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Adjacency preserving mappings

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Abstract. In a certain class of point-line geometries, called locally projective near polygons, which includes dual polar spaces and generalised $2n$-gons, surjective adjacency preserving mappings are already collineations.

Key words. Generalised polygon, near polygons, adjacency preserving mappings, dual polar space.

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1 Introduction

In [6] it is shown that every surjective adjacency preserving map on the set of maximal invariant subspaces of a null system is already an element of the basic group. Although this result sounds very 'metric', the proof given in [6] is actually rather geometric in spirit. A closer analysis of this proof suggested that it can be applied to a much more general, completely geometric setting. For this we introduce locally projective near polygons. This is a class of point-line geometries that contains generalised polygons but also dual polar spaces considered as point-line geometries.

We show that any surjective and adjacency preserving map between two locally projective near polygons with the same finite diameter and the same finite rank is already induced by an isomorphism of geometries.

The set of r-dimensional invariant subspaces and the set of $(r-1)$-dimensional invariant subspaces of a null system on a $(2r+1)$-dimensional projective space form a dual polar space, thus also a locally projective near polygon. Moreover, collineations of dual polar spaces related to polarities are elements of the associated linear group (see e.g. [3]). Hence the result given here is an extension of the main result in [6].

Even though some further generalisation of the result still might be possible, there is not too much room left for generalisation. Every bijection on the point set of a projective plane is adjacency preserving, but not necessarily induced by a collineation. Thus the result cannot be extended to arbitrary point-line geometries. On the other hand, weakening 'adjacency preserving' to 'collinearity preserving' produces also counterexamples ([7]).

2 Preliminaries

A *thick point-line geometry* $\mathfrak{G} = (\mathscr{P}, \mathscr{L}, \mathrm{I})$ consists of a non-empty set $\mathscr{P}$ of *points*, a non-empty set $\mathscr{L}$ of *lines* and an *incidence relation* I between points and lines such that every point is incident with at least three lines and such that every line is incident with at least three points. For a point p the set of lines incident with p is denoted $\mathscr{L}_p$. For a line l the set of points incident with l is denoted $\mathscr{P}_l$. For incidence relations we employ only the symbol I, hence most times we will omit it when naming a point-line geometry.

Suppose p, q are points of a point-line geometry. A *chain of length n from p to q* is an $n+1$ tuple $(x_0, x_1, \dots, x_n)$ of points with $x_0 = p$ and $x_n = q$ such that for all $i \in \{0, \dots, n-1\}$ the points x_i and x_{i+1} are different but incident with a common line. The *distance* $d(p,q)$ between two points p and q is the length of a shortest chain from p to q. If there is no chain from p to q we let $d(p,q) = \infty$. Furthermore, $d(p,p) = 0$ for any point p. In other words, d is the graph-theoretical distance in the collinearity graph of the geometry. Obviously, the triangle inequality holds for d. Two points p, q are *adjacent*, if $d(p,q) = 1$. For $p \in \mathscr{P}$ and $n \in \mathbb{N}$ the set of points at distance n to p is denoted $D_n(p)$. The number $\sup\{d(p,q) \mid p, q \in \mathscr{P}\} \in \mathbb{N} \cup \{\infty\}$ is called the *diameter* of the point-line geometry $\mathfrak{G}$.

We view projective spaces as incidence geometries with more objects than just points and lines (see. e.g. [5]). In particular, we work also with hyperplanes and, implicitly, with colines. A projective line is considered as a projective space of dimension 1.

A thick point-line geometry $\mathfrak{G} = (\mathscr{P}, \mathscr{L}, \mathrm{I})$ of finite diameter n is called a *locally projective near* $2n$*-gon of rank* $m \in \mathbb{N}\backslash\{0,1\}$ if the following requirements are met.

1. Any two different points are incident with at most one common line.
2. For any $(p, l) \in \mathscr{P} \times \mathscr{L}$ there exists a unique point $\pi(p, l)$ in $\mathscr{P}_l$ closest to p.
3. There is some $m = m(\mathfrak{G}) \in \mathbb{N}\backslash\{0,1\}$, called the *rank* of $\mathfrak{G}$, such that for every point $p \in \mathscr{P}$ the sets $\{S(p,q) \mid q \in \mathscr{P}\backslash\{p\}\}$, where
$$S(p,q) := \{l \in \mathscr{L}_p \mid \pi(q,l) \neq p\},$$
form a projective space of dimension $m-1$. We denote this projective space by $S(p)$.
4. If p and q are two different points with $d(p,q) < n$, then $S(p,q) \neq \mathscr{L}_p$. If $d(p,q) = n-1$, then $S(p,q)$ is a hyperplane of $S(p)$.

Remark. Thick point-line geometries satisfying the first two of the above requirements are known as near polygons ([4, 3.35]). The third axiom states that for every point p the line pencil $\mathscr{L}_p$ carries the structure of a projective space. This structure is induced by traces of the points different from p. The fourth axiom ensures that these traces behave reasonably well.

Next we provide some examples of locally projective near polygons.

2.1 Generalised polygons. Let $n \in \mathbb{N} \backslash \{0, 1\}$. An *ordinary n-gon*, $n \geqslant 2$, in some point-line geometry is a chain $x_0, \ldots, x_n$ of length n with $x_0 = x_n$ such that the lines joining x_i to x_{i+1} are pairwise different. The joining lines are also considered part of the ordinary n-gon. A *generalised n-gon* is a thick point-line geometry that contains no ordinary k-gon for $k < n$ and where any two elements of $\mathscr{P} \cup \mathscr{L}$ can be completed to an ordinary n-gon. An introduction to generalised polygons including alternative definitions of generalised polygons can be found in [8], while [9] offers an extensive discussion of generalised polygons.

Lemma 2.1. *A generalised* $2n$*-gon with* $n > 1$ *is a locally projective near* $2n$*-gon of rank* 2.

Proof. It follows straight from the definition that a generalised $2n$-gon has diameter n. If $n > 1$ then a generalised $2n$-gon contains no ordinary 2-gons. Hence any two different points are joined by at most one line.

Let $(p, l) \in \mathscr{P} \times \mathscr{L}$. Take an ordinary $2n$-gon containing p and l. There are two parts of this ordinary $2n$-gon joining p to l, one of which is shorter than the other. Let q denote the point in the shorter part incident with the line l. Then already within the ordinary $2n$-gon at hand, there is a chain from p to q whose length is less than n. Thus $c := d(p, q) < n$.

We claim $q = \pi(p, l)$. Suppose there is another point $r \in \mathscr{P}_l$ with $d(p, r) \leqslant c$. Then the shortest chain from p to r plus the shortest chain from q to p contains an ordinary k-gon where $k \leqslant 2c + 1 < 2n$. In a generalised $2n$-gon this is impossible. Thus $q = \pi(p, l)$ as claimed.

Let p, q denote two different points. If $d(p, q) = n$, then $\pi(q, l) \neq p$ for all $l \in \mathscr{L}_p$. If $d(p, q) = c < n$, then the shortest chain from p to q is unique, for otherwise two different chains of length c from p to q contain an ordinary k-gon, where $k \leqslant 2c < 2n$. This is not allowed. Let r denote the point in the shortest chain from p to q adjacent to p and let l denote the line incident with p and r. Then $S(p, q) = \{l\}$. Thus $S(p)$ is a projective line, that is, a projective space of dimension 1. And also $S(p, q)$ is a hyperplane of $S(p)$ if $0 < d(p, q) < n$. □

2.2 Dual polar spaces. Let σ denote a polarity on a projective space. A subspace W of the projective space is called *totally isotropic* if $W \subset W^\sigma$. The set of totally isotropic subspaces with symmetrised inclusion as incidence relation forms a *polar space*. The dual of such a geometry is a *dual polar space*. The (dual) polar spaces obtained in this fashion from a polarity are the classical (dual) polar spaces. There are also definitions of abstract (dual) polar spaces which define polar spaces as certain incidence geometries ([4, 3]).

Now consider again a projective space with a polarity σ. Take as points the totally isotropic subspaces of maximal dimension and as lines the totally isotropic subspaces of dimension one less. That is, we view the dual polar space associated with σ as a point-line geometry. In [2] Peter Cameron introduced a set of axioms that characterises abstract dual polar spaces as certain point-line geometries. We do not reproduce these somehow technical axioms here. However, we note that Theorem 1, Lemma 1 and Lemma 2 of [2] immediately imply the next result.

Lemma 2.2. *A dual polar space, considered as a point-line geometry, is a locally projective near polygon, where the diameter equals the rank.*

Polar spaces and dual polar spaces of rank 2 are precisely the generalised quadrangles. For an introduction to polar spaces see e.g. [1].

We now deduce some useful properties of locally projective near polygons.

Lemma 2.3. *Suppose $\mathfrak{G} = (\mathscr{P}, \mathscr{L})$ is a locally projective near $2n$-gon of rank m. Let p, q denote two different points. Then the following assertions hold.*

1. *If $d(p, q) = c < n$, then any chain $(p = x_0, \dots, x_c = q)$ of length c from p to q can be extended to a chain $(p = x_0, \dots, x_c, \dots, x_n)$ with $d(p, x_n) = n$.*
2. *There exists a point r with $d(p, r) = n > d(q, r)$.*
3. *If p, q are adjacent, there exists a point r with $d(p, r) = n = d(q, r) + 1$.*
4. *The point set of the projective space $S(p)$ coincides with the pencil $\mathscr{L}_p$.*
5. *For every hyperplane H of $S(p)$ there is some $r \in D_{n-1}(p)$ with $H = S(p, r)$.*

Proof. If $d(p, q) = c < n$, there is some line $l \in \mathscr{L}_q \backslash S(q, p)$. We know $\pi(p, l) = q$ for such a line. Let $x_{c+1} \in \mathscr{P}_l \backslash \{q\}$. Then $d(p, x_{c+1}) = c + 1$ and $d(q, x_{c+1}) = 1$. Repeating this step inductively, we obtain the required chain. This proves the first assertion.

For different points p, q with $d(p, q) < n$ the second assertion follows from the first. If $d(p, q) = n$ choose $l \in \mathscr{L}_q$ and $r \in \mathscr{P}_l \backslash \{q, \pi(p, l)\}$. Then $d(p, r) = n > 1 = d(q, r)$. This proves the second assertion.

To prove the third assertion, suppose p, q are adjacent points. By the second assertion there is some $r \in D_n(p)$ with $d(q, r) < n$. The triangle inequality implies $d(q, r) \geqslant d(p, r) - d(p, q) = n - 1$. Thus $d(q, r) = n - 1$.

Let $l \in \mathscr{L}_p$ and $r \in \mathscr{P}_l \backslash \{p\}$. Then $S(p, r) = \{l\}$. This proves the fourth assertion.

Finally, suppose H is a hyperplane of $S(p)$. There is some point s such that $H = S(p, s)$. Let $c := d(p, s) < n$. By the first assumption any chain of length c from p to s can be extended to a chain $(p, \dots, x_{n-1}, x_n)$ of length n, where $d(p, x_n) = n$. Let $r := x_{n-1}$. Then $d(p, r) = n - 1$ and $S(p, s) \subset S(p, r)$. Since $S(p, s)$ and $S(p, r)$ are hyperplanes we deduce $S(p, s) = S(p, r)$. □

2.3 Adjacency preserving maps. Let $\mathfrak{G} = (\mathscr{P}, \mathscr{L})$ and $\mathfrak{G}' = (\mathscr{P}', \mathscr{L}')$ denote two point-line geometries. A map $\phi : \mathscr{P} \to \mathscr{P}'$ is called *adjacency preserving*, if $d(p, q) = 1$ implies $d(p^\phi, q^\phi) = 1$. The following properties of adjacency preserving maps are obvious.

Lemma 2.4. *Let $\mathfrak{G} = (\mathscr{P}, \mathscr{L})$ and $\mathfrak{G}' = (\mathscr{P}', \mathscr{L}')$ denote two point-line geometries and $\phi : \mathscr{P} \to \mathscr{P}'$ an adjacency preserving map.*

1. *For any two points $p, q \in \mathscr{P}$ we have $d(p^\phi, q^\phi) \leqslant d(p, q)$.*
2. *If $\mathfrak{G}'$ contains no ordinary 2- and 3-gons, then ϕ induces a map from $\mathscr{L} \to \mathscr{L}'$ as follows: a line l is mapped to the line connecting p^ϕ to q^ϕ, where p and q are different points incident with l.*

3 Proof

In general, adjacency preserving maps can be rather wild. However, restrictions on the geometries involved together with surjectivity tame the beast. So, in this section we consider two locally projective near $2n$-gons $\mathfrak{G} = (\mathscr{P}, \mathscr{L})$ and $\mathfrak{G}' = (\mathscr{P}', \mathscr{L}')$ of common finite rank $m \geqslant 2$ and common finite diameter $n \geqslant 2$. Moreover $\phi : \mathscr{P} \to \mathscr{P}'$ is a surjective adjacency preserving map. By Lemma 2.4.2 there is an induced line mapping.

Lemma 3.1. *Let $p, q \in \mathscr{P}$ with $d(p,q) = n$ and $d(p^\phi, q^\phi) = n-1$. Then $(\mathscr{L}_p)^\phi \subset S(p^\phi, q^\phi)$.*

Proof. Let $l \in \mathscr{L}_p$ and $r := \pi(q,l)$. Then $d(r,q) = n-1$ and $d(p,r) = 1$. This implies $d(r^\phi, q^\phi) \leqslant n-1 = d(p^\phi, q^\phi)$ and $d(p^\phi, r^\phi) = 1$. If $d(r^\phi, q^\phi) = d(p^\phi, q^\phi) = n-1$, then there is some s incident with l^ϕ such that $d(s, q^\phi) = n-2$. This implies $\pi(q^\phi, l^\phi) = s \neq p^\phi$. If $d(r^\phi, q^\phi) < d(p^\phi, q^\phi)$ then $\pi(q^\phi, l^\phi) = r^\phi \neq p^\phi$. Hence $l^\phi \in S(p^\phi, q^\phi)$. □

Lemma 3.2. *Let $p, q \in \mathscr{P}$ with $d(p^\phi, q^\phi) = n-1$. Then $d(p,q) = n-1$.*

Proof. The only other possibility besides $d(p,q) = n-1$ is $d(p,q) = n$. So suppose $d(p,q) = n$. Choose points $r_1, \dots, r_{m-1}$ such that $d(p^\phi, r_i^\phi) = n-1$ for all $i \in \{1, \dots, m-1\}$ and $\{S(p^\phi, r_i^\phi) \mid i \in \{1, \dots, m-1\}\} \cup \{S(p^\phi, q^\phi)\}$ is a basis of the dual of the projective space $S(p^\phi)$. Then $n-1 \leqslant d(p, r_i) \leqslant n$ for all $i \in \{1, \dots, m-1\}$. Therefore $\bigcap_{i=1}^{m-1} S(p, r_i)$ contains at least one line $l \in \mathscr{L}_p$. If $d(p, r_i) = n$ then $l^\phi \in S(p^\phi, r_i^\phi)$ by Lemma 3.1. If $d(p, r_i) = n-1$ then $\pi(r_i^\phi, l^\phi) \neq p^\phi$ since $\pi(r_i, l) \neq p$. This implies $l^\phi \in S(p^\phi, r_i^\phi)$. By Lemma 3.1 also $l^\phi \in S(p^\phi, q^\phi)$. Thus $l^\phi \in \bigcap_{i=1}^{m-1} S(p^\phi, r_i^\phi) \cap S(p^\phi, q^\phi)$ which contradicts the fact that

$$\{S(p^\phi, r_i^\phi) \mid i \in \{1, \dots, m-1\}\} \cup \{S(p^\phi, q^\phi)\}$$

is a basis of the dual of $S(p^\phi)$. □

Lemma 3.3. *For every $p \in \mathscr{P}$ the induced map $\phi : \mathscr{L}_p \to \mathscr{L}'_{p^\phi}$ is surjective.*

Proof. Let $g \in \mathscr{L}'_{p^\phi}$. Choose points $r_1, \dots, r_{m-1} \in D_{n-1}(p^\phi)$ such that $\{g\} = \bigcap_{i=1}^{m-1} S(p^\phi, r_i)$. For every $i \in \{1, \dots, m-1\}$ choose q_i with $q_i^\phi = r_i$. Then $d(p, q_i) = n-1$ by Lemma 3.2. Thus there exists some $l \in \mathscr{L}_p$ with $l \in \bigcap_{i=1}^{m-1} S(p, q_i)$. But then $l^\phi = g$. □

Lemma 3.4. *Let $p, q \in \mathscr{P}$ with $d(p,q) = n$. Then $d(p^\phi, q^\phi) = n$.*

Proof. We show by induction that there is no $k \in \mathbb{N} \setminus \{0\}$ with $d(p^\phi, q^\phi) = n-k$. By Lemma 3.2 we have $d(p^\phi, q^\phi) \neq n-1$. Let $k \in \mathbb{N} \setminus \{0,1\}$ with $d(p^\phi, q^\phi) = n-k$ and $d(s^\phi, q^\phi) = n$ or $d(s^\phi, q^\phi) \leqslant n-k$ for all $s \in D_n(q)$. Let $g \in \mathscr{L}'_{p^\phi} \setminus S(p^\phi, q^\phi)$. By Lemma

3.3 there is some $l \in \mathcal{L}_p$ with $l^\phi = g$. Let r denote a point on l different from p and $\pi(q,l)$. Then $d(r,q) = d(p,q) = n$ and $d(r^\phi, q^\phi) = d(p^\phi, q^\phi) + 1 = n - (k-1)$, contradicting our assumptions. □

Lemma 3.5. *The map ϕ is injective.*

Proof. Let $p, q \in \mathcal{P}$ with $p \neq q$. By Lemma 2.3 there exists some $r \in \mathcal{P}$ with $d(p,r) = n > d(q,r)$. By Lemma 3.4 we know $d(p^\phi, r^\phi) = n$. We also know $d(q^\phi, r^\phi) \leqslant d(q,r) < n$. Hence $p^\phi \neq q^\phi$. □

Lemma 3.6. *Let $p, q \in \mathcal{P}$. Then $d(p,q) = 1$ if and only if $d(p^\phi, q^\phi) = 1$.*

Proof. Our main assumption on ϕ is that $d(p,q) = 1$ implies $d(p^\phi, q^\phi) = 1$. So we only have to show that $d(p^\phi, q^\phi) = 1$ implies $d(p,q) = 1$. So let $p, q \in \mathcal{P}$ with $d(p^\phi, q^\phi) = 1$. Then $d(p,q) < n$ by Lemma 3.4. By Lemma 2.3.1 there is some point $r \in \mathcal{P}$ with $n = d(p,r) = d(p,q) + d(q,r)$. Now Lemma 2.4.1, the triangle inequality and Lemma 3.4 imply

$$n = d(p,r) = d(p,q) + d(q,r) \geqslant d(p^\phi, q^\phi) + d(q^\phi, r^\phi) \geqslant d(p^\phi, r^\phi) = n.$$

Applying Lemma 2.4.1 once more yields $d(p,q) = d(p^\phi, q^\phi) = 1$. □

The last two lemmas are the key to our main result.

Theorem 3.7. *Let $\mathfrak{G} = (\mathcal{P}, \mathcal{L})$ and $\mathfrak{G}' = (\mathcal{P}', \mathcal{L}')$ denote two locally projective near $2n$-gons, $n \geqslant 1$, of finite rank $m \geqslant 2$. Then any surjective, adjacency preserving map $\phi : \mathcal{P} \to \mathcal{P}'$ induces an isomorphism of geometries.*

Proof. By Lemma 3.5 the map ϕ is injective. By Lemma 3.6 the inverse map is also adjacency preserving. Lemma 3.6 also implies that the induced map $\phi : \mathcal{L} \to \mathcal{L}'$ is a bijection. Moreover, by the construction of the induced map between the line sets, we have $p \mathrel{\mathrm{I}} l$ if and only if $p^\phi \mathrel{\mathrm{I}} l^\phi$, for any point p and any line l. In other words, ϕ is a geometric isomorphism between the geometries $\mathfrak{G}$ and $\mathfrak{G}'$. □

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