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A rank 3 tangent complex of $\mathrm{PSp}_4(q)$, q odd

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Abstract. Let $G = \mathrm{PSp}_4(q)$, $q = p^k$ odd. We show that the geometry of root subgroups of G is the tangent envelope of a system of conics that comprise the (q,q)-generalized quadrangle associated with G. The flags of this geometry form a rank 3 chamber complex in the sense of Tits [9], as one would expect from the theory of symmetric spaces for Lie groups. By way of application, we give an intrinsic interpretation of symplectic 2-transvections. We then show that the subgroup generated by a pair of short-root subgroups not contained in a p-Sylow is determined by the geometry. In particular, we describe the incidence conditions under which such pairs are contained in the maximal subgroups of G corresponding to the plus-point and minus-point stabilizers in the orthogonal construction of G ([2], xii).

Key words. Root subgroup geometry, generalized quadrangle, chamber complex, maximal subgroup.

1 Introduction

The significance of root subgroups in the study of groups of Lie type derives from two complementary interpretations: as groups of transvections on some natural module (extrinsic), and as points of some incidence geometry related to the lattice of maximal parabolic subgroups (intrinsic). There are natural comparisons with the theory of symmetric spaces for Lie groups. Let $G = \mathrm{PSp}_4(q)$, q odd. We construct an incidence geometry $\mathscr{G} = (\mathscr{P}, \mathscr{L})$ whose points are the root subgroups of G: long, short and virtual (defined in Section 2). Unlike the constructions in [6] and [8], no member of $\mathscr{L}$ consists entirely of long-root subgroups. Rather, $\mathscr{G}$ is generated by a collection of lines that can be viewed as the "tangent bundle" of the (q,q)-generalized quadrangle for G when this quadrangle is represented as a system of conics whose points are the long-root subgroups. It will then be easy to show that the flag complex of $\mathscr{G}$ is a rank 3 chamber complex.

In Section 3 we use $\mathscr{G}$ to construct the subgroups of index 2 in the centralizers of involutions of class $2A$ and $2C$. In Atlas notation $2A$ is central in a 2-Sylow of G and $C_G(2A) \simeq 2.L_2(q) \times L_2(q) : 2$, whereas $2C$ is an outer involution and $C_G(2C) \simeq L_2(q^2) : 2_2$. In orthogonal terminology these centralizers are the plus-point and minus-point stabilizers, respectively, and both are maximal subgroups of G.

2 The tangent bundle and chamber complex

Let $Z = Z(O_p(P))$, where the maximal parabolic subgroup P is the stabilizer of a maximal isotropic subspace of the natural module. Since Z is elementary abelian of order q^3 we will identify it with the vector space V of 2×2 symmetric matrices over $K = \mathrm{GF}(q)$. The points of $\mathscr{G}$ that belong to Z are the 1-dimensional subspaces of V, classified as follows: Subspaces consisting of singular matrices are long-root subgroups, those which contain a matrix with determinant -1 are short-root subgroups, and the remainder we call *virtual* root subgroups. Thus, as projective matrices the $q+1$ long-root subgroups in Z are

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u^2 & u \\ u & 1 \end{bmatrix}, \quad u \in K,$$

the $\binom{q+1}{2}$ short-root subgroups are

$$\begin{bmatrix} u & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} u^2 - \lambda^2 & u \\ u & 1 \end{bmatrix}, \quad u \in K, \ \lambda \in K^{\#},$$

and the $\binom{q}{2}$ virtual root subgroups are

$$\begin{bmatrix} u^2 - \varepsilon & u \\ u & 1 \end{bmatrix}, \quad u \in K, \ \varepsilon \notin K^2.$$

The above description is consistent with the representation $Z = \langle X_\beta, X_{\alpha+\beta}, X_{2\alpha+\beta} \rangle$ in terms of Chevalley generators, where X_γ is a root subgroup relative to a chosen split torus such that α is the fundamental short-root and β is the fundamental long root. The homogeneous triple $[\delta, u, v]$, where $\delta = 0$ or 1, represents the subgroup

$$\{X_\beta(\delta t) X_{\alpha+\beta}(ut) X_{2\alpha+\beta}(vt) \mid t \in K\}.$$

When this triple is identified with the corresponding projective matrix M in the obvious way the subgroup may be represented on a symplectic 4-space by the matrices

$$\begin{bmatrix} I_2 & M \\ 0 & I_2 \end{bmatrix}$$

which are identified with their negatives. The above representation presumes the canonical basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ with symplectic structure given by

$$\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix},$$

whereby the torus element $h(\zeta, \eta)$ is represented by

$$\mathrm{diag}(\zeta, \eta, \zeta^{-1}, \eta^{-1}) \quad \text{and} \quad n_\alpha h(\zeta, \eta) n_\alpha = h(\eta, \zeta)$$

See [1], §11.3.

We define the point set $\mathscr{P}$ for $\mathscr{G}$ to be the conjugates in G of the root subgroups described above. It is readily shown that G is transitive on the virtual root subgroups, as well. Thus $\mathscr{P}$ is partitioned into three orbits $\mathscr{P}_l, \mathscr{P}_s$ and $\mathscr{P}_v$. Let Π_Z denote the collection of root subgroups of Z. It follows that $\mathscr{P}_l \cap \Pi_Z$ is a conic Q in $\mathrm{PG}(2, q)$. The geometry whose point set is $\mathscr{P}_l$ and whose line set consists of all Q afforded by the conjugates of Z is $\mathscr{Q}$, the (q, q)-generalized quadrangle for G. Let $Q(x, y)$ be the conic determined by adjacent points x, y in $\mathscr{Q}$. Below we write Π_Q instead of Π_Z, and refer to Π_Q as the *focal plane* on Q.

To construct the line set $\mathscr{L}$ for $\mathscr{G}$ note that $\mathscr{P}_s \cap \Pi_Q$ consists of the non-absolute points (relative to the polarity induced by Q) in the envelope of tangents to Q. Let $\mathscr{L}_0$ be the orbit of the tangent lines under conjugation by G. For $x \in \mathscr{P}_l$, let $\mathscr{L}_0(x)$ be the collection of tangent lines on x and let Π_x be the set of points in $\mathscr{L}_0(x)$. We give Π_x the structure of $\mathrm{PG}(2, q)$ as follows. Since the conjugates of Z partition the short-root subgroups of G there are exactly two members of $\mathscr{L}_0$ on each $y \in \mathscr{P}_s$. Let $\mathscr{P}_l(y)$ denote the two points in $\mathscr{P}_l$ on the tangents through y. Let $y_1, y_2 \in \Pi_x \cap \mathscr{P}_s$ and suppose no member of $\mathscr{L}_0$ contains both y_1 and y_2. Then the group H generated by $\mathscr{P}_l(y_1) \cup \mathscr{P}_l(y_2) \backslash \{x\}$ contains $q + 1$ long-root subgroups z, each of which is adjacent to x in $\mathscr{Q}$. Let $L_x(z)$ be the member of $\mathscr{L}_0(z)$ that is tangent to $Q(x, z)$ and define the line on y_1 and y_2 by $y_1 y_2 = \Pi_x \cap (\bigcup_{z \in H} L_x(z))$. Denote by $\mathscr{L}_1$ the collection of all lines so constructed. Thus $|\mathscr{L}_1| = q^2(q + 1)(q^2 + 1)$. If $L \in \mathscr{L}_1$ then the stabilizer of L in G has order $\frac{1}{2} q^2(q - 1)(q^2 - 1)$ and so G is transitive on $\mathscr{L}_1$. We call the incidence system $(\mathscr{P}_l \cup \mathscr{P}_s, \mathscr{L}_0 \cup \mathscr{L}_1)$ the *tangent bundle* of $\mathscr{Q}$ and refer to Π_x as the *tangent plane* to $\mathscr{Q}$ at x.

The maximal flags of the tangent bundle are of type $(\mathscr{P}_l, \mathscr{L}_0, \Pi_x), (\mathscr{P}_s, \mathscr{L}_0, \Pi_x)$ or $(\mathscr{P}_s, \mathscr{L}_1, \Pi_x)$ and so the corresponding flag complex is not connected when adjacency of two maximal flags is defined by sharing a flag of rank 2. To remedy this situation we extend the line set to include the flags of the focal planes. The polar of $x \in \mathscr{P}_s \cap \Pi_Q$ is the secant line determined by $\mathscr{P}_l(x)$. The polar of $x \in \mathscr{P}_v \cap \Pi_Q$ is a line exterior to Q. It is readily shown that G is transitive on both sets of lines. Let $\mathscr{L}_2$ and $\mathscr{L}_3$ be the orbits of all secant and exterior lines, respectively, and let $\mathscr{L} = \bigcup_{i=0}^{3} \mathscr{L}_i$. If $x, y \in \mathscr{P}$ are adjacent in $\mathscr{G}$ we denote the member of $\mathscr{L}$ that they determine by xy. The flag complex of $\mathscr{G} = (\mathscr{P}, \mathscr{L})$ is residually connected and hence is a rank 3 chamber complex. In particular, the geometry whose points are the tangent planes and whose lines are the focal planes with two planes incident if they intersect in a line (necessarily a tangent) is isomorphic to $\mathscr{Q}$ via $\Pi_Q \mapsto Q$, $\Pi_x \mapsto x$.

We conclude this section with a theorem that describes the correspondence between $\mathscr{G}$ and the action of root elements in G on a symplectic module. Recall that the long-root subgroups and short-root subgroups of G are 1-parameter groups of transvections and 2-transvections, respectively. Specifically, the above matrices for the long-root subgroups in Z show that the transvections $\{\boldsymbol{v} \mapsto \boldsymbol{v} + t(\boldsymbol{e}, \boldsymbol{v})\boldsymbol{e} \mid t \in K\}$ correspond to the subgroup $[0,0,1]$ or $[1,u,u^2]$ where $\boldsymbol{e} = \boldsymbol{e}_1$ or $u\boldsymbol{e}_1 + \boldsymbol{e}_2$, respectively, and $(\boldsymbol{e},\boldsymbol{v})$ is the symplectic inner product. If $z \in \mathscr{P}_s \cap \Pi_Q$ is represented by $[0,1,u]$ then its polar is the secant line with homogeneous coordinates $[u,-2,0]$ and so $\mathscr{P}_l(z) = \left\{[0,0,1], \left[1,\frac{u}{2},\frac{u}{4}\right]\right\}$. It is readily verified that the transformations in the root subgroup z are the 2-transvections determined by $\boldsymbol{e} = \boldsymbol{e}_1$ and $\boldsymbol{f} = \frac{u}{2}\boldsymbol{e}_1 + \boldsymbol{e}_2$. Similarly, if $z = [1,u,u^2-\lambda^2]$ then its polar is $[u^2-\lambda^2,-2u,1]$, whereby $\mathscr{P}_l(z) = \{[1,u\pm\lambda,(u\pm\lambda)^2]\}$ and the 2-transvections in z are determined by $\boldsymbol{e} = (u+\lambda)\boldsymbol{e}_1 + \boldsymbol{e}_2$, $\boldsymbol{f} = (u-\lambda)\boldsymbol{e}_1 + \boldsymbol{e}_2$. Finally, let $\hat{Q}$ be the extension of Q over $\mathrm{GF}(q^2)$. If $z \in \mathscr{P}_v \cap \Pi_Q$ then the extension of the polar of z over $\mathrm{GF}(q^2)$ intersects $\hat{Q}$ in the two points $[1,u\pm\sqrt{\varepsilon},(u\pm\sqrt{\varepsilon})^2]$ of $\mathrm{PG}(2,q^2)$, whereas the transformations in z are of the form $\boldsymbol{v} \mapsto \boldsymbol{v} + t[(\boldsymbol{e},\boldsymbol{v})\boldsymbol{f} + (\boldsymbol{f},\boldsymbol{v})\boldsymbol{e}]$ with $\boldsymbol{e} = (u+\sqrt{\varepsilon})\boldsymbol{e}_1 + \boldsymbol{e}_2$, $\boldsymbol{f} = (u-\sqrt{\varepsilon})\boldsymbol{e}_1 + \boldsymbol{e}_2$. This proves the following theorem.

Theorem 2.1. *Let $x, y \in \mathscr{P}_l$ be adjacent as points of $\mathscr{Q}$ and let $Q = Q(x,y)$. If x and y are afforded respectively as groups of transvections by the orthogonal 1-spaces $\langle e\rangle$ and $\langle f\rangle$ then the short-root subgroup $L_x(y) \cap L_y(x)$ is the group of 2-transvections $\{v \mapsto v + t[(e,v)f + (f,v)e] \mid t \in K\}$. Further, if $z \in \mathscr{P}_v \cap \Pi_Q$, $\hat{L}$ is the extension of the polar of z over $\mathrm{GF}(q^2)$ and $\hat{Q}$ is the extension of Q, then the virtual root subgroup z is the group of 2-transvections determined by the orthogonal 1-spaces corresponding to $\hat{L} \cap \hat{Q}$ as t varies over K.*

3 Plus-point and minus-point stabilizers

In this section we determine the subgroup generated by a pair of short-root groups not contained in a common parabolic. Aspects of this determination have been addressed in [5] and [7] for $\mathrm{PSp}_{2n}(q)$, $n \geqslant 2$, but here we show directly that such a pair generates the subgroup of index 2 in either the centralizer of involution $2A$ or $2C$, using incidence relations in $\mathscr{G}$ to distinguish the cases. These centralizers are the plus-point and minus-point stabilizers, respectively, in the orthogonal construction of G. We represent the symmetric form that affords the quadratic form of index 2 by the matrix

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We will not require an explicit representation for the form of index 1.

With P and Q as in Section 2, let P^- be a parabolic opposite P and Q^- the corresponding opposite conic in $\mathscr{Q}$. Given $x \in \mathscr{P}_s \cap \Pi_Q$, each point in $\mathscr{P}_l(x)$ is adjacent in $\mathscr{Q}$ to a unique point of Q^-. These two points of Q^- comprise $\mathscr{P}_l(x^-)$ for some $x^- \in \mathscr{P}_s \cap \Pi_{Q^-}$, which we call the *opposite* of x in Π_{Q^-}.

Theorem 3.1. *Let Π_Q and Π_{Q^-} be opposite focal planes of $\mathscr{G}$ with $x \in \mathscr{P}_s \cap \Pi_Q$ and*

x^- its opposite in Π_{Q^-}. Let $y \in \mathscr{P}_s \cap \Pi_Q$ be distinct from x. Then, as a subgroup of G, $\langle x^-, y\rangle$ is contained in a conjugate of P if $xy \in \mathscr{L}_0$, whereas $\langle x^-, y\rangle < C_G(2A)$ if $xy \in \mathscr{L}_2$ and $\langle x^-, y\rangle < C_G(2C)$ if $xy \in \mathscr{L}_3$.

Proof. It is straightforward to classify the pairs $\{y, z\}$ with $y \in \mathscr{P}_s \cap \Pi_Q$ and $z \in \mathscr{P}_s \cap \Pi_{Q^-}$. In fact, since $P \cap P^-$ is transitive on $\mathscr{P}_s \cap \Pi_Q$ it suffices to fix $z = X_{-\alpha-\beta}$ and to describe the orbits of pairs as subsets of $\mathscr{P}_s \cap \Pi_Q$. We thus obtain the following $\frac{1}{2}(q+3)$ orbits where $\lambda, \mu \in K^\#$, $\mu^2 \neq 1$ and the number following the semicolon is the length of the orbit.

$$\begin{aligned}
O_A &= \{[0,1,0]\}; 1\\
O_B &= \{[0,1,\lambda],[1,\lambda,0]\}; 2(q-1)\\
O_C &= \{[1,0,-\lambda^2]\}; \tfrac{1}{2}(q-1)\\
O_\mu &= \{[1,\lambda,\lambda^2(1-\mu^2)]\}; q-1.
\end{aligned}$$

Now assume $z = x^-$ so that its opposite in Π_Q is $x = X_{\alpha+\beta}$. It follows that $\langle x^-, y\rangle$ is contained in the stabilizer of $Q(X_{2\alpha+\beta}, X_{-\beta})$ if $y \in O_B$, in which case xy is tangent to Q. If $y \in O_C$ then xy is a secant line provided $-1 \in K^2$, whereas xy is exterior to Q otherwise. Take $\lambda = 1$ and let $i = \sqrt{-1}$. Then the involution $\tau = h(i, -i)n_\alpha$ centralizes $\langle x^-, y\rangle$. If $y \in O_\mu$ for some μ then xy is a secant provided $1-\mu^2 \in K^2$ and an exterior line otherwise. Take $\lambda = 1$ and let $j_\mu = \sqrt{1-\mu^2}$. Then the involution $\tau_\mu = h(j_\mu^{-1}, j_\mu)n_\alpha$ centralizes $\langle x^-, y\rangle$. Thus τ and τ_μ are of class $2A$ when xy is a secant, and of class $2C$ when xy is an exterior line.

Corollary 3.2. *If $q \neq 3$, the subgroup of G generated by a pair of short-root subgroups not contained in a common parabolic is isomorphic to either $L_2(q^2)$ or $2.L_2(q) \times L_2(q)$.*

Proof. If $q = 3$ there is no orbit of type O_μ, so assume $y \in O_C$. Then there is the possibility by Dickson's theorem ([3], page 44) that $\langle x^-, y\rangle \simeq L_2(5)$. That this is the case follows by setting $a = X_\beta(1)X_{2\alpha+\beta}(-1)$, $b = X_{-\alpha-\beta}(1)$ and $c = babab$. Then $c^2 = b^3 = (cb)^5 = 1$, whereas $a = [b, c]b$. Now suppose $q \neq 3$ and consider the products $X_{-\alpha-\beta}(t)X_\beta(u)X_{\alpha+\beta}(u)X_{2\alpha+\beta}(j_\mu^2 u)$ and $X_{-\alpha-\beta}(t)X_\beta(u)X_{2\alpha+\beta}(-u)$. Direct computation using the representation in Section 2 easily demonstrates that the square of such products is never diagonal for non-zero values of t and u. Thus when xy is an exterior line it follows that $\langle x^-, y\rangle \simeq L_2(q^2)$ since the conditions of Dickson's theorem are subsumed. When xy is a secant line $\langle x^-, y\rangle$ is seen to be all of $2.L_2(q) \times L_2(q)$ as follows. Let

$$J_-(i,t) = \begin{pmatrix} 1 & 0 & 0 & 0\\ -it & 1 & 0 & 0\\ -it & 0 & 1 & 0\\ t^2 & it & it & 1 \end{pmatrix}, \quad J_-(j_\mu, t) = \begin{pmatrix} 1 & 0 & 0 & 0\\ -j_\mu t & 1 & 0 & 0\\ j_\mu^{-1}t & 0 & 1 & 0\\ t^2 & -j_\mu^{-1}t & j_\mu t & 1 \end{pmatrix}$$

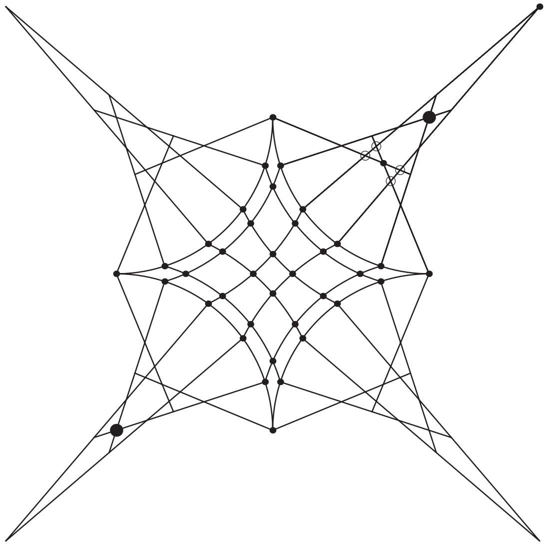

Figure 1: The tangent bundle of the symplectic quadrangle over GF(5). For opposite short-root subgroups (large dots) the group $\langle x^-, y\rangle$ is determined by whether xy is a secant (large circle), exterior line (small circle) or tangent of Q.

and

$$J_+(t) = \begin{pmatrix} 1 & -t & -t & -t^2 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then as t varies over K either the $J_-(i,t)$ or the $J_-(j_\mu,t)$ together with the $J_+(t)$ generate the group of matrices J such that $J^T BJ = B$ where J^T is the transpose of J. Let $y \in O_C$ be represented by $[1,0,1]$. Then the map $J_-(i,t) \mapsto X_{-\alpha-\beta}(t)$, $J_+(t) \mapsto X_\beta(t)X_{2\alpha+\beta}(t)$ induces a homomorphism onto $\langle x^-, y\rangle$ with kernel $\pm I$. In case $y \in O_\mu$ we represent y by $[1, j_\mu^{-1}, 1]$. Then the corresponding homomorphism onto $\langle x^-, y\rangle$ is $J_-(j_\mu,t) \mapsto X_{-\alpha-\beta}(t), (J_-(j_\mu,t))^T J_+(t) \mapsto X_\beta(t)X_{\alpha+\beta}(j_\mu^{-1}t)X_{2\alpha+\beta}(t)$.

Remark 3.3. Even though the number of orbits of pairs of short-root subgroups not contained in a common parabolic is a function of the field, the corollary shows that the subgroup generated by such a pair $\{x, y\}$ is determined by the relation between x and y as points of $\mathcal{G}$. See Figure 1.

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