vietoris–Begle mapping theorem. In the last section we will consider smooth unitals (see Definition 4.1). By using transversality arguments we will prove additional regularity properties for such unitals. In this way, we will see that certain assumptions, which had to be included in the previous two sections, are always satisfied for smooth unitals.

In [9], H. Löwe, R. Löwen, and E. Soytürk investigated unitals from a different point of view. Their definition of a unital is contained in our notion of a spherical unital (cf. Definition 2.7). They concentrate on unitals of codimension 1 in the point spaces of topological translation planes and show—among other topics—that every compact ovoid in $\mathbb{R}^{2l}$ is a unital in any topological affine translation plane defined on $\mathbb{R}^{2l}$, where $l \in \{1, 2, 4, 8\}$. In our forthcoming paper [7] we will consider sets of absolute points of continuous (or smooth) polarities in compact, connected (or smooth) projective planes and will investigate in which respect such sets are similar to classical unitals.

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1 Preliminaries and notation

Let $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ be a projective plane. We call the elements of P *points*, the elements of $\mathscr{L}$ *lines* and the elements of the incidence relation $\mathscr{F} \subseteq P \times \mathscr{L}$ *flags*. If (p, L) is a flag, we say that the point p and the line L are incident, or that p lies on L, or that L passes through p. The set of points incident with a line L is called a *point row* and is denoted by P_L or also by L, if no confusion can arise. The set $\mathscr{L}_p$ of lines incident with a point p is called a *line pencil*. The incidence structure $\mathscr{P}^*$ obtained by interchanging the rôles of P and $\mathscr{L}$ is called the *dual plane* of $\mathscr{P}$.

Definition 1.1. A *compact, connected projective plane* is a projective plane $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ whose point space P and line space $\mathscr{L}$ are compact, connected topological spaces such that joining points and intersecting lines are continuous operations on their respective domains, i.e. the *join map* $\vee : P \times P \setminus \Delta(P) \to \mathscr{L}$ and the *intersection map* $\wedge : \mathscr{L} \times \mathscr{L} \setminus \Delta(\mathscr{L})$ are continuous, where Δ denotes the diagonal: $\Delta(X) = \{(x,x) \mid x \in X\}$. For later use we define $\vee_p : P\setminus\{p\} \to \mathscr{L}_p : q \mapsto p \vee q$ for any $p \in P$.

By [10], Corollary 41.5, the flag space $\mathscr{F}$ of a compact projective plane $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ is closed in $P \times \mathscr{L}$. For the sake of simplicity, we will always assume here that point rows are topological manifolds. In fact, the topological properties of compact, connected projective planes are close to those of manifolds, see [10], Chapter 5, and no examples of compact, connected projective planes are known for which this assumption would not be satisfied. In some cases, however, this assumption is unnecessary or would become unnecessary if we replaced statements on the homeomorphism type of certain sets (cf. Definitions 2.5, 2.7) by statements on their (co-)homological properties. This would seem quite artificial in the present context, but will become natural in our forthcoming paper [7], where we will investigate sets of absolute points of continuous polarities.

By the continuity of the join map and the intersection map, also P, $\mathscr{L}$, and $\mathscr{F}$ are topological manifolds. Furthermore, point rows and line pencils are homeomorphic to l-dimensional spheres, where $l \in \{1,2,4,8\}$, the point space P and the line space $\mathscr{L}$ have dimension $2l$, and $\mathscr{F}$ has dimension $3l$. Throughout this paper, the letter l will always be used in this sense. The sphere of dimension m will be denoted by $\mathbb{S}_m$. For every line $L \in \mathscr{L}$ the set P/L (with the quotient topology) is homeomorphic to $\mathbb{S}_{2l}$. All these statements can be found in [10], 52.1, 3, 5, and 6 (b).

Definition 1.2. A *smooth projective plane* is a projective plane whose point space and line space are differentiable manifolds such that the join map and the intersection map are differentiable.

Both ‘differentiable’ and ‘smooth’ will be used in the sense of C^∞ in this paper. By a smooth submanifold we will always mean a smoothly embedded submanifold. The classical planes $\mathscr{P}_2\mathbb{F}$, $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, are smooth projective planes and smooth projective planes are compact, connected projective planes. The point space and the line space of a smooth projective plane are homeomorphic to the point space of the classical projective plane of the same dimension, see [8]. Point rows of smooth projective planes are smoothly embedded submanifolds of the point space, and any two distinct point rows L, K intersect transversally in P, i.e. $\mathrm{T}_pP_L + \mathrm{T}_pP_K = \mathrm{T}_pP$ for $p = L \wedge K$, see [1], 2.6 and 2.13. A dual statement holds for line pencils instead of point rows.

2 The concept of a unital

Throughout this section, $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ will always denote a compact, connected projective plane of dimension $2l$ with manifold lines.

Definition 2.1. In analogy to the definition of unitals in finite geometry, cf. [4], 3.3.7, we define a *unital of finite type* as a subset U of the point space of a projective plane, such that every line intersects U in precisely 0, 1 or n points, where $n \geq 2$ is a fixed number, and for every point $p \in U$ there is exactly one line T_p which intersects U only in p. Unitals of finite type with $n = 2$ are called *ovals*.

The unique line T_p with $U \cap T_p = \{p\}$ is called the *tangent* to U at the point p. Lines which intersect U in more than one point are called *secants*, and lines which have no point in common with U are called *exterior lines*. The same notions will be used also in connection with other types of unitals, which will appear later in this paper.

By [3], Theorem 2.6, a closed oval O in a compact, connected projective plane $\mathscr{P}$ is a *topological oval* in the sense of [3], Definition 2.4. This implies that the maps

$$\psi_y : O \to \mathscr{L}_y : x \mapsto \begin{cases} x \vee y & \text{if } x \neq y \\ T_y & \text{if } x = y \end{cases}$$

are homeomorphisms for each $y \in U$. In particular, we have a (trivial) covering with covering space $O \backslash \{y\}$, base space $\mathscr{L}_y \backslash T_y$ and projection $\vee_y|_{O\backslash\{y\}}$. This motivates the following definition: a unital of finite type U in a compact, connected projective plane $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ is called a *topological* unital of finite type if U is closed in P and the maps $\xi_y : U\backslash\{y\} \to \mathscr{L}_y\backslash\{T_y\} : x \mapsto x \vee y$ are covering projections for each $y \in U$. Since each secant intersects U in exactly n points, $U\backslash\{y\}$ is then an $(n-1)$-fold covering of $\mathscr{L}_y\backslash\{T_y\}$.

Proposition 2.2. *There are no topological unitals of finite type other than topological ovals.*

Proof. That topological ovals are topological unitals of finite type has already been shown above. So let U be a topological unital of finite type and let $y \in U$. Since $\xi_y : U\backslash\{y\} \to \mathscr{L}_y\backslash\{T_y\}$ is a covering projection and $\mathscr{L}_y\backslash\{T_y\}$ is homeomorphic to the simply connected space $\mathbb{R}^l$, the space $U\backslash\{y\}$ is homeomorphic to $n-1$ disjoint copies of $\mathbb{R}^l$. Thus, U itself is homeomorphic to a bouquet of $n-1$ spheres, where the common point of all spheres corresponds to y. Now choose $z \in U$ distinct from y. By the above arguments, U is also homeomorphic to a bouquet of $n-1$ spheres, where the common point of the spheres corresponds to z. This yields a contradiction except for $n-1 = 1$. Thus, U is a closed oval.

Definition 2.3. A *polarity* of a projective plane $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ is a map π of the disjoint union $P \cup \mathscr{L}$ onto itself with $\pi^2 = \mathrm{id}$ which exchanges points and lines and preserves incidence, i.e. $(p, L) \in \mathscr{F} \Leftrightarrow (L^\pi, p^\pi) \in \mathscr{F}$. A point $p \in P$ which lies on p^π is called an *absolute point*.

2.1 A brief account of classical unitals. We call the sets of absolute points of continuous polarities of classical projective planes *classical unitals*. Every continuous polarity of a classical plane $\mathscr{P}_2\mathbb{F}$ ($\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$) is conjugate to the *standard elliptic polarity*, the *standard planar polarity*, or the *standard hyperbolic polarity*, see

[10], 13.18 and 18.29. For $\mathbb{F} = \mathbb{R}$, there is no planar polarity. As a general reference for polarities of classical projective planes we mention Sections 13 and 18 of [10], in particular 13.18, 18.32 and 18.33. Some of the properties of classical unitals discussed here will be proved later by more theoretical arguments, others will provide the motivation for certain assumptions in propositions and theorems in the following sections.

In order to describe polarities of classical planes, we recall that the $\mathbb{R}$-algebras $\mathbb{F}$ may be constructed inductively by means of the Cayley–Dickson process (cf. [10], 11.1, or [5], Chapter 9), where $\mathbb{F}_0 = \mathbb{R}$, $\mathbb{F}_1 = \mathbb{C}$, $\mathbb{F}_2 = \mathbb{H}$ and $\mathbb{F}_3 = \mathbb{O}$. For $m \geq 1$, these algebras have an involutory antiautomorphism $x \mapsto \bar{x}$, called *conjugation*, and an involutory automorphism ε which leaves $\mathbb{F}_{m-1}$ fixed and commutes with conjugation (cf. [10], 18.28). By the way, ε induces on $\mathscr{P}_2\mathbb{F}_m$, $m \in \{1, 2, 3\}$, an involutory automorphism which fixes pointwise the subplane $\mathscr{P}_2\mathbb{F}_{m-1}$. For $m = 0$ there is no such involutory automorphism, and conjugation is defined to be the identity.

Except for $\mathbb{F} = \mathbb{O}$, the standard polarities on $\mathscr{P}_2\mathbb{F}$ may by described by the corresponding sesquilinear forms, which in homogeneous coordinates are given as follows:

$$f(x, y) = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \bar{x}_3 y_3 \quad \textit{standard elliptic polarity}$$

$$g(x, y) = \bar{x}_1 y_1 + \bar{x}_2 y_2 - \bar{x}_3 y_3 \quad \textit{standard hyperbolic polarity}$$

$$h(x, y) = \bar{x}_1^{\varepsilon} y_1 + \bar{x}_2^{\varepsilon} y_2 + \bar{x}_3^{\varepsilon} y_3 \quad \textit{standard planar polarity}$$

The standard elliptic polarity on $\mathscr{P}_2\mathbb{F}$ has no absolute points. As can be seen from the definition of the Hermitian form g, the unital $U_{\mathrm{hyp}}(\mathbb{F})$ consisting of the absolute points of the standard hyperbolic polarity of $\mathscr{P}_2\mathbb{F}$ has exterior lines. In affine coordinates we have $U_{\mathrm{hyp}}(\mathbb{F}) = \{(x_1, x_2) \in \mathbb{F}^2 \mid |x_1|^2 + |x_2|^2 = 1\}$. The latter is true also in the case $\mathbb{F} = \mathbb{O}$, see 18.21. In particular, $U_{\mathrm{hyp}}(\mathbb{F})$ is always homeomorphic to a sphere of dimension $2l - 1$ ($l = \dim \mathbb{F}$). For $\mathbb{F} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$, the unital $U_{\mathrm{pla}}(\mathbb{F})$ which corresponds to the set of absolute points of the standard planar polarity is a sphere of dimension $\frac{3}{2}l - 1$, cf. [10], 18.32. This can be proved in the following way: the Lie group of collineations of $\mathscr{P}_2\mathbb{F}$ which commute with the standard planar polarity (the *planar motion group*) acts doubly transitively on $U = U_{\mathrm{pla}}(\mathbb{F})$. Using this smooth group action, one can show that for each $p \in U$ we have a locally trivial fibration with total space $U \backslash \{p\}$, base space $\mathscr{L}_p \backslash \{T_p\}$, and projection $\xi_p : U \backslash \{p\} \to \mathscr{L}_p \backslash \{T_p\} : x \mapsto x \vee p$. Here, T_p is the image of p under the standard planar polarity. For a suitable choice of $p \in U$ and $K \in \mathscr{L}_p \backslash \{T_p\}$, we see by easy calculations that the fibre $(K \cap U) \backslash \{p\}$ is homeomorphic to $\mathbb{R}^{(l/2)-1}$. The proof is then completed as in Proposition 2.6 below. We summarize here some further consequences: the planar unital U is a smooth submanifold of P, there is exactly one tangent at each point $p \in U$, and each secant intersects U transversally in a submanifold homeomorphic to $\mathbb{S}_{(l/2)-1}$.

Also the *hyperbolic motion group* (defined analogously) acts doubly transitively on $U_{\mathrm{hyp}}(\mathbb{F})$, cf. [10], 18.23. We obtain that for each point $q \in U_{\mathrm{hyp}}(\mathbb{F})$ there is exactly one tangent at q to $U_{\mathrm{hyp}}(\mathbb{F})$, and each secant intersects $U_{\mathrm{hyp}}(\mathbb{F})$ transversally in a smooth submanifold homeomorphic to $\mathbb{S}_{l-1}$.

The following definition of unitals is motivated by the properties of classical unitals established above.

Definition 2.4. Let $\mathcal{P} = (P, \mathcal{L}, \mathcal{F})$ be a compact, connected projective plane. A subset U of P is called a *unital* if the following axioms are satisfied:

(U1) For every $x \in U$ there is exactly one tangent $T_x \in \mathcal{L}_x$.

(U2) Every secant intersects U in a set homeomorphic to $\mathbb{S}_k$ ($k \geq 0$ fixed).

Throughout this paper, the letter k will always be used in the sense of Definition 2.4. We will denote the set of tangents by U^* and the set of tangents through a point $p \in P \backslash U$ by $\mathcal{T}_p$. The set of points in which these tangents touch the unital U, also called the *set of feet* of p, will be denoted by B_p.

For each $x \in U$, the map $\xi_x : U \backslash \{x\} \to \mathcal{L}_x \backslash \{T_x\}$ is surjective, and for each $L \in \mathcal{L}_x \backslash \{T_x\}$ the set $\xi_x^{-1}(L)$ is homeomorphic to $\mathbb{R}^k$. It would be natural to require that for each point $x \in U$ the maps ξ_x are locally trivial fibrations. In fact, this is the case for the classical unitals, see 2.1. However, this condition is unnecessarily strong. The next proposition shows that already under weaker assumptions U is homeomorphic to a sphere.

Proposition 2.5. *Let U be a unital in a compact, connected projective plane $\mathcal{P} = (P, \mathcal{L}, \mathcal{F})$. Then each of the following two conditions implies that U is homeomorphic to a sphere of dimension $k + l$:*

(i) *U is closed in P and there is a point $p \in U$ such that the map ξ_p (defined above) is a locally trivial fibration.*

(ii) *There are two distinct points $p, q \in U$ such that the maps ξ_p and ξ_q are locally trivial fibrations.*

Proof. The punctured line pencil $\mathcal{L}_p \backslash \{T_p\}$ is homeomorphic to $\mathbb{R}^l$. Since $\mathbb{R}^l$ is contractible, we infer that the locally trivial fibration induced by ξ_p is in fact trivial. Hence, $U \backslash \{p\}$ is homeomorphic to $\mathbb{R}^l \times \mathbb{R}^k$. If U is compact as required in (i), we conclude that U is homeomorphic to $\mathbb{S}_{k+l}$. In the case (ii), both $U \backslash \{p\}$ and $U \backslash \{q\}$ are homeomorphic to $\mathbb{R}^{k+l}$. Thus U is a $(k + l)$-dimensional topological manifold. Choose a homeomorphism $\psi : U \backslash \{p\} \to \mathbb{R}^{k+l}$ and identify $\mathbb{S}_{k+l}$ with the one-point compactification of $\mathbb{R}^{k+l}$. Then the map

$$U \to \mathbb{S}_{k+l} : x \mapsto \begin{cases} \psi(x) & \text{for } x \neq p \\ \infty & \text{for } x = p \end{cases}$$

is a continuous bijection and thus a homeomorphism by domain invariance, see [6], p. 82, Exercise (18.10).

Proposition 2.5 and the results on classical unitals motivate the following.

Definition 2.6. A unital U in a compact, connected projective plane is called a *spherical unital* if U is homeomorphic to a sphere of dimension $k + l$.

Remark. Note that topological ovals are spherical unitals by [3], Theorem (3.7). For spherical unitals, we have $k \le l - 1$. Otherwise U would be an open submanifold of P. Since U is compact and P is connected we would get $U = P$, a contradiction.

The following proposition characterizes spherical unitals U whose sets of tangents U^* are spherical unitals in the dual plane. As a consequence of this proposition, we then have $U^{**} = U$, since U^{**} consists of those points p with $|\mathcal{L}_p \cap U^*| = 1$.

Proposition 2.7. *Let U be a spherical unital in a compact, connected projective plane $\mathcal{P} = (P, \mathcal{L}, \mathcal{F})$. Then U^* is a spherical unital in $\mathcal{P}^*$ if and only if U^* is closed in P and for each $x \in P \backslash U$ there are either no tangents at all through x or the set B_x of feet is homeomorphic to $\mathbb{S}_k$.*

Proof. Let U be a spherical unital with the properties described above. By the compactness of U^* and $\mathcal{F}$, we see that the map $U \mapsto U^* : x \mapsto T_x$ is a homeomorphism, i.e. $U^* \approx \mathbb{S}_{k+l}$. For $q \in U$, we have $\mathcal{L}_q \cap U^* = \{T_q\}$, where T_q is the unique tangent at q. For $p \in P \backslash U$, there are no tangents to U through p at all, i.e. $\mathcal{L}_p \cap U^* = \emptyset$, or B_p is homeomorphic to $\mathbb{S}_k$. Then the continuous map $\vee_p$ induces a homeomorphism between B_p and $\mathcal{T}_p$, i.e. $\mathcal{L}_p \cap U^* \approx \mathbb{S}_k$. Hence, U^* satisfies the axioms of a spherical unital.

Assume now that U^* is a spherical unital in $\mathcal{P}^*$. Then U^* is homeomorphic to $\mathbb{S}_{k'+l}$ for some $k' \ge 0$, and using again the homeomorphism $U \mapsto U^* : x \mapsto T_x$ we see that $k' = k$. Choose $x \in P \backslash U$. For each $L \in U^*$ there is exactly one point $y \in P$ with $\mathcal{L}_y \cap U^* = \{L\}$, namely the point y determined by $P_L \cap U = \{y\}$. In particular, the line pencil $\mathcal{L}_x$ cannot have exactly one line in common with U^*. Hence, $\mathcal{L}_x \cap U^*$ is empty or homeomorphic to $\mathbb{S}_k$. We have to show that in the latter case B_x is homeomorphic to $\mathbb{S}_k$. Since U is compact and $\mathcal{F}$ is closed in $P \times \mathcal{L}$, the map $\mathcal{T}_x \to U : K \mapsto y_K$ with $P_K \cap U = \{y_K\}$ is continuous. Hence, B_x is homeomorphic to $\mathcal{T}_x = \mathcal{L}_x \cap U^*$ and therefore to $\mathbb{S}_k$. This completes the proof.

The results on spherical unitals in the following proposition are analogous to results on topological ovals in [3], Theorem (3.7).

Proposition 2.8. *Let U be a spherical unital. Then there are secants through every point. Moreover, $P \backslash U$ consists of exactly two connected components if U has codimension* 1 *in P, and $P \backslash U$ is connected otherwise. Assume in addition that the set of secants is open in $\mathcal{L}$. Then exterior lines exist if and only if the codimension of U in P is equal to* 1.

Proof. For $k = 0$, U is a closed oval. In this case, the above statements (and much more) have been proved in [3]. So we may assume that $k > 0$, $l > 1$. Assume that there exists a point p such that there are no secants through p. Then we have $p \notin U$, and the map $\vee_p$ induces a homeomorphism between U and $\mathcal{T}_p \subseteq \mathcal{L}_p$. Since $U \approx \mathbb{S}_{k+l}$ and $\mathcal{L}_p \approx \mathbb{S}_l$ we conclude by [6], Theorem 18.3, that $k + l \le l$, a contradiction.

Because of $l > 1$, we have $H_1(P; \mathbb{Z}) = 0$ by [10], 52.14, and $H_0(P, P \backslash U; \mathbb{Z}) \cong \overline{H}^{2l}(U; \mathbb{Z}) = 0$ by Alexander duality. The exact homology sequence of the pair $(P, P \backslash U)$ then yields a short exact sequence

$$0 \to H_1(P, P \backslash U; \mathbb{Z}) \to H_0(P \backslash U; \mathbb{Z}) \to H_0(P; \mathbb{Z}) \to 0.$$

This sequence splits because of $H_0(P;\mathbb{Z}) \cong \mathbb{Z}$. Again by Alexander duality, we have $H_1(P, P\backslash U;\mathbb{Z}) \cong \overline{H}^{2l-1}(U;\mathbb{Z})$. Hence, $H_0(P\backslash U;\mathbb{Z})$ is isomorphic to $\mathbb{Z}^2$ for $k = l-1$ and isomorphic to $\mathbb{Z}$ in the other cases. This proves the second statement.

By the compactness of U and $\mathcal{F}$ we see that the set $\mathcal{E}$ of exterior lines is open in $\mathcal{L}$. Since, by assumption, the set $\mathcal{S}$ of secants is open, too, we conclude that U^* is closed in $\mathcal{L}$. Then an easy compactness argument shows that the map which assigns to each point $x \in U$ the tangent T_x is continuous and hence a homeomorphism onto U^*. In particular, U^* is homeomorphic to $\mathbb{S}_{k+l}$. As above, we see that $\mathcal{L}\backslash U^*$ consists of exactly two connected components if the codimension of U^* in $\mathcal{L}$ is equal to 1 and that $\mathcal{L}\backslash U^*$ is connected otherwise. The set $\mathcal{S} = \bigvee(U \times U\backslash\Delta(U))$ is connected because the codimension of $\Delta(U)$ in $U \times U$ is at least 2. Since $\mathcal{L}\backslash U^*$ is the disjoint union of the open, connected set $\mathcal{S}$ and the open set $\mathcal{E}$, we conclude that exterior lines exist if and only if the codimension of U^* in $\mathcal{L}$, and hence of U in P, is equal to 1.

Remark. For smooth unitals U (see Definition 4.1), we will see in Section 4 that the set of secants is always open in $\mathcal{L}$. Furthermore, we will show by means of a Wang sequence that there are tangents through every point of $P\backslash U$ if the codimension of U in P is bigger than 1. In our forthcoming paper [7] we will prove more detailed results on the existence of secants, tangents and exterior lines through various points for sets of absolute points of smooth polarities.

3 Possible dimensions of unitals

In this section we will investigate the possible dimensions of spherical unitals. Under natural assumptions (see (R1) and (R2)) we will show that dimensions which do not occur for classical unitals are impossible also in this more general setting.

Let U be a spherical unital in a compact, connected projective plane $\mathcal{P} = (P, \mathcal{L}, \mathcal{F})$. Since unitals of codimension 1 exist in the classical projective planes, we may assume that the codimension of U in P is bigger than 1. Then the following regularity assumption is natural (cf. Propositions 2.7, 2.8):

(R1) There exists a point $p \in P\backslash U$ such that every line through p intersects the unital, and the set B_p of feet (or the set $\mathcal{T}_p$ of tangents through p) is homeomorphic to a sphere of dimension k.

Note that for $B_p \approx \mathbb{S}_k$ the map $\vee_p$ induces a homeomorphism between B_p and $\mathcal{T}_p$. Conversely, if $\mathcal{T}_p$ is homeomorphic to $\mathbb{S}_k$ we see that $B_p \approx \mathbb{S}_k$ by using the compactness of U and $\mathcal{F}$, cf. the proof of Proposition 2.7.

Proposition 3.1. *Let U be a spherical unital which satisfies condition* (R1). *Then we have* $l > 1$, *and the dimension of U cannot exceed* $\frac{3}{2}l - 1$, *the dimension of the classical planar unital in the classical projective plane of the same dimension.*

Proof. For $l = 1$, the unital U would be a closed oval which satifies condition (R1), in contradiction to [3], Theorems 2.6 and 3.7 (a). Hence, we have $l > 1$. The continuous, surjective map $\rho_p : U\backslash B_p \to \mathcal{L}_p\backslash\mathcal{T}_p : x \mapsto x \vee p$ is closed since U is compact and

$\vee_p^{-1}(\mathscr{T}_p) = B_p$. For each $L \in \mathscr{L}_p \backslash \mathscr{T}_p$, we have $\rho_p^{-1}(L) \approx \mathbb{S}_k$ and hence $\tilde{\bar{H}}^q(\rho_p^{-1}(L); \mathbb{Z}) = 0$ for all $q < k$. By the Vietoris–Begle mapping theorem (see [11], p. 344, Theorem 15), the map $\rho_p^* : \bar{H}^q(\mathscr{L}_p \backslash \mathscr{T}_p; \mathbb{Z}) \to \bar{H}^q(U \backslash B_p; \mathbb{Z})$ is an isomorphism for $q < k$. Now assume that $k > (l/2) - 1$. Then we have $l - k - 1 < k$ and hence $\bar{H}^{l-k-1}(U \backslash B_p; \mathbb{Z}) \cong \bar{H}^{l-k-1}(\mathscr{L}_p \backslash \mathscr{T}_p; \mathbb{Z})$. By [6], Theorem 18.3, the homology groups of $U \backslash B_p$ and $\mathscr{L}_p \backslash \mathscr{T}_p$ are isomorphic to those of spheres of dimensions $(k + l) - k - 1 = l - 1$ and $l - k - 1$, respectively. Thus, for $k < l - 1$ we have $H_{l-k-1}(U \backslash B_p; \mathbb{Z}) = 0$ (because of $0 < l - k - 1 < l - 1$) and $H_{l-k-1}(\mathscr{L}_p \backslash \mathscr{T}_p; \mathbb{Z}) = \mathbb{Z}$, a contradiction. For $k = l - 1$, we get a contradiction, too: we then have $H_0(U \backslash B_p; \mathbb{Z}) = \mathbb{Z}$ (because of $l > 1$) and $H_0(\mathscr{L}_p \backslash \mathscr{T}_p; \mathbb{Z}) = \mathbb{Z}^2$.

Using the fact that closed ovals do not exist in 8-dimensional compact, connected projective planes (hence $k > 0$, see [3], Theorem 3.5), we get the following

Corollary 3.2. *Let U be a spherical unital in an* 8*-dimensional compact, connected projective plane, such that condition* (R1) *is satisfied. Then U has dimension* 5, *i.e. it has the same dimension as the classical planar unital in* $\mathscr{P}_2\mathbb{H}$.

If we assume that in addition to (R1) the following regularity assumption (R2) is satisfied, we can improve our result obtained in Proposition 3.1.

(R2) $U \backslash B_p$ is a fibration over $\mathscr{L}_p \backslash \mathscr{T}_p$ with projection $\rho_p : U \backslash B_p \to \mathscr{L}_p \backslash \mathscr{T}_p : x \mapsto x \vee p$.

Theorem 3.3. *Let U be a spherical unital which satisfies conditions* (R1) *and* (R2). *Then we have* $l > 1$, *and U has dimension* $(3/2)l - 1$, *i.e. it has the same dimension as the classical planar unital in the corresponding classical plane.*

Proof. By Proposition 3.1 we have $k < l - 1$. In the sequel we assume that $k \neq (l/2) - 1$. Then we have $k > 0$ since closed ovals do not exist in 8- and 16-dimensional compact projective planes (see [3], Theorems 2.6 and 3.5). The essential tool in this proof is the Gysin exact sequence associated with the fibration $(U \backslash B_p, \mathscr{L}_p \backslash \mathscr{T}_p, \rho_p)$ (see, e.g., [11], p. 260, Theorem 11). We use homology with coefficients in $\mathbb{Z}_2$. Then the homology groups of the fibres $\rho_p^{-1}(L) \approx \mathbb{S}_k$ have trivial automorphism groups because of $k > 0$. Thus the fibration is orientable, and the Gysin sequence may be applied (cf. the remark below). By [6], Theorem 18.3, the homology groups of $U \backslash B_p$ and $\mathscr{L}_p \backslash \mathscr{T}_p$ are isomorphic to those of spheres of dimensions $l - 1$ and $l - k - 1$, respectively. Hence we have $H_{l-2k-2}(\mathscr{L}_p \backslash \mathscr{T}_p; \mathbb{Z}_2) = 0$ and $H_{l-k-1}(U \backslash B_p; \mathbb{Z}_2) = 0$ because of $0 < k < l - 1$. The Gysin sequence

$$\cdots \to H_{l-k-1}(U \backslash B_p; \mathbb{Z}_2) \to H_{l-k-1}(\mathscr{L}_p \backslash \mathscr{T}_p; \mathbb{Z}_2) \to H_{l-2k-2}(\mathscr{L}_p \backslash \mathscr{T}_p; \mathbb{Z}_2) \to \cdots$$

then yields $H_{l-k-1}(\mathscr{L}_p \backslash \mathscr{T}_p; \mathbb{Z}_2) = 0$, a contradiction.

Remark. We did not use homology with coefficients in $\mathbb{Z}$ since the Gysin exact sequence applies only to orientable fibrations. Here, a fibration is called orientable if the fundamental group of the base space acts trivially on the homology of the fibre (cf. [11], p. 476). For homology with coefficients in $\mathbb{Z}$, the non-vanishing homology groups of the fibres $\rho_p^{-1}(L) \approx \mathbb{S}_k$ are isomorphic to $\mathbb{Z}$ for $k > 0$. Since the automorphism group of $\mathbb{Z}$ is abelian, the action of the fundamental group of $\mathscr{L}_p \backslash \mathscr{T}_p$ on the homology of $\mathbb{S}_k$ reduces to an action of the abelianization of this group, i.e. to an action of $H_1(\mathscr{L}_p \backslash \mathscr{T}_p; \mathbb{Z})$. Hence, the fact that $\mathscr{T}_p$ might be wildly embedded in $\mathscr{L}_p$ causes no problem in our case, but if $\mathscr{T}_p$ has codimension 2 in $\mathscr{L}_p$, then it is not clear if the fibration is orientable because we then have $H_1(\mathscr{L}_p \backslash \mathscr{T}_p; \mathbb{Z}) \cong \mathbb{Z}$ by [6], Theorem 18.3.

4 Unitals in smooth projective planes

In this section we will show that certain regularity assumptions in the previous two sections (cf. Proposition 2.8 and Theorem 3.3) are satisfied for smooth unitals in smooth projective planes $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$.

Definition 4.1. Let $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ be a smooth projective plane. A *smooth unital U* is a spherical unital which is a smooth submanifold of the point space P such that every secant intersects U transversally.

Remark. Note that in virtue of 2.4 all classical unitals are smooth unitals.

In Propositions 4.3 and 4.4 we will establish additional geometric properties for smooth unitals. The next lemma contains two results on smooth projective planes which will be used in the proofs of these propositions.

Lemma 4.2. *Let $\mathscr{P} = (P, \mathscr{L}, \mathscr{F})$ be a smooth projective plane. Then the join map $\vee$ is a submersion onto $\mathscr{L}$ and the map $\vee_p$ is a submersion onto $\mathscr{L}_p$ for each $p \in P$.*

Proof. Let $p \in P$ and $x \in P \backslash \{p\}$. We want to show that the differential $(\mathrm{D}\vee_p)_x$ is surjective. For an arbitrary line $L \in \mathscr{L}_x \backslash \{x \vee p\}$ the perspectivity $P_L \to \mathscr{L}_p : z \mapsto z \vee p$ is a diffeomorphism with inverse $\mathscr{L}_p \to P_L : K \mapsto K \wedge L$. Thus the map $(\mathrm{D}\vee_p)_x|_{\mathrm{T}_x P_L} : \mathrm{T}_x P_L \to \mathrm{T}_{x \vee p} \mathscr{L}_p$ is an isomorphism and $(\mathrm{D}\vee_p)_x$ is surjective. Hence, the map $\vee_p$ is a submersion.

Now let $x, y \in P$, $x \neq y$ and put $L := x \vee y$. The differential of $\vee$ in (x, y) is the map $\mathrm{D}\vee_{(x,y)} : \mathrm{T}_x P \times \mathrm{T}_y P \to \mathrm{T}_L \mathscr{L} : (u, v) \mapsto (\mathrm{D}\vee_y)_x(u) + (\mathrm{D}\vee_x)_y(v)$. The images of $(\mathrm{D}\vee_y)_x$ and $(\mathrm{D}\vee_x)_y$ are $\mathrm{T}_L \mathscr{L}_y$ and $\mathrm{T}_L \mathscr{L}_x$, respectively. Since line pencils intersect transversally, the surjectivity of $\mathrm{D}\vee_{(x,y)}$ follows. Thus $\vee$ is a submersion.

Proposition 4.3. *The set of secants of a smooth unital U is open in $\mathscr{L}$ and the set U^* of tangents is compact.*

Proof. The set of secants of U is the image of the the join map restricted to the submanifold $(U \times U) \backslash \Delta(U)$ of $(P \times P) \backslash \Delta(P)$. We want to show that $\vee|_{(U \times U) \backslash \Delta(U)}$ is still a submersion. Choose $x, y \in U$, $x \neq y$ and put $L := x \vee y$. Since $\vee_x$ maps $P_L \backslash \{x\}$ onto $\{L\}$, the tangent space $\mathrm{T}_y P_L$ is contained in the kernel of $(\mathrm{D}\vee_x)_y$. Analogously, we get $(\mathrm{D}\vee_y)_x(\mathrm{T}_x P_L) = \{0\}$. Hence we have $\mathrm{D}\vee_{(x,y)}(\mathrm{T}_x P_L \times \mathrm{T}_y P_L) = (\mathrm{D}\vee_y)_x(\mathrm{T}_x P_L) \times$

$(\mathrm{D}\vee_x)_y(\mathrm{T}_y P_L) = \{0\}$. Because of $(\mathrm{T}_x U \times \mathrm{T}_y U) + (\mathrm{T}_x P_L \times \mathrm{T}_y P_L) = \mathrm{T}_x P \times \mathrm{T}_y P$ we conclude that $\mathrm{D}\vee_{(x,y)}(\mathrm{T}_x U \times \mathrm{T}_y U) = \mathrm{T}_L \mathscr{L}$, see Lemma 4.2. Thus the restriction of $\vee$ to $(U \times U)\backslash\Delta(U)$ is a submersion and hence an open map. In particular, the image of $(U \times U)\backslash\Delta(U)$ is open in $\mathscr{L}$. Finally, U^* is compact being the complement of the union of the open set of secants and the open set of exterior lines in the compact space $\mathscr{L}$.

Proposition 4.4. *Let U be a smooth unital of codimension bigger than* 1 *in P. Then there are no exterior lines. Furthermore, there are secants and tangents through every point of $P\backslash U$.*

Proof. For $k = 0$, the unital U is a closed oval and the above statements follow by [3], Theorems 3.5 and 3.7. So we may assume that $k > 0$, $l > 1$. Since the set of secants is open, the first statement and the fact that there are secants through every point of $P\backslash U$ are consequences of Proposition 2.8.

Assume that there is a point $p \in P\backslash U$ such that there are only secants through p. We first show that the restriction of $\vee_p$ to U is a submersion. So let $x \in U$ and let $L := x \vee p$. Because of $(\mathrm{D}\vee_p)_x(\mathrm{T}_x P_L) = \{0\}$, we have $(\mathrm{D}\vee_p)_x(\mathrm{T}_x U) = (\mathrm{D}\vee_p)_x \cdot (\mathrm{T}_x U + \mathrm{T}_x P_L) = (\mathrm{D}\vee_p)_x(\mathrm{T}_x P) = \mathrm{T}_L \mathscr{L}_p$. Here we have used Lemma 4.2 and the transversal intersection of U and P_L. Hence, $\vee_p|_U$ is a submersion. This map is also proper since U is compact. By the fibration theorem of Ehresmann ([2], Theorem 8.12), we conclude that U is a locally trivial fibration over $\mathscr{L}_p$ with projection $\vee_p|_U$ and fibres homeomorphic to $\mathbb{S}_k$. Because of $0 < k < l - 1$ we have $H_{k+1-l}(\mathbb{S}_k; \mathbb{Z}_2) = 0$ and $H_k(U) = 0$. Thus the Wang sequence

$$\cdots \to H_{k+1-l}(\mathbb{S}_k; \mathbb{Z}_2) \to H_k(\mathbb{S}_k; \mathbb{Z}_2) \to H_k(U) \to \cdots$$

associated with this fibration (see, e.g., [11], p. 456, Corollary 6) yields $H_k(\mathbb{S}_k; \mathbb{Z}_2) = 0$, a contradiction.

The following corollary is a consequence of Theorem 3.3 in the case of smooth unitals.

Corollary 4.5. *Let U be a smooth unital such that condition* (R1) *is satisfied. Then U has the same dimension as one of the classical unitals.*

Proof. It suffices to show that also condition (R2) is satisfied. Then we can apply Theorem 3.3. The map ρ_p (defined in (R2)) is proper, since it extends to $\vee_p|_U : U \to \mathscr{L}_p$. As in the proof of the preceding Proposition we see that ρ_p is a submersion. Hence, by the fibration theorem of Ehresmann (see [2], Theorem 8.12), $U\backslash B_p$ is a locally trivial fibre bundle over $\mathscr{L}_p\backslash\mathscr{T}_p$ with projection ρ_p, i.e. condition (R2) is satisfied.

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