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Extending extremal contractions from an ample section

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Abstract. Let $\mathscr{E}$ be an ample vector bundle of rank r on a complex projective manifold X such that there exists a section $s \in \Gamma(\mathscr{E})$ whose zero locus, $Z = (s = 0)$, is a smooth submanifold of the expected dimension $\dim X - r$. We study the problem of extending birational contractions of Z to the ambient variety proving an extension property for blow-ups and we apply our results to classify X as above when Z is a $\mathbb{P}$-bundle on a surface with nonnegative Kodaira dimension.

Key words. Vector bundle, extremal ray, Fano–Mori contraction.

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1 Introduction

All through the paper we will work in the following

Setup 1.1. Let X be a smooth complex projective variety of dimension n and $\mathscr{E}$ an ample vector bundle of rank r on X such that there exists a section $s \in \Gamma(\mathscr{E})$ whose zero locus, $Z = (s = 0)$, is a smooth submanifold of the expected dimension $\dim Z = \dim X - r = n - r$.

A classical and natural problem is to exploit the geometric properties of Z to get information on the geometry of X; for an account of the results in case $r = 1$, i.e. when Z is an ample divisor, see [6, Chapter 5]. In [3] we considered the problem from the point of view of Mori theory, posing the following question: assume that Z is not minimal, i.e. Z has at least one extremal ray in the negative part of the Mori cone; does this ray (or the associated extremal contraction) determine a ray (or a contraction) in X, and if so, does this new ray determine the structure of X?

Through the paper we will assume that Z is not minimal, we assume also that $\dim Z \geqslant 2$; if $\dim Z = 1$ then $Z \simeq \mathbb{P}^1$ and this case is treated in [13], where the problem of special sections of an ample vector bundle was studied first.

We showed in [3] that there is always an extremal face F_Z of $NE(Z)$ that determines an extremal face F_X of $NE(X)$; now we slightly improve our results, proving that, if $N_1(Z) \simeq N_1(X)$ (which is always true if $\dim Z \geqslant 3$, see 2.8) then there is always an extremal face F_Z of $NE(Z)$ which coincides with an extremal face of $NE(X)$ and in this case we say that the face is *liftable* to X; this is the context of Theorem 3.2 and Corollary 3.4.

Note that, a priori, the liftability of a face does not imply the extendability of the associated contraction; namely the contraction associated to F_X on X restricted to Z is not necessarily the contraction associated to F_Z (see Remark 3.5); if this is the case we say that the face is *extendable*. If F_Z corresponds to a fiber type contraction on Z, then F_Z is extendable and the contraction of the face F_X in X is again a fiber type contraction [3, 3.12 and 3.13].

In this paper we study the extendability problem for birational contractions: a general result is given in 3.8. Then we prove an extension property for blow-ups, namely:

Theorem 1.2. *Let R_Z be an extremal ray on Z, whose associated contraction $\varphi : Z \to Z'$ is the blow-up of a smooth subvariety C of codimension $m \geqslant 3$.*

If R_Z is liftable to R_X then it is extendable; moreover, if $\phi : X \to X'$ is the contraction associated to R_X, then X' is smooth, Z' is isomorphic to a subvariety of X', and ϕ is the blow-up of C.

A straightforward corollary of this theorem is the following

Corollary 1.3. *Assume that $\varphi : Z \to Z'$ is the blow-up of a smooth minimal variety ($K_{Z'}$ is nef) along a smooth subvariety of codimension $m \geqslant 3$. Then X is the blow-up of a smooth variety X' along a smooth subvariety of codimension $m + r$.*

Another application of the above theorem concerns the problem to determine whether a blow-up of a projective space $\mathbb{P}^{n-r}$ along a linear subspace can be the zero locus of a section of an ample vector bundle in a projective manifold. We discuss this problem in the second part of Section 4.

We finally apply our results to prove the following

Theorem 1.4. *Assume that Z is a $\mathbb{P}^d$-bundle on a surface S of nonnegative Kodaira dimension. Then X is a $\mathbb{P}^{r+d}$-bundle on S.*

In the case $r = 1$ this theorem, together with previous existing results, completes the proof of a long lasting general conjecture by A. Sommese for $n = 4$, see [6, Conjecture 5.5.1]; the same theorem was proved in a different way (only for the case $r = 1$ and $n = 4$) recently in [16].

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2 Notations and preliminaries

We use the standard notation from algebraic geometry, in particular it is compatible with that of [11] and of [12]. This paper is a sequel of [3] to which we refer constantly.

In the paper X will always stand for a smooth complex projective variety of dimension n and K_X will be its canonical divisor; the famous *Cone Theorem* of Mori says that the closure of the cone of effective 1-cycles into the real vector space of 1-cyles modulo numerical equivalence, $\overline{NE(X)} \subset N_1(X)$, is polyhedral in the part contained in the set $\{Z \in N_1(X) : K_X.Z < 0\}$. An extremal face F (or F_X) of X is a face of this polyhedral part; an extremal ray is a face of dimension 1. To every extremal face is associated a morphism to a normal variety; namely we have the following *Base point free theorem* of Kawamata and Shokurov.

Theorem 2.1. *Let X and F be as above. Then there exists a projective morphism $\varphi : X \to W$ from X onto a normal variety W which is characterized by the following properties*:

i) *For an irreducible curve C in X, $\varphi(C)$ is a point if and only if the class of C is in F.*

ii) *φ has only connected fibers.*

Definition 2.2. The map φ of the above theorem is usually called the *Fano–Mori contraction* (or the *extremal contraction*) associated to the face F. A Cartier divisor H such that $H = \varphi^*(A)$ for an ample divisor on W is called a *good supporting divisor* of the map φ (or of the face F).

The contraction is of fiber type if $\dim W < \dim X$, otherwise it is birational. We usually denote with $E = E(\varphi) := \{x \in X : \dim(\varphi^{-1}\varphi(x)) > 0\}$ the exceptional locus of φ; if φ is of fiber type then of course $E := X$.

Remark 2.3. Note that a good supporting divisor for a Fano–Mori contraction is of the form $H = K_X + rL$, where r is a positive integer and L is an ample line bundle. In fact if H is a good supporting divisor then $H - K_X$ is an ample line bundle by Kleiman's criterion.

On the other hand note also that any nef but not ample line bundle H of the form $H = K_X + rL$, with r a positive integer and L an ample line bundle, defines (or is associated to) an extremal face $F := \{Z \in \overline{NE(X)} : H.Z = 0\}$.

Example 2.4. Let $\varphi : X := \mathrm{Bl}_Y(X') \to X'$ be the blow-up of a projective manifold X' along a submanifold $Y \subset X'$ of codimension m; let also $E \subset X$ be the exceptional divisor. This is a Fano–Mori contraction and a good supporting divisor for this contraction is $H = K_X + (m-1)L$, where $L = -E + \varphi^*(A)$ for an ample divisor A on X'.

Let W be a projective manifold and let $\mathcal{G}$ be a rank m vector bundle on W; then $\varphi : X := \mathbb{P}(\mathcal{G}) \to W$ is a Fano–Mori contraction, called a $\mathbb{P}$-bundle contraction or a (classical) scroll over W; to avoid possible confusion, we will use always the first denomination. If $\xi_{\mathcal{G}}$ denotes the tautological bundle on X then a good supporting divisor for the contraction φ is $H = K_X + mL$, where $L = \xi_{\mathcal{G}} + \varphi^*(A)$ for an ample divisor A on W.

More generally a fiber type contraction $\varphi : X \to W$ of a projective manifold X onto a normal projective variety W supported by a divisor of the type $H = K_X + (\dim X - \dim W + 1)L$, with L an ample line bundle, is a Fano–Mori contraction and it is called an adjunction theoretic scroll. In a neighborhood of a generic fiber an adjunction theoretic scroll is a $\mathbb{P}$-bundle (this is a theorem of Fujita); however there can be special fibers of dimension greater than $(\dim X - \dim W)$.

Another important result of Mori, see [15], is the existence of rational curves in the extremal rays. Namely if X has an extremal ray R then there exists a rational curve C on X such that $R = \mathbb{R}[C]$ and $0 < -K_X \cdot C \leqslant n+1$. Such a curve C is called an *extremal curve*.

Definition 2.5. Let R be an extremal ray on X. We define the positive integer l as

$$l = l(R) := \min\{-K_X.C : C \text{ is a rational curve in } R\}.$$

l is called the *length of the ray* while a rational curve C in the ray R such that $l = -K_X.C$ is called a *minimal extremal curve*.

The importance of this integer comes from the following proposition, proved by Ionescu and Wisniewski.

Proposition 2.6 ([19]). *Let φ be an extremal contraction associated to the extremal ray R; let S be an irreducible component of a non-trivial fiber of φ. The following formula holds*

$$\dim S + \dim E(\varphi) \geqslant \dim X + l(R) - 1.$$

The zero locus of a section of an ample vector bundle has a lot of good properties, we will frequently use the following two:

Proposition 2.7 ([3, 2.18]). *Let X and Z be as in* 1.1 *and let Y be a subvariety of X of dimension $\geqslant r$. Then* $\dim Z \cap Y \geqslant \dim Y - r$.

The other is a very strong result, a Weak Lefschetz type theorem for ample vector bundles, proved by Sommese in [17] and subsequently with slightly weaker assumptions in [13].

Theorem 2.8. *Let X, $\mathscr{E}$ and Z be as in* 1.1 *and let $i : Z \hookrightarrow X$ be the embedding. Then*

(2.8.1) $\mathrm{H}^i(i) : \mathrm{H}^i(X, \mathbb{Z}) \to \mathrm{H}^i(Z, \mathbb{Z})$ *is an isomorphism for* $i \leqslant \dim Z - 1$.

(2.8.2) $\mathrm{H}^i(i)$ *is injective and its cokernel is torsion free for* $i = \dim Z$.

(2.8.3) $\mathrm{Pic}(i) : \mathrm{Pic}(X) \to \mathrm{Pic}(Z)$ *is an isomorphism for* $\dim Z \geqslant 3$.

(2.8.4) $\mathrm{Pic}(i)$ *is injective and its cokernel is torsion free for* $\dim Z = 2$.

The following lemma is probably well known but we will provide anyway a proof for the interested reader, since we did not find a good reference for it.

Lemma 2.9. *Let $\varphi : X \to X'$ be the blow-up of a smooth variety along a smooth subvariety $Y \subset X'$ with exceptional divisor $E(\varphi)$ and $\mathscr{E}$ a* rank r *vector bundle on X. Assume that $\mathscr{E}_F \simeq \oplus^r \mathscr{O}_{\mathbb{P}}(1)$ for every fiber of φ. Then there exists a* rank r *vector bundle $\mathscr{E}'$ on X' such that*

$$\mathscr{E} \otimes E(\varphi) = \varphi^* \mathscr{E}'.$$

Proof. The vector bundle $\tilde{\mathscr{E}} := \mathscr{E} \otimes E(\varphi)$ is trivial along any fiber of φ. We have to prove that $f_*(\tilde{\mathscr{E}}) =: \mathscr{E}'$ is a locally free sheaf of rank r. This is a local question at any point $y \in Y$, and we can apply the Theorem on Formal Functions, see [10, Theorem III.11.1], which says that

$$f_*(\tilde{\mathscr{E}})_y = \varprojlim \mathrm{H}^0(F_n, \tilde{\mathscr{E}}_n)$$

(with $F_n = X \times_Y \mathrm{Spec}(\mathscr{O}_y/m_y^n)$). Since $\tilde{\mathscr{E}}$ is trivial on $F = \varphi^{-1}(z)$ there are r linearly independent non-zero sections of $\tilde{\mathscr{E}}_{|F} = F \times \mathbb{C}^r$; the same proof of the Castelnuovo criterion for blow-up, as for instance in [5, Proposition 2.4] or [10, Theorem V.5.7], gives that the sections actually extend to r non-zero and linearly independent sections of the $\varprojlim \mathrm{H}^0(F_n, \tilde{\mathscr{E}}_n)$ (i.e. they extend in a formal neighbourhood of F). This gives that $f_*(\tilde{\mathscr{E}})_y$ is locally free of rank r. □

In the setup of the previous lemma, it is useful to find conditions which ensure the ampleness of $\mathscr{E}'$:

Lemma 2.10. *In the situation of the previous lemma if $\mathscr{E}$ is ample and Y is a point then also $\mathscr{E}'$ is ample; the same is true if $\mathscr{E}$ is ample and there exists a surjection $N_Y^* \to -L \to 0$ with L a nef line bundle on Y.*

Proof. The first part has been proved in [14, Lemma 5.1]. To prove the second part we will apply [7, Lemma 5.7]; let $E = \mathbb{P}(N_Y^*)$ be the exceptional divisor of φ and consider the following diagram:

$$\begin{array}{ccc} \mathbb{P}(\mathscr{E}') & \xleftarrow{\tilde{\varphi}} & \mathbb{P}(\varphi^*\mathscr{E}') \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p} \\ X' & \xleftarrow{\varphi} & X \end{array}$$

where $\tilde{\varphi}$ is the blow-up of $\mathbb{P}(\mathscr{E}')$ along $p_1^{-1}(Y)$. Let $\xi_{\mathscr{E}'}$ be the tautological line bundle of $\mathbb{P}(\mathscr{E}')$; we have

$$\tilde{\varphi}^* \xi_{\mathscr{E}'} = \xi_{\varphi^*\mathscr{E}'} = \xi_{\mathscr{E} \otimes \mathscr{O}(E)} = \xi_{\mathscr{E}} + p^* E$$

so that $\tilde{\varphi}^* \xi_{\mathscr{E}'} - p^* E = \xi_{\mathscr{E}}$ is ample on $\mathbb{P}(\varphi^*\mathscr{E}') = \mathbb{P}(\mathscr{E})$.

Moreover $\xi_{\mathscr{E}'}$ is ample on $p_1^{-1}(Y)$. In fact, let $Y' \subset E$ be the section corresponding to the surjection $N_Y^* \to -L \to 0$; Y' is mapped isomorphically onto Y by φ and $E_{Y'} = L_{Y'}$, hence we have

$$(\mathscr{E} \otimes L)_{|Y'} = (\mathscr{E} \otimes E)_{|Y'} = \varphi^* \mathscr{E}'_{|Y'} = \mathscr{E}'_{|Y}$$

and $\mathscr{E}'$ is ample on Y. We can thus apply the quoted lemma to get the ampleness of $\xi_{\mathscr{E}'}$. □

3 Lifting of birational contractions

Let $X, \mathscr{E}$ and Z be as in 1.1; in this section we will improve some general results in [3]; we start with a definition which was not stated there.

Definition 3.1. Assume that $N_1(Z) \simeq N_1(X)$, which is always the case if $\dim Z \geqslant 3$ by Theorem 2.8, and let F_Z be an extremal face in $NE(Z)$. If under the above identification $N_1(Z) \simeq N_1(X)$ the face F_Z is an extremal face F_X in $NE(X)$, then we will say that the face F_Z is *liftable* to F_X.

The following is a refinement of [3, Theorem 3.4]:

Theorem 3.2 (Lifting of extremal faces). *Assume that Z is not minimal in the sense of Mori theory, i.e. K_Z is not nef. Let F_Z be an extremal face of Z, $D_Z = (K_Z + \tau H_Z)$ a good supporting $\mathbb{Q}$-Cartier divisor of F_Z and H the line bundle on X which restricts to H_Z. If H is ample on X, the $\mathbb{Q}$-divisor $D = K_X + \det \mathscr{E} + \tau H$ is nef, not ample and defines an extremal face F_X of X.*

Moreover, if $N_1(Z) \simeq N_1(X)$, F_Z is liftable to F_X.

Proof. The first part of the theorem has been proved in [3, Theorem 3.4], so we have to prove only the last assertion. Since under the identification $N_1(Z) \simeq N_1(X)$ we have $NE(Z) \subset NE(X)$ and $NE(Z)_{K<0} \subset NE(X)_{K<0}$ it is enough to show that, for every extremal ray R_X in the face F_X, there is a curve in R_X lying on Z.

Let R_X be an extremal ray of F_X and $\varphi_R : X \to T$ the associated extremal contraction; since the contraction of F_X is supported by $K_X + \det \mathscr{E} + \tau H$, this divisor is zero on the curves in R_X, yielding $l(R_X) \geqslant r + \tau$; if the contraction is birational, then, using 2.6, for a non-trivial fiber F of φ_R, $\dim F \geqslant r + \tau$, hence $\dim F \geqslant r + 1$ and we are done by 2.7.

In the same way we get our result if the contraction is of fiber type and has a fiber of dimension $r + 1$; so we are left with the case of an equidimensional fiber type contraction with r-dimensional fibers; note that in this case, the φ_R-ample line bundle H has intersection number one with the extremal rational curve generating the ray by [3, Proposition 2.7], so that, letting $H' = H + \varphi_R^* A$, with A ample on T, the divisor $K_X + (r + 1)H'$ is a good supporting divisor for φ_R, which, by [8, 2.12] is thus a $\mathbb{P}$-bundle contraction.

As in the proof of the first part of [3, Theorem 3.4], we get that Z is a regular section of this bundle, a contradiction with Theorem 2.8. □

Proposition 3.3. *Assume that $N_1(Z) \simeq N_1(X)$ and that the extremal face F_Z is liftable to F_X. Denote by φ and ϕ the contractions associated to F_Z and F_X; let $K_Z + \tau H_Z$ be a good supporting divisor for F_Z and let H be the divisor on X which restricts to H_Z.*

Then, up to replacing H with $H' = H + \phi^*A$, with A a sufficiently ample line bundle, we can assume that that H' is ample on X and that φ is supported by $K_Z + \tau H'_Z$.

Proof. The line bundle H is ϕ-ample and thus H' is ample. Moreover $K_Z + \tau H'_Z = K_Z + \tau H_Z + \tau(\phi^*A)_Z$ is a good supporting divisor of F_Z since $\tau(\phi^*A)_Z$ is nef and it is zero on the curves of F_Z.

Proposition 3.4. *If Z is not minimal there exists at least one extremal face F_Z which is liftable to X.*

Proof. Let L be an ample line bundle on X; the restriction of this line bundle to Z, L_Z, is ample on Z, so, if K_Z is not nef there exist a rational number $\sigma > 0$ such that the divisor $K_Z + \sigma L_Z$ is nef but not ample and it defines an extremal face G_Z. This face satisfies the assumptions of Theorem 3.2 and so it is liftable to an extremal face G_X. □

Remark 3.5. Let us note that, a priori, the fact that an extremal face of $\overline{NE(Z)}$ is liftable to an extremal face of $\overline{NE(X)}$ does not imply that the restriction ϕ_Z of the extremal contraction ϕ associated to F_X coincides with the extremal contraction φ associated to F_Z; as explained in [3], we have a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\ i\ } & Z \\ {\scriptstyle\phi}\downarrow & \overset{\phi_Z}{\swarrow} & \downarrow{\scriptstyle\varphi} \\ Y & \xleftarrow{\ \pi\ } & W \end{array} \tag{3.6}$$

where $\pi : W \to \phi_Z(Z)$ is a finite morphism.

To complete the lifting process, we introduce the following definition:

Definition 3.7. In the above notation, if π is a isomorphism onto its image, that is if the restriction ϕ_Z coincides with the extremal contraction φ of F_Z, then we will say that the face F_Z, or the associated contraction φ, is *extendable*.

In [3] we proved that if F_Z is a liftable face associated to a fiber type contraction then it is extendable and moreover π is the identity. Now we will deal with birational contractions.

Proposition 3.8. *Assume that there exists an extremal ray R_Z, whose associated contraction, $\varphi : Z \to W$, is birational, which is liftable to an extremal ray R_X. We can assume that φ is supported by $K_Z + \tau H_Z$ with $\tau \geqslant 1$ (Remark 2.3) and that H_Z is the restriction of an ample line bundle H on X (Proposition 3.3).*

If $\tau > 1$ then ϕ is birational, ϕ_Z has connected fibers and $\pi : W \to \phi_Z(Z)$ is the normalization morphism. If $\tau = 1$ then ϕ can be either birational or of fiber type; in the first case ϕ_Z has connected fibers and $\pi : W \to \phi_Z(Z)$ is the normalization morphism while in the second ϕ is an adjunction theoretic scroll contraction onto W (see 2.4) and R_Z is extendable.

Proof. If $\tau > 1$ the proof is as in [3, Propositions 3.13, 3.14], observing that the intersection of Z with any non-trivial fiber F of ϕ has dimension $\dim(Z \cap F) \geqslant 1$.

If $\tau = 1$, by Theorem 3.2 the contraction ϕ is supported by $K_X + \det \mathscr{E} + H$, and so its length $l(\phi)$ is $\geqslant r + 1$.

If ϕ is birational then again the proof of [3, Propositions 3.13] applies since $\dim(Z \cap F) \geqslant 1$.

If ϕ is of fiber type, by Inequality 2.6 we have that all its fibers have dimension $\geqslant r$; if the generic fiber has dimension $\geqslant r + 1$ then it has non-trivial intersection with Z, and this is impossible as in the proof of [3, Proposition 3.14]. So the generic fiber of ϕ is r-dimensional, $l(\phi) = r + 1$ and, if C is a minimal extremal rational curve in a fiber of ϕ with $-K_X.C = l(\phi)$, then $H.C = 1$ and $\det \mathscr{E}.C = r$. In particular $K_X + (r+1)H$ is a good supporting divisor for ϕ, which thus is an adjunction theoretic scroll.

We have to prove now that R_Z is extendable; for this we will first prove that ϕ_Z has connected fibers and then that $\phi_Z(Z)$ is normal. Since ϕ_Z contracts only the curves whose numerical class is in R_Z, outside of the exceptional locus $E(\varphi)$ ϕ_Z is finite-to-one; in particular, if f is a fiber of ϕ which does not contain curves of $E(\varphi)$ then f is r-dimensional, and thus is a projective space $\mathbb{P}^r$.

Since $\det \mathscr{E}.C = r$ for a minimal extremal rational curve, for every line in f, $(\det \mathscr{E})_l \simeq \mathcal{O}_{\mathbb{P}^1}(r)$, $\mathscr{E}_f = \oplus^r \mathcal{O}(1)$, and $Z \cap f$ is one point, thus it is connected (note that we have proved that, outside of $\phi^{-1}(\phi(E(\varphi)))$ ϕ is a projective bundle and Z is a regular section).

On the other hand, the non-trivial fibers of ϕ_Z are connected since they are intersections of Z with fibers of ϕ and [3, 3.13] applies again. Thus ϕ_Z has connected fibers and $\phi_Z(Z) = \phi(X)$, which is normal; so $\pi : W \to \phi_Z(Z)$ is the identity and R_Z is extendable. □

Example 3.9. Let us note that the last case of the above proposition is effective: let $X = \mathbb{P}^2 \times \mathbb{P}^1$ and Z be a $\mathbb{F}_1$-surface in the linear system $\mathcal{O}_{\mathbb{P}\times\mathbb{P}}(1,1)$; the contraction of the (-1) curve of Z lifts to the $\mathbb{P}$-bundle contraction onto $\mathbb{P}^2$.

Proposition 3.10. *In the setup of the above proposition if φ and ϕ are both birational, then $E(\varphi) = E(\phi) \cap Z$.*

Proof. If $x \in E(\varphi)$, then there exists a curve $C \subset Z$ which contains x and is contracted by φ; but, on Z, φ and ϕ_Z contract the same curves, therefore $x \in C$ is contained in $E(\phi)$.

On the other hand, if $x \in E(\phi) \cap Z$ we consider the unsplit family V of deformations of a minimal extremal rational curve contracted by ϕ (see [12, IV.2]). If $\mathrm{Locus}(V, 0 \to x)$ denotes the locus of the curves in V which pass through x, by [12, IV.2.6], $\dim \operatorname{Locus}(V, 0 \to x) \geqslant r + \tau$, hence $\dim(\operatorname{Locus}(V, 0 \to x) \cap Z) \geqslant 1$, so that x lies in a curve contracted by ϕ, and so by φ. $\square$

Remark 3.11. Actually, the proof of the last proposition shows that the fibers of φ are exactly the intersections of the fibers of ϕ with Z.

4 Blow-ups

Proof of Theorem 1.2. Let $D_Z = K_Z + (m-1)H_Z$ be a good supporting divisor of $\varphi : Z \to Z'$, where H_Z is an ample line bundle on Z which restricts to $\mathcal{O}_{\mathbb{P}}(1)$ on every non-trivial fiber of φ. By the Proposition 3.3 we can assume that the extension of H_Z to X, namely H, is ample.

Let as usual ϕ be the contraction associated to the ray R_X to which R_Z is liftable; it is supported by $D = K_X + \det \mathcal{E} + (m-1)H$ and, by 3.8, it is birational.

The non-trivial fibers of ϕ have dimension $\leqslant r + m - 1$ by Proposition 2.7 (see also Remark 3.11); on the other hand, by Proposition 2.6 the dimension of any non-trivial fiber is exactly $r + m - 1$ and $l(\phi) = r + m - 1$.

We can apply [2, Theorem 5.2] to deduce that $\phi : X \to X'$ is the blow-up of a smooth subvariety of codimension $r + m - 1$. Let us point out also that the restriction of $\det \mathcal{E}$ to every line in a fiber of ϕ is $\mathcal{O}_{\mathbb{P}^1}(r)$, and so $\mathcal{E}$ splits on the fibers of ϕ as $\oplus^r \mathcal{O}_{\mathbb{P}}(1)$; thus by Lemma 2.9, we have that $\mathcal{E} \otimes [-E(\phi)] = \phi^* \mathcal{E}'$.

We want to prove now that $Z' \to X'$ is a closed embedding, that is that R_Z is extendable. For this, in the spirit described in the introduction of the paper [4] we now consider a local situation: choose a point $z \in \phi_Z(E(\varphi))$, an affine neighbourhood U of z in X' and consider the restrictions of ϕ and φ to the inverse images of U; to simplify the notation denote again by X, X' and Z the new spaces and by ϕ, ϕ_Z and φ the restricted maps.

In this affine situation and in the notation of the Lemma 2.9 we have that $\mathcal{E}'$ is trivial and in particular $\mathcal{E}$ splits as $\oplus^r L$, where $L = -E(\phi)$; note that $K_X + (r + m - 1)L$ is a good supporting divisor for ϕ. We will now use the horizontal slicing procedure ([4, Lemma 2.6]): let L_i, with $i = 1, \dots, r$, be general smooth sections of L and let $X_i = \bigcap_{j=1,\dots,i} L_j$; note that $X_0 = X$ and that $X_r \simeq Z$; we have a chain of surjections

$$\mathrm{H}^0(X, D) \twoheadrightarrow \mathrm{H}^0(X_1, D_{X_1}) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{H}^0(Z, D_Z)$$

and this implies (see the proof of [4, Lemma 2.6]) that $\pi : Z' \to X'$ is a closed embedding. $\square$

As mentioned in the introduction we have also the following application.

Proposition 4.1. *Assume that Z is the blow-up of a projective space $\mathbb{P}^{n-r}$ along a linear space Y of codimension $m \geqslant 3$. Then X is a projective bundle on $\mathbb{P}^{m-1}$; namely $g : X = \mathbb{P}(\mathcal{G}) \to \mathbb{P}^{m-1}$ for some vector bundle $\mathcal{G}$ on $\mathbb{P}^{m-1}$ with*

$$0 \to \mathcal{V} \to \mathcal{G} \to \oplus^{n-r-m-1} \mathcal{O}_{\mathbb{P}} \oplus \mathcal{O}_{\mathbb{P}}(1) \to 0$$

and $\mathcal{E} = \xi_{\mathcal{G}} \otimes g^* \mathcal{V}^{\check{}}$, where $\xi_{\mathcal{G}}$ is the tautological line bundle of $\mathcal{G}$.

Proof. The variety Z has two extremal rays: the blow-down contraction to $\mathbb{P}^{n-r}$ and a $\mathbb{P}$-bundle contraction on $\mathbb{P}^{m-1}$; by Proposition 3.4 one of these rays is liftable to X; if the fiber type ray is liftable, then we are done by [3, Corollary 4.2]; as stated in that corollary the existence of the sequence is a known fact about vector bundles (see for instance [9, B.5.6]).

We will now show that the birational ray cannot be liftable. Suppose, by contradiction, that this is the case: by Theorem 1.2 we have that the associated contraction is extendable to a contraction $\phi : X \to X'$ which gives a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\;i\;} & \mathrm{Bl}_Y(\mathbb{P}^{n-r}) \\ \phi\downarrow & & \downarrow\varphi \\ X' & \xleftarrow{\;j\;} & \mathbb{P}^{n-r} \end{array}$$

We also know that $\mathscr{E}_E = \oplus^r \mathscr{O}(1)$; hence there exists a vector bundle $\mathscr{E}'$ on X' such that $\mathscr{E} = \phi^*\mathscr{E}' \otimes (-E)$; this vector bundle is ample, by Lemma 2.10, and $\mathbb{P}^{n-r}$ is the zero locus of a section of it. This implies, by [13, Theorem A], that X' is a projective space $\mathbb{P}^n$ and $\mathscr{E}'$ decomposes as $\oplus^r \mathscr{O}_{\mathbb{P}^n}(1)$, but this contradicts the ampleness of $\mathscr{E}$. $\square$

Remark 4.2. Let us note that a blow-up of a projective space $\mathbb{P}^{n-r}$ along a linear space Y of codimension $m \geqslant 3$ cannot be an ample section of a line bundle or of a vector bundle which is a direct sum of line bundles; this follows from [7, Proposition 5.8]. Therefore there exists no example for the above proposition if $\mathscr{E}$ is a line bundle (that is if $\mathscr{V}$ is a line bundle), or a direct sum of line bundles.

For general vector bundles $\mathscr{E}$ however this can happen, as the following example will show.

Example 4.3. On $X = \mathbb{P}^k \times \mathbb{P}^2 = \mathbb{P}(\oplus^{k+1}\mathscr{O}_{\mathbb{P}^2})$, with $k \geqslant 3$, consider the line bundles $\pi_1^*(\mathscr{O}(a)) \otimes \pi_2^*(\mathscr{O}(b)) =: (a,b)$, where π_i are the projections. For $m \gg 0$, let $D_1 \in |(1,m)|$ and $D_2 \in |(1,m+1)|$ be sufficiently general divisors. They correspond to sections of $\oplus^{k+1}\mathscr{O}_{\mathbb{P}^2}(m)$ and $\oplus^{k+1}\mathscr{O}_{\mathbb{P}^2}(m+1)$ on $\mathbb{P}^2$ and therefore they will give an injective morphism of vector bundles, with cokernel V:

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-m-1) \oplus \mathscr{O}_{\mathbb{P}^2}(-m) \to \oplus^{k+1}\mathscr{O}_{\mathbb{P}^2} \to V \to 0.$$

We notice that V is actually a vector bundle since, if $k \geqslant 3$, the two sections can be taken linearly independent at each point of $\mathbb{P}^2$.

Moreover we have that V is ample; in fact the tautological bundle of $\mathbb{P}(V)$ is the restriction of $\xi = \pi_1^*(\mathscr{O}(1))$, the tautological bundle of X, to $\mathbb{P}(V)$ and therefore our claim follows if we show that the restriction of π_1 to $\mathbb{P}(V)$ is a finite-to-one map onto $\mathbb{P}^k$, by a general choice of the sections. Since $\mathbb{P}(V) = D_1 \cap D_2$, this can be proved applying twice the next lemma; the first time to $\pi_1 : X \to \mathbb{P}^k$ and $L = |(1,m)|$, the second time to $\pi_1 : D_1 \to \mathbb{P}^k$ and $L = |(1,m+1)|_{D_1}$. Note in fact that, for $m \gg 0$, $\pi_{1*}((1,m)) = S^m(\oplus^3\mathscr{O}_{\mathbb{P}^k})(1)$ is a spanned vector bundle on $\mathbb{P}^k$ of rank $> k$.

Lemma 4.4. *Let $p : X \to Y$ be a flat morphism of projective manifolds and let L be an ample and spanned line bundle on X. Suppose moreover that p_*L is spanned by global sections on Y and that* $\operatorname{rank}(p_*L)$ *is bigger than* $\dim Y$. *Then the restriction of p to a general $D \in |L|$ is equidimensional, hence flat.*

Proof. It is enough to show that D meets any fiber of the map p properly. In fact D is ample and therefore it meets any fiber; if a fiber $p^{-1}(x)$ is contained in D then it means that the section corresponding to D in p_*L will vanish at the point x, but this is impossible since the assumptions imply that a general section of p_*L does not vanish anywhere. □

Now, dualizing the sequence we have constructed on $\mathbb{P}^2$ and twisting it by $\mathcal{O}(-m)$ we get

$$0 \to V\check{}(-m) \to \oplus^{k+1}\mathcal{O}_{\mathbb{P}^2}(-m) \to \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2} \to 0.$$

If we set $\mathcal{V} := V\check{}(-m)$ and $\mathcal{G} := \oplus^{k+1}\mathcal{O}_{\mathbb{P}^2}(-m)$ then this is a sequence as in Proposition 4.1: in fact $\xi_{\mathcal{G}} \otimes \pi_2^*\mathcal{V}\check{} = \pi_1^*\mathcal{O}(1) \otimes \pi_2^* V$ is an ample vector bundle.

Let therefore $X = \mathbb{P}(\mathcal{G}) = \mathbb{P}(\oplus^{k+1}\mathcal{O}_{\mathbb{P}^2}(-m))$, $g(= \pi_2) : X \to \mathbb{P}^2$ and $\mathcal{E} = \xi_{\mathcal{G}} \otimes g^*\mathcal{V}\check{}$. Then $\mathcal{E}$ is an ample vector bundle on X with a section s, which corresponds to the composite of the duals of the canonical map $g^*(\mathcal{G}) \to \xi_{\mathcal{G}}$ and of $g^*\mathcal{V} \to g^*\mathcal{G}$, whose zero locus is $Z := \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2})$, which is the blow-up of $\mathbb{P}^3$ at a point.

5 $\mathbb{P}^d$-bundles on surfaces with $\kappa > 0$ and their Mori cone

Proposition 5.1. *Let $p : Z \to S$ be a $\mathbb{P}^d$-bundle over a smooth surface such that $\kappa(S) \geqslant 0$; assume that Z has an extremal ray R different from the one associated to the $\mathbb{P}$-bundle contraction. Then the associated contraction φ_R is a blow-down $\varphi_R : Z \to Z_1$ of a divisor $E = p^{-1}(C)$, with C an exceptional (-1)-curve on S, such that $E \simeq \mathbb{P}^1 \times \mathbb{P}^d$ and $E_E \simeq \mathcal{O}(0, -1)$.*

Moreover, Z_1 has a $\mathbb{P}^d$-bundle structure on S_1, where S_1 is the surface obtained contracting the exceptional curve C on S, and $\varphi_R(E)$ is a fiber of $p_1 : Z_1 \to S_1$.

Proof. Suppose that Z has an extremal ray, R, different from the bundle contraction; there exists a rational curve C_0 ($[C_0] \in R$) such that $-K_Z.C_0 > 0$ and $p(C_0)$ is not a point. Let $C = p(C_0)$, let $\nu : \mathbb{P}^1 \to C$ be the normalization of C and consider the fiber product

$$\begin{array}{ccc} Z \times_S \mathbb{P}^1 & \xrightarrow{\bar{\nu}} & Z \\ \bar{p}\downarrow & & \downarrow p \\ \mathbb{P}^1 & \xrightarrow{\nu} & S \end{array} \tag{5.2}$$

The map $\bar{p} : Z_C := Z \times_S \mathbb{P}^1 \to \mathbb{P}^1$ is a $\mathbb{P}$-bundle on $\mathbb{P}^1$; let C_0 be a minimal section of $\bar{p}$; the proof of [18, Lemma 1.5] applies and we get $K_S.C < 0$.

Since on S there is only a finite number of curves which have negative intersection with K_S (the (-1)-curves) we deduce that the image of the exceptional locus of φ_R is C, and C is a (-1)-curve.

Moreover, since the fibers of different extremal contractions can meet only in points, we have that all the fibers of φ_R have dimension one; combining these facts we get that φ_R is a divisorial contraction. By [1] φ_R is a smooth blow-down contraction.

The exceptional locus of φ_R, E, is thus $p^{-1}(C)$ and carries two different $\mathbb{P}$-bundle structures, it is so forced to be $\mathbb{P}^1 \times \mathbb{P}^d$; the description of E_E is clear observing that the lines in one ruling are extremal curves for the blow-up, while those in the other ruling are contained in fibers of the bundle projection.

Let $\mathscr{F}$ be a rank $d+1$ vector bundle on S such that $Z = \mathbb{P}_S(\mathscr{F})$; the restriction of $\mathscr{F}$ to C is, up to twist, $\bigoplus^{d+1}\mathcal{O}_{\mathbb{P}^1}$; therefore if we denote by $\sigma : S \to S_1$ the contraction of C, by Lemma 2.9 there exists a rank $d+1$ vector bundle $\mathscr{F}_1$ on S_1 such that $\mathscr{F} = \sigma^*\mathscr{F}_1$. Consider the commutative diagram

$$\begin{array}{ccc} Z = \mathbb{P}(\mathscr{F}) & \xrightarrow{\bar{\sigma}} & Z_1 = \mathbb{P}(\mathscr{F}_1) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_1} \\ S & \xrightarrow{\sigma} & S_1 \end{array}$$

The map $\bar{\sigma}$ is a good contraction which contracts exactly the curves in R, so it coincides with φ_R. $\square$

6 $\mathbb{P}^d$-bundles on surfaces as ample sections

Proof of Theorem 1.4. By Proposition 5.1 the extremal rays of Z are the ray corresponding to the $\mathbb{P}^d$-bundle fibration and, possibly, other rays of birational type; such a ray corresponds to a blow-down $\beta : Z \to Z_1$ which contracts $\mathbb{P}^d \times \mathbb{P}^1$ to $Y \simeq \mathbb{P}^d$.

By Proposition 3.4 we have that at least one extremal ray of Z is liftable to X; if this ray is the fiber type one, then, by [3, Corollary 4.2] X is a $\mathbb{P}^{r+d}$-bundle on S and we are done.

Suppose now that the ray that is liftable is a birational one, corresponding to a blow-down $\beta : Z \to Z_1$; by Proposition 5.1 Z_1 has a $\mathbb{P}^d$-bundle structure over a smooth surface S_1, $p_1 : Z_1 \to S_1$, obtained contracting a (-1)-curve of S to a point s_1.

Now β is supported by $K_Z + H_Z$ (e.g. taking $H_Z = -E(\beta)$, where $E(\beta)$ is the exceptional divisor of β) and, by Proposition 3.3 we can assume that the line bundle H which restricts to H_Z is ample on X. By Proposition 3.8 if β is liftable to a fiber type ray, ϕ, then ϕ is a scroll contraction $\phi : X \to Z_1$; the proof of Proposition 3.8 also shows that, outside of $\phi^{-1}(\phi(E(\beta)))$ ϕ is a projective bundle and Z is a regular section.

Choose a smooth non-rational curve B in S_1 which does not contain s_1; $T = p_1^{-1}(B)$ is a $\mathbb{P}$-bundle on B, and it is not contained in $\phi(E(\beta))$. Denote by U the inverse image of T via ϕ, $U = \phi^{-1}(T)$; U is a $\mathbb{P}$-bundle on T and $Z \cap U$ is a regular section. Therefore $Z \cap U$ is isomorphic to T and thus $\rho(Z \cap U) = 2$; on the other hand $Z \cap U$ is the zero locus of a section of the ample vector bundle $\mathscr{E}_U$, thus, by Theorem 2.8 $\rho(Z \cap U) \geqslant 3$, a contradiction.

So β is liftable to a birational ray, corresponding to a contraction $\bar{\beta} : X \to X_1$; by the Theorem (1.2) β is extendable and $\bar{\beta}$ is a smooth blow-up of $Y_1 \subset X_1$, such that the restriction of $E(\bar{\beta})$ to Z is $E(\beta)$; thus by Lemma 2.9 there exists a vector bundle $\mathscr{E}_1$ on X_1 such that

$$\mathscr{E} \otimes \mathscr{O}_X(E) = \Pi^* \mathscr{E}_1$$

and moreover, by Lemma 2.10, $\mathscr{E}_1$ is ample.

Summing up, we have replaced the starting triple $(X, \mathscr{E}, Z)$ with a new triple $(X_1, \mathscr{E}_1, Z_1)$ satisfying the assumptions of the theorem and such that $\rho(Z_1) = \rho(Z) - 1$. Thus we can repeat the above procedure, i.e. one of the extremal contractions of Z_1 is liftable. Since $\rho(Z)$ is finite, at some point of this process we must find some triple $(X_k, \mathscr{E}_k, Z_k)$ such that the $\mathbb{P}^d$-bundle contraction of Z_k is extendable to a $\mathbb{P}^{r+d}$-bundle contraction of X_k and $\mathscr{E}_{k|F} \simeq \oplus^r \mathscr{O}(1)$ for every fiber of the bundle contraction.

Let $\bar{\beta}_k : X_{k-1} \to X_k$ be the last blow-down contraction and let F be the fiber of the $\mathbb{P}^{r+d}$-bundle contraction of X_k which contains Y_k, the center of $\bar{\beta}_k$ (which is $F \cap Z_k \simeq \mathbb{P}^d$). Let l' be a line in F which meets Y_k transversally and let l be its strict transform in X_{k-1}. We have

$$(\mathscr{E}_{k-1})_l \simeq (\bar{\beta}_k^* \mathscr{E}_k)_l \otimes \mathscr{O}(-E_k)_l \simeq (\oplus^r \mathscr{O}_{\mathbb{P}^1}(1)) \otimes \mathscr{O}_{\mathbb{P}^1}(-1) \simeq \oplus^r \mathscr{O}_{\mathbb{P}^1},$$

contradicting the ampleness of $\mathscr{E}_{k-1}$. $\square$

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