Adv. Geom. **3** (2003), 433–452

Advances in Geometry

Complex geometry of generalized annuli

Chiara de Fabritiis

(Communicated by G. Gentili)

Abstract. We study the complex geometry of a class of domains in $\mathbb{C}^n$ which generalize the annuli in $\mathbb{C}$, i.e., which are quotients of the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ for the action of a group generated by a hyperbolic element of $\operatorname{Aut}\mathbb{B}^n$. In particular, we prove that the degree of holomorphic maps between two such domains is bounded by a constant which depends on the "radii" of the domains only and we give some results on the existence of complex geodesics for the Kobayashi distance in these domains.

Key words. Holomorphic maps, annuli, degree of a map, complex geodesics.

2000 Mathematics Subject Classification. Primary 32A10, 32A40; secondary 32H15, 32A30

1 Introduction

The aim of this paper is to study the complex geometry of a class of domains in $\mathbb{C}^n$ which are quotients of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ for the action of a group generated by a hyperbolic element of $\operatorname{Aut}\mathbb{B}^n$ (see Section 2 for definitions). Since the annuli in $\mathbb{C}$ are obtained as quotients of the unit disk $\Delta = \{\xi \in \mathbb{C} : |\xi| < 1\}$ for the action of a group generated by a hyperbolic element of $\operatorname{Aut}\Delta$, the domains we study can be seen as a generalization of annuli to several complex variables.

In Section 2 we give some definitions we need in the sequel of the paper and we recall some statements concerning this class of domains, which were introduced in [3]. In Section 3 we generalize to several complex variables a result which is due to Schiffer in the one-dimensional case: the degree of a holomorphic map between two annuli is bounded by a constant which depends on the moduli of the annuli only.

In Section 4 we study the geometry of extremal mappings and complex geodesics for the Kobayashi distance in this class of domains. In particular we prove that there always exists an extremal mapping through two given points of a "generalized annulus" and we give several results on existence and non-existence of complex geodesics according to the "radius" of the domain and to other parameters which classify these domains.

2 Preliminaries and statements

We denote the unit ball for the Euclidean metric in $\mathbb{C}^n$ by $\mathbb{B}^n$; the following result is well known.

Theorem 2.1. *Any holomorphic automorphism γ of $\mathbb{B}^n$ can be extended holomorphically to a neighborhood of the closure of $\mathbb{B}^n$; if γ has no fixed points in $\mathbb{B}^n$, then its extension has either one or two fixed points in $\partial\mathbb{B}^n$.*

From now on we shall denote by the same symbol a holomorphic automorphism of $\mathbb{B}^n$ and its extension to the closure of $\mathbb{B}^n$.

Definition 2.2. Let $\gamma \in \operatorname{Aut}\mathbb{B}^n$: if γ has at least one fixed point in $\mathbb{B}^n$, then γ is said to be *elliptic*; if γ has no fixed points in $\mathbb{B}^n$ and has one fixed point in $\partial\mathbb{B}^n$, it is said to be *parabolic*; if γ has no fixed points in $\mathbb{B}^n$ and has two fixed points in $\partial\mathbb{B}^n$, it is said to be *hyperbolic*.

To generalize the construction of annuli to several complex variables, we will focus our attention to the action of hyperbolic elements on $\mathbb{B}^n$. First of all, we recall a result which is due to de Fabritiis and Gentili (see [5]).

Proposition 2.3. *Let γ be a hyperbolic element in* $\operatorname{Aut}\mathbb{B}^n$; *then there exist $T \in \mathbb{R}^*$ and $\theta_2, \dots, \theta_n \in \mathbb{R}$ such that γ is conjugate to*

$$\gamma_0 : z \mapsto \left(\frac{z_1 \cosh T + \sinh T}{z_1 \sinh T + \cosh T}, \frac{e^{i\theta_2} z_2}{z_1 \sinh T + \cosh T}, \dots, \frac{e^{i\theta_n} z_n}{z_1 \sinh T + \cosh T}\right). \tag{2.1}$$

In the sequel it will be useful to consider the problem on the Siegel half-space $\mathbb{H}^n = \{w \in \mathbb{C}^n \mid \Im w_1 > |w_2|^2 + \dots + |w_n|^2\}$ which is biholomorphic to $\mathbb{B}^n$ via the Cayley transform $\mathscr{C}$ (see e.g. Rudin [7] or Abate [1]), so we also give the form of hyperbolic elements in $\operatorname{Aut}\mathbb{H}^n$.

Corollary 2.4. *Let $\mu \in \operatorname{Aut}\mathbb{H}^n$ be hyperbolic; then there exist $\lambda \in \mathbb{R}^+ \setminus \{1\}$ and $\theta_2, \dots, \theta_n \in \mathbb{R}$ such that μ is conjugate to*

$$\mathscr{C} \circ \gamma_0 \circ \mathscr{C}^{-1} = \mu_0 : w \mapsto (\lambda^2 w_1, \lambda e^{i\theta_2} z_2, \dots, \lambda e^{i\theta_n} z_n), \tag{2.2}$$

where $\lambda = e^T$.

This result enables us to consider the quotients of $\mathbb{B}^n$ (resp. $\mathbb{H}^n$) for the action of the group Γ (M) generated by a hyperbolic element $\gamma \in \operatorname{Aut}\mathbb{B}^n$ $(\mu \in \operatorname{Aut}\mathbb{H}^n)$. Since the quotients $\mathbb{H}^n/M_1$ and $\mathbb{H}^n/M_2$ are biholomorphic iff M_1 and M_2 are conjugate in $\operatorname{Aut}\mathbb{H}^n$, then it is enough to consider the case of a group generated by an element of the form (2.2). As we are interested in the group M generated by μ_0, rather than in the element μ_0 itself, we can always suppose that $\lambda > 1$, that is $T > 0$. Let

$\ln : \mathbb{H}^1 \to \mathbb{R} \times (0,\pi)$ be a branch of the logarithm, set $b = 1/\ln\lambda = 1/T$ and consider the holomorphic map $\mathrm{pr} : \mathbb{H}^n \to \mathbb{C}^n$ given by

$$\mathrm{pr}(w) = (e^{ib\pi \ln w_1}, e^{-b(\ln\lambda + i\theta_2)\ln w_1/2} w_2, \dots, e^{-b(\ln\lambda + i\theta_n)\ln w_1/2} w_n). \tag{2.3}$$

In [3] and [4] the following result is proved:

Theorem 2.5. *Let μ_0 be given by* (2.2) *and M be the group generated by μ_0. Then the map* $\mathrm{pr} : \mathbb{H}^n \to \mathbb{C}^n$ *given by* (2.3) *factors on $\mathbb{H}^n/M$ giving a biholomorphism between $\mathbb{H}^n/M$ and the bounded domain*

$$\Omega(r, \theta_2, \dots, \theta_n) = \left\{ \xi \in \mathbb{C}^n \,\middle|\, r < |\xi_1| < 1, \sum_{j=2}^{n} |\xi_j|^2 |\xi_1|^{\theta_j/\pi} < \sin\left(\frac{\pi \ln|\xi_1|}{\ln r}\right) \right\},$$

where $r = e^{-\pi^2/\ln\lambda} = e^{-\pi^2/T} \in (0,1)$, $\theta_2, \dots, \theta_n \in \mathbb{R}$. In particular $\mathbb{H}^n/M$ is a Stein manifold which is biholomorphic to a bounded domain in $\mathbb{C}^n$.

As a consequence of the previous theorem, the domains $\Omega(r, \theta_2, \dots, \theta_n)$ can be seen as a generalization to several complex variables of the annuli, which are the quotients of the unit disk $\Delta = \mathbb{B}^1$ for the action of the groups generated by hyperbolic automorphisms of Δ. In fact $\Omega(r) = \{\xi \in \mathbb{C} \mid r < |\xi| < 1\}$ is an annulus in $\mathbb{C}$ and for this reason we shall often call these domains "generalized annuli".

To simplify notation, when no confusion can arise we only write Ω instead of $\Omega(r, \theta_2, \dots, \theta_n)$. Of course, via the Cayley transform we can also study the problem on $\mathbb{B}^n$: in this case the covering will be given by $(\mathbb{B}^n \xrightarrow{\chi} \Omega)$ where $\chi = \mathrm{pr} \circ \mathscr{C}$.

Remark 2.6. Notice that the domain $\Omega(r, \theta_2, \dots, \theta_n)$ retracts by deformation on the annulus $\Omega(r)$, in particular it is doubly connected.

From now on, we shall denote by Ω_1 the domain $\Omega(r_1, \theta_2, \dots, \theta_{n_1}) \subset \mathbb{C}^{n_1}$ and by Ω_2 the domain $\Omega(r_2, \vartheta_2, \dots, \vartheta_{n_2}) \subset \mathbb{C}^{n_2}$; in order to simplify the notation, the symbol $\mathscr{H}$ will stand for $\mathrm{Hol}(\Omega_1, \Omega_2) = \{f : \Omega_1 \to \Omega_2 \mid f \text{ holomorphic}\}$. For all $f \in \mathscr{H}$, we will denote the degree of f by $d(f)$.

Definition 2.7. Let $f, g \in \mathscr{H}$. We say that f and g are *homotopic* if there exists a continuous map $F : [0,1] \times \Omega_1 \to \Omega_2$ such that

(i) $F(0,\cdot) = f$, $F(1,\cdot) = g$;

(ii) $F(t,\cdot) \in \mathscr{H}$ for all $t \in [0,1]$.

Of course, the fact of being homotopic is an equivalence relation on $\mathscr{H}$; we shall denote the homotopy class of $f \in \mathscr{H}$ by $[f]$.

At last, if D is a domain in $\mathbb{C}^n$, we shall denote by $k_D(\cdot,\cdot)$ the *Kobayashi distance on D* and by $\kappa_D(\cdot,\cdot)$ the *Kobayashi metric on D* (for a comprehensive reference on this topic see [1] or [6]).

3 Holomorphic maps between generalized annuli

In this section we generalize the estimate of the degree of a holomorphic map between two annuli which is due to Schiffer (see [8]) in the one-dimensional case.

Theorem 3.1. *If* $f \in \mathrm{Hol}(\Omega(r_1), \Omega(r_2))$, *then* $|d(f)| \leqslant [\ln r_2/\ln r_1]$. *Moreover, if equality holds there exists* $\theta \in \mathbb{R}$ *such that for all* $\xi \in \Omega(r_1)$

$$f(\xi) = \begin{cases} e^{i\theta}\xi^{d(f)} & \text{if } d(f) > 0, \\ r_2 e^{i\theta}\xi^{d(f)} & \text{if } d(f) < 0. \end{cases}$$

Let $\Omega_1 = \Omega(r_1, \theta_2, \dots, \theta_{n_1}) \subset \mathbb{C}^{n_1}$ and $\Omega_2 = \Omega(r_1, \vartheta_2, \dots, \vartheta_{n_2}) \subset \mathbb{C}^{n_2}$ be two generalized annuli; since the fundamental group of both Ω_1 and Ω_2 is isomorphic to $\mathbb{Z}$, we can define the degree $d(f) \in \mathbb{Z}$ of a holomorphic map $f \in \mathscr{H}$ by choosing generators $\tilde{\alpha}_j$ of $\pi_1(\Omega_j)$ represented by $\alpha_j(t) = (\sqrt{r_j}e^{2\pi i t}, 0, \dots, 0)$ for $j = 1, 2$ and setting $f_*(\tilde{\alpha}_1) = d(f)\tilde{\alpha}_2$.

Theorem 3.2. *If* $f \in \mathscr{H}$, *then* $|d(f)| \leqslant [\ln r_2/\ln r_1]$.

Proof. Let $S_n = \{w \in \mathbb{C}^n \mid \Im w_1 \in (0, \pi), \Im e^{w_1} > |w_2|^2 + \cdots + |w_n|^2\}$; it is easily verified that the map $E : S_n \ni w \to (e^{w_1}, w_2, \dots, w_n) \in \mathbb{H}^n$ is a biholomorphism.

Then we can consider the coverings $(S_{n_j} \xrightarrow{q_j} \Omega_j)$ for $j = 1, 2$, where the maps q_1 and q_2 are given by

$$q_1(w) = (e^{ib_1\pi w_1}, e^{-b_1(\ln\lambda_1 + i\theta_2)w_1/2}w_2, \dots, e^{-b_1(\ln\lambda_1 + i\theta_{n_1})w_1/2}w_{n_1}),$$
$$q_2(w) = (e^{ib_2\pi w_1}, e^{-b_2(\ln\lambda_2 + i\vartheta_2)w_1/2}w_2, \dots, e^{-b_2(\ln\lambda_2 + i\vartheta_{n_2})w_1/2}w_{n_2}),$$

with $b_j = -\ln r_j/\pi^2$ and $\ln\lambda_j = 1/b_j$. The group of deck-transformations of the coverings $(S_{n_j} \xrightarrow{q_j} \Omega_j)$ is generated by $v_j = E^{-1} \circ \mu_j \circ E : S_{n_j} \to S_{n_j}$ given by

$$v_1(w) = (w_1 + 2\ln\lambda_1, \lambda_1 e^{i\theta_2}w_2, \dots, \lambda_1 e^{i\theta_{n_1}}w_{n_1}),$$
$$v_2(w) = (w_1 + 2\ln\lambda_2, \lambda_2 e^{i\vartheta_2}w_2, \dots, \lambda_2 e^{i\vartheta_{n_2}}w_{n_2}),$$

respectively. Since the domain S_{n_1} is simply connected, there exists a continuous map $\tilde{f} : S_{n_1} \to S_{n_2}$ such that $q_2 \circ \tilde{f} = f \circ q_1$; the maps q_1 and q_2 being local biholomorphisms, we immediately obtain that $\tilde{f}$ is holomorphic. Interpreting the degree of f via the isomorphism between the fundamental groups of Ω_j and the groups of deck-transformations of the coverings, we obtain the following equality

$$\tilde{f} \circ v_1 = v_2^{d(f)} \circ \tilde{f}; \tag{3.1}$$

the comparison between (3.1) and the contracting property of the Kobayashi distance will yield the conclusion. Let us consider the points $w^0 = (i\pi/2, 0, \dots, 0)$ and

$w^1 = v_1(w^0) = (2\ln\lambda_1 + i\pi/2, 0, \dots, 0) \in S_{n_1}$; if k_D denotes the Kobayashi distance on D, we have the following chain of inequalities

$$\begin{aligned} k_{S_{n_2}}(\tilde{f}(w^0), \tilde{f}(w^1)) \leqslant k_{S_{n_1}}(w^0, w^1) &= k_{\mathbb{H}^{n_1}}(E(w^0), E(w^1)) \\ &= k_{\mathbb{H}^1}(i, \lambda_1^2 i) = \ln\lambda_1 = -\pi^2/\ln r_1, \end{aligned}$$

where the second equality is due to the fact that $E(w^0), E(w^1) \in \mathbb{H}^1 \times \{0\}$ which is a holomorphic retract of $\mathbb{H}^n$. Since

$$k_{S_{n_2}}(\tilde{f}(w^0), \tilde{f}(w^1)) = k_{S_{n_2}}(\tilde{f}(w^0), \tilde{f}(v_1(w^0))) = k_{S_{n_2}}(\tilde{f}(w^0), v_2^{d(f)}(\tilde{f}(w^0))),$$

considering the projection on the first component and denoting $f_1(w^0)$ by c, the above inequality and the contracting property of the Kobayashi distance yield

$$k_{S_1}(c, c + 2d(f)\ln\lambda_2) \leqslant k_{S_{n_2}}(\tilde{f}(w^0), v_2^{d(f)}(\tilde{f}(w^1))) \leqslant -\pi^2/\ln r_1.$$

Via the biholomorphism $E : S_1 \to \mathbb{H}^1$, we can evaluate $k_{S_1}(c, c + 2d(f)\ln\lambda_2)$ obtaining

$$k_{S_1}(c, c + 2d(f)\ln\lambda_2) = k_{\mathbb{H}^1}(e^c, \lambda_2^{2d(f)} e^c) \geqslant -\pi^2 |d(f)|/\ln r_2;$$

as $r_2 < 1$, this implies $|d(f)| \leqslant \ln r_2/\ln r_1$. □

This means that the ratio of the logarithms of "inner radii" (that is, the generalization of the ratio of the moduli) bounds the degree of holomorphic maps between generalized annuli. The following proposition gives an even deeper interest to the above theorem, since it tells us that the degree is a complete homotopy invariant.

Proposition 3.3. *Let $f, g \in \mathscr{H}$. Then $d(f) = d(g)$ if and only if $[f] = [g]$, that is if and only if f and g are homotopic.*

Proof. The "if" part is obvious. In order to prove the converse implication, we write $f = (f_1, \dots, f_{n_2})$ and $g = (g_1, \dots, g_{n_2})$; using the retraction by deformation of Ω_2 onto $\Omega(r_2) \times \{0\}$ given by $\rho(t, \xi) = (\xi_1, t\xi_2, \dots, t\xi_n)$, it is easily seen that we can limit ourselves to the case $n_2 = 1$.

Setting $d = d(f) = d(g)$, we then have

$$\begin{aligned} &\frac{1}{2\pi i}\int_{\sqrt{r_1}S^1} \frac{\partial f}{\partial \xi_1}(\xi_1, 0, \dots, 0)/f(\xi_1, 0, \dots, 0)\, d\xi_1 \\ &\qquad = \frac{1}{2\pi i}\int_{\sqrt{r_1}S^1} \frac{\partial g}{\partial \xi_1}(\xi_1, 0, \dots, 0)/g(\xi_1, 0, \dots, 0)\, d\xi_1 = d. \end{aligned}$$

Let $\varphi, \psi : \Omega_1 \to \mathbb{C}^*$ be given by $\varphi(\xi) = \xi_1^{-d} f(\xi)$ and $\psi(\xi) = \xi_1^{-d} g(\xi)$; it is easily verified that

$$\frac{1}{2\pi i}\int_{\sqrt{r_1}S^1}\frac{\partial\varphi}{\partial\xi_1}(\xi_1,0,\dots,0)/\varphi(\xi_1,0,\dots,0)\,d\xi_1$$
$$=\frac{1}{2\pi i}\int_{\sqrt{r_1}S^1}\frac{\partial\psi}{\partial\xi_1}(\xi_1,0,\dots,0)/\psi(\xi_1,0,\dots,0)\,d\xi_1=0.$$

Consider the maps $\varphi_*,\psi_*:\pi_1(\Omega_1)\to\pi_1(\mathbb{C}^*)$; the above equality implies that both these maps are trivial and therefore φ,ψ can be lifted to continuous maps $\tilde\varphi,\tilde\psi:\Omega_1\to\mathbb{C}$ such that $\exp\circ\tilde\varphi=\varphi$ and $\exp\circ\tilde\psi=\psi$. Since $\exp:\mathbb{C}\to\mathbb{C}^*$ is a local biholomorphism and both φ and ψ are holomorphic, the two maps $\tilde\varphi$ and $\tilde\psi$ are holomorphic. Then we have found holomorphic maps $\tilde\varphi,\tilde\psi:\Omega_1\to\mathbb{C}$ such that $f(\xi)=\xi_1^d e^{\tilde\varphi(\xi)}$ and $g(\xi)=\xi_1^d e^{\tilde\psi(\xi)}$ for all $\xi\in\Omega_1$. Now set

$$H(t,\xi)=\xi_1^d e^{\tilde\varphi(\xi)+t(\tilde\psi(\xi)-\tilde\varphi(\xi))};$$

obviously H is continuous, and the map $H(t,\cdot)$ is holomorphic for all $t\in[0,1]$; moreover $H(0,\cdot)=f$ and $H(1,\cdot)=g$, so we are left to verify that $H([0,1]\times\Omega_1)\subset\Omega(r_2)$. Fix $\xi\in\Omega_1$ and consider the map

$$\beta:[0,1]\ni t\mapsto|H(t,\xi)|=|\xi_1^d|e^{\Re\tilde\varphi(\xi)+t\Re(\tilde\psi(\xi)-\tilde\varphi(\xi))}\in\mathbb{R};$$

the map β is monotonic and we have $r_2<|f(\xi)|=\beta(0)<1$, $r_2<|g(\xi)|=\beta(1)<1$. Thus $r_2<|H(t,\xi)|=\beta(t)<1$ for all $t\in[0,1]$; so $H([0,1]\times\Omega_1)\subset\Omega(r_2)$ and this concludes the proof of the assertion, since H is the required homotopy between the maps f and g. □

As a consequence of the proof of Proposition 3.3 we obtain the following

Corollary 3.4. *Let $f\in\mathscr{H}$, then there exists a holomorphic map $u:\Omega_1\to\mathbb{C}$ such that $f_1(\xi)=\xi_1^{d(f)}e^{u(\xi)}$ for all $\xi\in\Omega_1$.*

Moreover, gathering Theorem 3.2 and Proposition 3.3, we obtain that, if the "hole" in Ω_1 is smaller than the "hole" in Ω_2, any holomorphic map from Ω_1 to Ω_2 is homotopic to a constant map.

Corollary 3.5. *Let $f\in\mathscr{H}$; if $r_1<r_2$ then f is homotopic to a constant.*

Now we turn to the study of the homotopy classes $\mathscr{H}_d=\{f\in\mathscr{H}\mid d(f)=d\}$. If $|d|<\ln r_2/\ln r_1$ the following remark and proposition ensure that the family $\mathscr{H}_d$ is very "ample".

Remark 3.6. For all $f\in\mathscr{H}_0$, there exists a holomorphic map $\tilde f:\Omega_1\to\mathbb{H}^{n_2}$ such that $\mathrm{pr}_2\circ\tilde f=f$. Vice versa, for all holomorphic map $\tilde f:\Omega_1\to\mathbb{H}^{n_2}$, the map $\mathrm{pr}_2\circ\tilde f$ belongs to $\mathscr{H}_0$.

Proof. Since $\mathbb{H}^{n_1}$ is simply connected, the existence of a continuous $\tilde f:\Omega_1\to\mathbb{H}^{n_2}$

such that $\mathrm{pr}_2 \circ \tilde{f} = f$ is equivalent to the triviality of the map f_* (that is to $d(f) = 0$). As pr_2 is a local biholomorphism, the assertion follows. $\square$

This simple remark classifies all elements of $\mathscr{H}_0$; moreover the boundedness of Ω_1 ensures that there exists a huge family of holomorphic maps from Ω_1 to $\mathbb{H}^{n_2}$; in this sense we can say that $\mathscr{H}_0$ is "big".

Now let us consider the case when $0 < |d| < \ln r_2/\ln r_1$. First of all, if $n_2 \geqslant 2$ we set $L = \max\{|\vartheta_j|/\pi \mid j = 2, \dots, n_2\}$. If $0 < d < \ln r_2/\ln r_1$, choose $C \in \mathbb{R}$ such that $r_2 < Cr_1^d < r_1^d < C$; if $0 < -d < \ln r_2/\ln r_1$, choose $C \in \mathbb{R}$ such that $r_2 < Cr_2 < r_2 r_1^d < Cr_2 r_1^d < 1$ and set

$$\delta = \begin{cases} \min\{(1-C)/2, (Cr_1^d - r_2)/2\} & \text{if } d > 0, \\ \min\{(Cr_2 - r_2)/2, (1 - Cr_1^d r_2)/2\} & \text{if } d < 0, \end{cases}$$

$$K = \begin{cases} \min\left\{\sin\left(\frac{\pi\ln((1+C)/2)}{\ln r_2}\right), \sin\left(\frac{\pi\ln((Cr_1^d + r_2)/2)}{\ln r_2}\right)\right\} & \text{if } d > 0, \\ \min\left\{\sin\left(\frac{\pi\ln((Cr_2+r_2)/2)}{\ln r_2}\right), \sin\left(\frac{\pi\ln((1+Cr_1^d r_2)/2)}{\ln r_2}\right)\right\} & \text{if } d < 0. \end{cases}$$

Then we can describe an ample set of the elements belonging to $\mathscr{H}_d$:

Proposition 3.7. *Let $d \in \mathbb{Z}$ be such that $0 < |d| < \ln r_2/\ln r_1$ and C, δ, L, K be as above. For any $\theta \in \mathbb{R}$, any holomorphic function $\sigma : \Omega_1 \to \mathbb{C}$ such that $\|\sigma\|_\infty \leqslant \delta$ and any holomorphic map $h : \Omega_1 \to \mathbb{C}^{n_2 - 1}$ such that $\sum_{j=1}^{n_2-1} |h_j(\xi)|^2 < r_2^L K$ for all $\xi \in \Omega_1$, the map*

$$f : \Omega_1 \ni \xi \mapsto \begin{cases} (Ce^{i\theta}\xi_1^d + \sigma(\xi), h(\xi)) \in \Omega_2, & \text{if } d > 0, \\ (Cr_2 e^{i\theta}\xi_1^d + \sigma(\xi), h(\xi)) \in \Omega_2, & \text{if } d < 0, \end{cases}$$

belongs to $\mathscr{H}_d$.

Proof. We shall perform the proof in the case $d > 0$, the case $d < 0$ can be obtained from this by minor changes. First of all, let us prove that the map f belongs to $\mathscr{H}$. Since it is obvious that f is a holomorphic map, we are left to prove that it maps Ω_1 into Ω_2. Let $\xi \in \Omega_1$; the choice of C and δ implies that the first component of f satisfies the following inequality

$$r_2 < \frac{Cr_1^d + r_2}{2} < |f_1(\xi)| < \frac{C+1}{2} < 1, \tag{3.2}$$

which yields

$$0 < \frac{\pi\ln((C+1)/2)}{\ln r_2} < \frac{\pi\ln|f_1(\xi)|}{\ln r_2} < \frac{\pi\ln((Cr_1^d + r_2)/2)}{\ln r_2} < \pi.$$

The map $\sin : [0, \pi] \to \mathbb{R}$ is concave and therefore the choice of K entails

$$K \leqslant \sin\left(\frac{\pi \ln|f_1(\xi)|}{\ln r_2}\right) \tag{3.3}$$

for all $\xi \in \Omega_1$. By (3.2) and (3.3) we then have

$$\sum_{j=2}^{n_2} |f_j(\xi)|^2 |f_1(\xi)|^{\vartheta_j/\pi} \leqslant r_2^{-L} \sum_{j=1}^{n_2-1} |h_j(\xi)|^2 \leqslant K < \sin\left(\frac{\pi \ln|f_1(\xi)|}{\ln r_2}\right)$$

and therefore f maps Ω_1 to Ω_2. At last, we prove that $d(f) = d$, that is $f \in \mathscr{H}_d$. Let $H : [0,1] \times \Omega_1 \to \mathbb{C}^{n_2}$ be given by $H(t,\xi) = (Ce^{i\theta}\xi_1^d + t\sigma(\xi), th(\xi))$. The map H is of course continuous and by the above reasoning we have $H([0,1] \times \Omega_1) \subset \Omega_2$; moreover $H(1,\cdot) = f$ and $H(0,\xi) = (Ce^{i\theta}\xi_1^d, 0, \dots, 0)$. Since the degree of $H(0,\cdot)$ is equal to d, we are done. □

Now we consider the case when $|d(f)| = \ln r_2/\ln r_1$ which of course can occur iff $\ln r_2/\ln r_1 \in \mathbb{N}$. The following results describe the maps $f \in \mathscr{H}_d$ when $|d| = \ln r_2/\ln r_1$: the restriction of f to $\Omega(r_1) \times \{0\}$ has to follow a prescribed pattern, while outside this annulus the behaviour of f can be quite "free".

Theorem 3.8. *Let $f \in \mathscr{H}_d$ where $|d| = \ln r_2/\ln r_1$. Then there exist $\theta \in \mathbb{R}$ and $p \in \mathrm{Hol}(\Omega_1, \mathbb{C})$ such that $p|_{\Omega(r_1)\times\{0\}} \equiv 0$, $\frac{\partial p}{\partial \xi_j}\big|_{\Omega(r_1)\times\{0\}} \equiv 0$ for all $j = 2, \dots, n_1$ and*

$$f_1(\xi) = \begin{cases} e^{i\theta}\xi_1^d e^{p(\xi)} & \text{if } d > 0, \\ e^{i\theta} r_2 \xi_1^d e^{p(\xi)} & \text{if } d < 0. \end{cases} \tag{3.4}$$

Proof. Since we are interested in the behaviour of f_1 only and Ω_2 retracts by deformation onto $\Omega(r_2) \times \{0\}$, it is enough to consider the case $n_2 = 1$. Moreover we shall perform the proof only in the case $d > 0$, the case $d < 0$ being obtained from this by a few minor changes.

By Corollary 3.4 we can find a holomorphic map $u : \Omega_1 \to \mathbb{C}$ such that

$$f(\xi) = \xi_1^d e^{u(\xi)}. \tag{3.5}$$

Let us consider the map $\tau : \Omega(r_1) \ni \xi_1 \mapsto f(\xi_1, 0, \dots, 0) \in \Omega(r_2)$; since $\Omega(r_1)$ is a deformation retract of Ω_1 the degree of τ is equal to d. As $S_1 = \mathbb{R} \times (0,\pi)$ is simply connected and $q_2 : S_1 \to \Omega(r_2)$ is a local biholomorphism, there exists a holomorphic lifting $\tilde{\tau}$ of $\tau \circ q_1$, that is a holomorphic map $\tilde{\tau} : S_1 \to S_1$ such that $q_2 \circ \tilde{\tau} = \tau \circ q_1$. Interpreting the degree of τ via the action of the group of deck-transformations yields $\tilde{\tau} \circ v_1 = v_2^d \circ \tilde{\tau}$, that is

$$\tilde{\tau}(z + 2\ln\lambda_1) = \tilde{\tau}(z) + 2d\ln\lambda_2$$

for all $z \in S_1$. Taking $z = i\pi/2$ and using the equality $d\ln\lambda_2 = \ln\lambda_1$ we obtain

$$\tilde{\tau}\left(\frac{i\pi}{2} + 2\ln\lambda_1\right) = \tilde{\tau}\left(\frac{i\pi}{2}\right) + 2\ln\lambda_1. \tag{3.6}$$

Transferring the problem on $\mathbb{H}^1$, we obtain as a consequence of the Schwarz lemma that $\tilde{\tau}$ is an automorphism of S_1 of the form $\tilde{\tau} : z \mapsto z + x$ for a suitable $x \in \mathbb{R}$.

In fact, let $\sigma = E \circ \tilde{\tau} \circ E^{-1} : \mathbb{H}^1 \to \mathbb{H}^1$ and set $\tilde{\tau}(i\pi/2) = x + iy$. Since $\sigma(i) = \exp(x + iy)$ and $\sigma(i\lambda_1^2) = \lambda_1^2 \exp(x + iy)$, by the contracting property of the Kobayashi distance we obtain

$$k_{\mathbb{H}^1}(\sigma(i), \sigma(i\lambda_1^2)) = \tanh^{-1}\left(\frac{\lambda_1^2 - 1}{\sqrt{\lambda_1^4 + 1 - 2\lambda_1^2 \cos 2y}}\right) \leqslant k_{\mathbb{H}^1}(i, i\lambda_1^2) = \ln \lambda_1.$$

Now the fact that tanh is increasing yield

$$\frac{\lambda_1^2 - 1}{\sqrt{\lambda_1^4 + 1 - 2\lambda_1^2 \cos 2y}} \leqslant \frac{\lambda_1^2 - 1}{\lambda_1^2 + 1}$$

and hence $\cos 2y = -1$, that is $y = \pi/2$. Moreover the equality at one point of the Kobayashi distance yields, by the Schwarz lemma, that σ is an automorphism of $\mathbb{H}^1$; since $\sigma(i) = ie^x$ and $\sigma(i\lambda_1^2) = ie^x\lambda_1^2$, we have $\sigma(w) = e^x w$ for all $w \in \mathbb{H}^1$, that is $\tilde{\tau}(z) = z + x$ for a suitable $x \in \mathbb{R}$. Then there exists $\theta \in \mathbb{R}$ such that $\tau(\xi) = e^{i\theta}\xi^d$ for all $\xi \in \Omega(r_1)$ and we have proved that $f(\xi_1, 0, \dots, 0) = e^{i\theta}\xi_1^d$ for all $\xi_1 \in \Omega(r_1)$. Comparing this equality with (3.5) we infer that u is a holomorphic map which takes values in $i\theta + 2\pi i\mathbb{Z}$ on $\Omega(r_1) \times \{0\}$ and hence, setting $p(\xi) = u(\xi) - u(\sqrt{r_1}, 0, \dots, 0)$, we obtain that $p(\xi_1, 0, \dots, 0) = 0$ for all $\xi_1 \in \Omega(r_1)$.

In order to prove that $\frac{\partial p}{\partial \xi_j}\big|_{\Omega(r_1)\times\{0\}} \equiv 0$ for all $j = 2, \dots, n_1$, it is enough to consider the map $\Omega(r_1, \theta_j) \ni (\xi_1, \xi_j) \mapsto f(\xi_1, 0, \dots, 0, \xi_j, 0, \dots, 0) \in \Omega(r_2)$ and hence we can limit ourselves to the case $n_1 = 2$.

Since for any $\xi_1 \in \Omega(r_1)$ and any $R < r_1^{|\theta_2|/2\pi}(\sin(\pi \ln|\xi_1|/\ln r_1))^{1/2}$ the set $\{\xi_1\} \times \Delta_R$ is contained in Ω_1, we can find holomorphic maps $a_1 : \Omega(r_1) \to \mathbb{C}$ and $a_2 : \Omega_1 \to \mathbb{C}$ such that $p(\xi) = a_1(\xi_1)\xi_2 + a_2(\xi)\xi_2^2$. Our last assumption is therefore equivalent to $a_1 \equiv 0$. As f maps Ω_1 in $\Omega(r_2)$, using the form of f we have that for all $\xi \in \Omega_1$ the following inequality holds:

$$\ln r_2 - d \ln|\xi_1| < \Re p(\xi) < -d \ln|\xi_1|. \tag{3.7}$$

Fix $\xi_1^0 \in \Omega(r_1)$ and for any $0 < \varepsilon < \sin(\pi \ln|\xi_1^0|/\ln r_1)$ denote by R the number $r_1^{|\theta_2|/2\pi}(\sin(\pi \ln|\xi_1^0|/\ln r_1) - \varepsilon))^{1/2}$. Then for all $\xi_2 \in \overline{\Delta_R}$ the point $\xi = (\xi_1^0, \xi_2)$ belongs to Ω_1. Now let us set $\rho = R/2$; then we have

$$\rho|a_1(\xi_1^0)| \leqslant \rho \max_{|\xi_2| \leqslant \rho} |a_1(\xi_1^0) + a_2(\xi)\xi_2| = \rho \max_{|\xi_2| = \rho} |a_1(\xi_1^0) + a_2(\xi)\xi_2| = \max_{|\xi_2| = \rho} |p(\xi)|.$$

The Borel–Carathéodory theorem and (3.7) imply that

$$\max_{|\xi_2| = \rho} |p(\xi)| \leqslant \frac{R}{R - \rho} \max_{|\xi_2| = \rho} \Re p(\xi) = 2 \max_{|\xi_2| = \rho} \Re p(\xi) \leqslant -2d \ln|\xi_1^0|,$$

and therefore $|a_1(\xi_1^0)| \leqslant -4r_1^{-|\theta_2|/2\pi} d\ln|\xi_1^0|(\sin(\pi\ln|\xi_1^0|/\ln r_1) - \varepsilon)^{-1/2}$; the arbitrariness of ε yields that for all $\xi_1^0 \in \Omega(r_1)$ the following inequality holds

$$|a_1(\xi_1^0)| \leqslant \frac{-4r_1^{-|\theta_2|/2\pi} d\ln|\xi_1^0|}{\sqrt{\sin(\pi\ln|\xi_1^0|/\ln r_1)}}. \tag{3.8}$$

It is easily seen that the right hand term of (3.8) goes to 0 when $|\xi_1^0| \to 1^-$, yielding $\lim_{|\xi_1^0|\to 1^-}|a_1(\xi_1^0)| = 0$. The same reasoning can be performed for $|\xi_1^0| \to r_1^+$; also in this case we obtain that $\lim_{|\xi_1^0|\to r_1^+}|a_1(\xi_1^0)| = 0$. Then we can invoke the maximum modulus principle and we are done. □

Now we consider the behaviour of the last $n_2 - 1$ components of f on the annulus $\Omega(r_1) \times \{0\}$.

Proposition 3.9. *Let $f \in \mathscr{H}_d$ where $|d| = \ln r_2/\ln r_1$; then $f_j(\xi_1, 0, \ldots, 0) = 0$ for all $\xi_1 \in \Omega(r_1)$ and all $j = 2, \ldots, n_2$.*

Proof. First of all note that it is enough to prove the assertion for $n_1 = 1$ and $n_2 = 2$; in fact for any $j \in \{2, \ldots, n_2\}$ consider the map

$$g : \Omega(r_1) \ni \xi_1 \mapsto (f_1(\xi_1, 0, \ldots, 0), f_j(\xi_1, 0, \ldots, 0)) \in \Omega(r_2, \vartheta_j);$$

it is obvious that f and g have the same degree, thus we can limit ourselves to study the case $n_1 = 1$, $n_2 = 2$. Moreover, as in the previous theorem, we perform the proof for $d > 0$ only, leaving the proof of the case $d < 0$ to the reader.

Theorem 3.8 tells us that $|f_1(\xi)| = |\xi|^d$; since f maps $\Omega(r_1)$ into $\Omega(r_2, \vartheta_j)$ we have that for all $\xi \in \Omega(r_1)$ the following inequality holds

$$|f_2(\xi)|^2|\xi|^{d\vartheta_j/\pi} = |f_2(\xi)|^2|f_1(\xi)|^{\vartheta_j/\pi} < \sin(\pi\ln|f_1(\xi)|/\ln r_2) = \sin(\pi\ln|\xi|/\ln r_1),$$

that is

$$|f_2(\xi)|^2 < |\xi|^{-d\vartheta_j/\pi}\sin(\pi\ln|\xi|/\ln r_1) \tag{3.9}$$

for all $\xi \in \Omega(r_1)$. Taking the limit of the right hand side both for $|\xi| \to 1^-$ and $|\xi| \to r_1^+$ we obtain that

$$\lim_{|\xi|\to r_1^+} |f_2(\xi)|^2 = \lim_{|\xi|\to 1^-} |f_2(\xi)|^2 = 0;$$

the maximum modulus principle yields the conclusion. □

The big difference in $\mathscr{H}_d$ between the one-dimensional and the multi-dimensional case arises just when $|d| = \ln r_2/\ln r_1$. In the one-dimensional case $\mathscr{H}_d$ is a one-dimensional topological space which is isomorphic to S^1, while in the multidimensional case it contains an open set in an infinite-dimensional Fréchet space. In fact, provided $n_1 + n_2 \geqslant 3$, the following proposition gives a "large" family of maps belonging to

$\mathscr{H}_d$ where $d = \ln r_2/\ln r_1$ (and of course the same can be done, mutatis mutandis, for $d = -\ln r_2/\ln r_1$).

Since the domain $\Omega(r, \theta_2, \dots, \theta_{n_1})$ is biholomorphic to $\Omega(r, \theta_2 + 2\pi l, \dots, \theta_{n_1} + 2\pi l)$ for all $l \in \mathbb{Z}$ (see [4] for a proof), in order to simplify computations we can suppose that $\theta_j \leqslant 0$ for all $j = 2, \dots, n_1$ and $\vartheta_j \geqslant 0$ for all $j = 2, \dots, n_2$. Now set

$$K = \frac{-r_2 \ln r_2}{\pi} \quad \text{and} \quad K_1 = \frac{\ln(1 + K/(n_1 - 1))}{d} \quad \text{if } n_1 \geqslant 2.$$

For any $\theta \in \mathbb{R}$, $\sigma_{jk} \in \mathrm{Hol}(\Omega_1, \mathbb{C})$ for $j, k = 2, \dots, n_1$ and $h_{jk} \in \mathrm{Hol}(\Omega_1, \mathbb{C})$ for $j = 2, \dots, n_1$, $k = 2, \dots, n_2$, set $\tau(\xi) = \sum_{j,k=2}^{n_1} \xi_j \xi_k \sigma_{jk}(\xi)$.

Theorem 3.10. *If*

$$\sum_{j,k=2}^{n_1} |\sigma_{jk}(\xi)| \leqslant K_1 \quad \textit{and} \quad \sum_{j,k=2}^{n_1, n_2} |h_{jk}(\xi)|^2 \leqslant 1 + \frac{\pi(n_1 - 1)K_1}{\ln r_1} - \frac{\pi^2 K_1^2}{2 \ln^2 r_1}, \tag{3.10}$$

then the map

$$f : \Omega_1 \ni \xi \mapsto \left(e^{i\theta} \xi_1^d e^{d\tau(\xi)}, \sum_{j=2}^{n_1} \xi_j h_{j2}(\xi), \dots, \sum_{j=2}^{n_1} \xi_j h_{jn_2}(\xi) \right) \in \Omega_2$$

belongs to $\mathscr{H}_d$ where $d = \ln r_2/\ln r_1$.

Proof. By definition the map f is obviously holomorphic. Now we prove that f maps Ω_1 into Ω_2; after that a simple remark will show that the degree of f is equal to d and therefore f belongs to $\mathscr{H}_d$. First of all, we give two estimates of τ which will be useful in the sequel. By the definition of τ we have $|\tau(\xi)| \leqslant \sum_{j,k=2}^{n_1} |\xi_j \xi_k \sigma_{jk}(\xi)|$; since $\theta_k \leqslant 0$ for $k = 2, \dots, n_1$, then we obtain $|\xi_k| < 1$ for all $\xi \in \Omega_1$ and for $k = 2, \dots, n_1$, and therefore

$$|\tau(\xi)| \leqslant \sum_{j,k=2}^{n_1} |\sigma_{jk}(\xi)| \leqslant K_1 \tag{3.11}$$

for all $\xi \in \Omega_1$. Moreover for all $\xi \in \Omega_1$ the following bound on τ also holds

$$\begin{aligned} |\tau(\xi)| &\leqslant \left(\sum_{j,k=2}^{n_1} |\xi_j|\,|\xi_k| \right) \left(\sum_{j,k=2}^{n_1} |\sigma_{jk}(\xi)| \right) \leqslant K_1 \left(\sum_{j=2}^{n_1} |\xi_j| \right)^2 \\ &\leqslant (n_1 - 1) K_1 \sum_{j=2}^{n_1} |\xi_j|^2 \leqslant (n_1 - 1) K_1 \sum_{j=2}^{n_1} |\xi_j|^2 |\xi_1|^{\theta_j/\pi} \\ &\leqslant (n_1 - 1) K_1 \sin\left(\frac{\pi \ln|\xi_1|}{\ln r_1} \right). \end{aligned} \tag{3.12}$$

Let us notice that $f_1(\xi)$ can be written as $e^{i\theta} \xi_1^d + e^{i\theta} \xi_1^d q(\xi)$, where

$$q(\xi) = e^{d\tau(\xi)} - 1 = \sum_{l>0} \frac{(d\tau(\xi))^l}{l!}.$$

Now we estimate $q(\xi)$: by (3.11) and (3.12) we obtain

$$\begin{aligned}
|q(\xi)| &\leqslant \sum_{l>0} \frac{|d\tau(\xi)|^l}{l!} \leqslant d|\tau(\xi)| \left(\sum_{l>0} \frac{(dK_1)^{l-1}}{l!} \right) \\
&\leqslant d(n_1 - 1)K_1 \left(\sum_{l>0} \frac{(dK_1)^{l-1}}{l!} \right) \sin\left(\frac{\pi \ln|\xi_1|}{\ln r_1} \right) \\
&\leqslant (n_1 - 1)(e^{dK_1} - 1) \sin\left(\frac{\pi \ln|\xi_1|}{\ln r_1} \right) = K \sin\left(\frac{\pi \ln|\xi_1|}{\ln r_1} \right).
\end{aligned}$$

As for any $\xi \in \Omega_1$ we have $|\xi_1| \leqslant 1$, the following inequalities yield

$$|\xi_1|^d - K \sin\left(\frac{\pi \ln|\xi_1|}{\ln r_1} \right) \leqslant |f_1(\xi)| \leqslant |\xi_1|^d + K \sin\left(\frac{\pi \ln|\xi_1|}{\ln r_1} \right). \tag{3.13}$$

Now consider the functions $\Phi, \Psi : [r_1, 1] \to \mathbb{R}$ given by

$$\Phi(t) = t^d - K \sin\left(\frac{\pi \ln t}{\ln r_1} \right), \quad \Psi(t) = t^d + K \sin\left(\frac{\pi \ln t}{\ln r_1} \right);$$

it is easily seen that $\Phi(r_1) = \Psi(r_1) = r_2$, that $\Phi(1) = \Psi(1) = 1$ and that both of them are increasing (in fact their derivatives on $[r_1, 1]$ are always positive due to the choice of K). Then (3.13) implies that $r_2 < |f_1(\xi)| < 1$ for all $\xi \in \Omega_1$.

In order to prove that f maps Ω_1 to Ω_2, we have to check the second condition, namely that $\sum_{j=2}^{n_2} |f_j(\xi)|^2 |f_1(\xi)|^{\vartheta_j/\pi} < \sin\left(\frac{\pi \ln|f_1(\xi)|}{\ln r_2} \right)$ for all $\xi \in \Omega_1$. The definition of f_1 and the relation $d \ln r_1 = \ln r_2$ entail

$$\begin{aligned}
\sin\left(\frac{\pi \ln|f_1(\xi)|}{\ln r_2} \right) &= \sin\left(\frac{\pi \ln|\xi_1^d e^{d\tau(\xi)}|}{\ln r_2} \right) \\
&= \sin\left(\frac{\pi \ln|\xi_1|}{\ln r_1} \right) \cos\left(\frac{\pi \Re\tau(\xi)}{\ln r_1} \right) + \cos\left(\frac{\pi \ln|\xi_1|}{\ln r_1} \right) \sin\left(\frac{\pi \Re\tau(\xi)}{\ln r_1} \right) \\
&\geqslant \sin\left(\frac{\pi \ln|\xi_1|}{\ln r_1} \right) \left(1 - \frac{\pi^2 |\tau(\xi)|^2}{2 \ln^2 r_1} \right) + \frac{\pi |\tau(\xi)|}{\ln r_1};
\end{aligned}$$

by (3.11) and (3.12) we obtain that

$$\sin\left(\frac{\pi \ln|f_1(\xi)|}{\ln r_2} \right) \geqslant \sin\left(\frac{\pi \ln|\xi_1|}{\ln r_1} \right) \left(1 - \frac{\pi^2 K_1^2}{2 \ln^2 r_1} + \frac{\pi (n_1 - 1) K_1}{\ln r_1} \right). \tag{3.14}$$

Since ϑ_k is non-negative for $k = 2, \dots, n_2$, then for all $\xi \in \Omega_1$ we have $|f_1(\xi)|^{\vartheta_k/\pi} \leqslant 1$ for $k = 2, \dots, n_2$, therefore the second inequality in (3.10) yields that for all $\xi \in \Omega_1$ the following chain of inequalities holds

$$\begin{aligned}
\sum_{k=2}^{n_2} |f_k(\xi)|^2 |f_1(\xi)|^{\vartheta_k/\pi} &\leqslant \sum_{k=2}^{n_2} |f_k(\xi)|^2 = \sum_{k=2}^{n_2} \left| \sum_{j=2}^{n_1} \xi_j h_{jk}(\xi) \right|^2 \\
&\leqslant \sum_{k=2}^{n_2} \left(\sum_{j=2}^{n_1} |\xi_j|^2 \right) \left(\sum_{j=2}^{n_1} |h_{jk}(\xi)|^2 \right) \\
&\leqslant \left(\sum_{j=2}^{n_1} |\xi_j|^2 |\xi_1|^{\theta_j/\pi} \right) \left(\sum_{j,k=2}^{n_1,n_2} |h_{jk}(\xi)|^2 \right) \\
&< \sin\left(\frac{\pi \ln|\xi_1|}{\ln r_1} \right) \left(1 + \frac{\pi(n_1 - 1)K_1}{\ln r_1} - \frac{\pi^2 K_1^2}{2 \ln^2 r_1} \right); \qquad (3.15)
\end{aligned}$$

together with (3.14) this ensures that $f(\Omega_1)$ is contained in Ω_2.

Now consider the map

$$H : [0,1] \times \Omega_1 \ni \xi \mapsto \left(e^{i\theta} \xi_1^d e^{d\tau(\xi)}, t \sum_{j=2}^{n_1} \xi_j h_{j2}(\xi), \dots, t \sum_{j=2}^{n_1} \xi_j h_{jn_2}(\xi) \right) \in \mathbb{C}^{n_2};$$

since for any $t \in [0,1]$ the inequalities contained in (3.10) are both satisfied, we obtain that H maps $[0,1] \times \Omega_1$ into Ω_2. Moreover $H(t,\cdot)$ is holomorphic for any $t \in [0,1]$, and $H(1,\cdot) = f$, while it is easily seen that $H(0,\cdot)$ has degree d and therefore $\deg(f) = d$, which concludes the proof. □

Even if in the general case the bounds given by (3.10) can be non-optimal, there is at least one case in which they are optimal. If $\Omega_1 = \Omega(r_1, 0)$ and $\Omega_2 = \Omega(r_2, 0)$ then the following corollary holds.

Corollary 3.11. *For any $\theta \in \mathbb{R}$ and $h \in \mathrm{Hol}(\Omega_1, \bar{\Delta})$ the map*

$$f : \Omega_1 \ni \xi \mapsto (e^{i\theta} \xi_1^d, \xi_2 h(\xi)) \in \Omega_2$$

belongs to $\mathscr{H}_d$ where $d = \ln r_2 / \ln r_1$. Vice versa, for any map f in $\mathscr{H}_d$ of the form $f(\xi) = (f_1(\xi_1), f_2(\xi))$ there exist $\theta \in \mathbb{R}$ and $h \in \mathrm{Hol}(\Omega_1, \bar{\Delta})$ such that the equality $f(\xi) = (e^{i\theta} \xi_1^d, \xi_2 h(\xi))$ holds for all $\xi \in \Omega_1$.

Proof. Since the sufficiency of the condition can be obtained by direct computation as in the proof of the previous theorem, we are left to prove its necessity. By Theorem 3.8 there exists $\theta \in \mathbb{R}$ such that $f_1(\xi_1, 0) = e^{i\theta} \xi_1^d$; as f_1 does not depend on ξ_2 we have that $f_1(\xi) = e^{i\theta} \xi_1^d$ for any $\xi \in \Omega_1$. By Proposition 3.9 there exists a holomorphic function $h : \Omega_1 \to \mathbb{C}$ such that $f_2(\xi) = \xi_2 h(\xi)$. Since $f(\Omega_1) \subset \Omega_2$ we obtain that for all $\xi \in \Omega_1$

$$|\xi_2 h(\xi)|^2 < \sin\left(\frac{\pi \ln|e^{i\theta}\xi_1^d|}{\ln r_2}\right) = \sin\left(\frac{\pi \ln|\xi_1|}{\ln r_1}\right).$$

If $\xi^0 = (\xi_1^0, \xi_2^0) \in \Omega_1$ choose $\varepsilon > 0$ such that $|\xi_2^0|^2 \leqslant \sin(\pi \ln|\xi_1^0|/\ln r_1) - \varepsilon$ and set $R = (\sin(\pi \ln|\xi_1^0|/\ln r_1) - \varepsilon)^{1/2}$; then the following chain of inequalities holds

$$\begin{aligned} |h(\xi^0)|^2 &\leqslant \max_{|\xi_2| \leqslant R} |h(\xi_1^0, \xi_2)|^2 = \max_{|\xi_2| = R} |h(\xi_1^0, \xi_2)|^2 = R^{-2} \max_{|\xi_2| = R} |f_2(\xi_1^0, \xi_2)|^2 \\ &= \frac{\max_{|\xi_2|=R} |f_2(\xi_1^0, \xi_2)|^2}{\sin(\pi \ln|\xi_1^0|/\ln r_1) - \varepsilon} \leqslant \frac{\sin(\pi \ln|\xi_1^0|/\ln r_1)}{\sin(\pi \ln|\xi_1^0|/\ln r_1) - \varepsilon}. \end{aligned}$$

Letting ε go to 0 we obtain that $|h(\xi^0)| \leqslant 1$ and then we are done. □

4 Complex geodesics for generalized annuli

In this section we prove some results on complex geodesics in generalized annuli.

Definition 4.1. Given $\xi, \zeta \in D$ an *extremal map φ through ξ and ζ* is a holomorphic map $\varphi : \Delta \to D$ for which there exist $t, s \in \Delta$ such that $\varphi(t) = \xi$, $\varphi(s) = \zeta$ and $k_D(\xi, \zeta) = k_\Delta(t, s)$. A *complex geodesic* for the domain D is a holomorphic isometry $\varphi : \Delta \to D$ with respect to the Kobayashi distance of Δ and D (that is, a holomorphic map which is extremal through any point of its image).

Analogous definitions can be given replacing the Kobayashi distance with the Kobayashi metric: in this case we speak of an *infinitesimal extremal map* and of an *infinitesimal complex geodesic*. Recall that for any $\xi, \zeta \in \Omega$ and for any $z_0 \in \chi^{-1}(\xi)$, $w_0 \in \chi^{-1}(\zeta)$ we have

$$k_\Omega(\xi, \zeta) = \inf\{k_{\mathbb{B}^n}(z_0, \tilde{w}) : \tilde{w} \in \chi^{-1}(\zeta)\} = \inf\{k_{\mathbb{B}^n}(z_0, \gamma_0^j(w_0)) : j \in \mathbb{Z}\}; \quad (4.1)$$

this equality yields both the existence of extremal maps through any couple of points in generalized annuli and a characterization of complex geodesics which will be useful in order to solve some problems concerning existence of complex geodesics in Ω.

Remark 4.2. For any $\xi, \zeta \in \Omega$ there exists an extremal map through ξ and ζ.

Proof. Consider the covering $(\mathbb{B}^n \xrightarrow{\chi} \Omega)$: since $\mathbb{B}^n$ is complete hyperbolic we can choose $z, w \in \mathbb{B}^n$ such that $\chi(z) = \xi$, $\chi(w) = \zeta$ and $k_\Omega(\xi, \zeta) = k_{\mathbb{B}^n}(z, w)$. Let $\tilde{\varphi} : \Delta \to \mathbb{B}^n$ be a complex geodesic through z and w ($\tilde{\varphi}$ does exist since $\mathbb{B}^n$ is a strictly convex bounded domain in $\mathbb{C}^n$) and set $\varphi = \chi \circ \tilde{\varphi}$. Setting $t = \tilde{\varphi}^{-1}(z)$ and $s = \tilde{\varphi}^{-1}(w)$, it is easily seen that φ is an extremal map through ξ and ζ. □

Proposition 4.3. *Let $\tilde{\varphi} : \Delta \to \mathbb{B}^n$ be a complex geodesic in $\mathbb{B}^n$; then the holomorphic map $\varphi = \chi \circ \tilde{\varphi} : \Delta \to \Omega$ is a complex geodesic in Ω iff*

$$k_{\mathbb{B}^n}(\tilde{\varphi}(t), \tilde{\varphi}(s)) = \inf\{k_{\mathbb{B}^n}(\tilde{\varphi}(t), \gamma_0^j(\tilde{\varphi}(s))) : j \in \mathbb{Z}\} \tag{4.2}$$

for any $t, s \in \Delta$. Vice versa, if φ is a complex geodesic in Ω then any lifting $\tilde{\varphi}$ of φ to $\mathbb{B}^n$ is a complex geodesic in $\mathbb{B}^n$ for which (4.2) *holds for any $t, s \in \Delta$.*

Proof. If $\tilde{\varphi} : \Delta \to \mathbb{B}^n$ is a complex geodesic for which (4.2) holds for any $t, s \in \Delta$, then (4.1) and (4.2) imply that $k_\Delta(t, s) = k_\Omega(\chi \circ \tilde{\varphi}(t), \chi \circ \tilde{\varphi}(s))$ for all $t, s \in \Delta$ and therefore $\varphi = \chi \circ \tilde{\varphi}$ is a complex geodesic in Ω.

Vice versa, if $\varphi : \Delta \to \Omega$ is a complex geodesic, then $k_\Delta(t, s) = k_\Omega(\varphi(t), \varphi(s))$ holds for any $t, s \in \Delta$. Fix $s_0 \in \Delta$, choose $a_0 \in \mathbb{B}^n$ such that $\chi(a_0) = \varphi(s_0)$ and let $\tilde{\varphi} : \Delta \to \mathbb{B}^n$ be the lifting of φ through a_0, i.e. the unique holomorphic map from Δ to $\mathbb{B}^n$ such that $\varphi = \chi \circ \tilde{\varphi}$ and $\tilde{\varphi}(s_0) = a_0$. Since the Kobayashi distance is contracted by holomorphic maps, we then have

$$k_\Delta(t, s) = k_\Omega(\varphi(t), \varphi(s)) = k_\Omega(\chi \circ \tilde{\varphi}(t), \chi \circ \tilde{\varphi}(s)) \leqslant k_{\mathbb{B}^n}(\tilde{\varphi}(t), \tilde{\varphi}(s)) \leqslant k_\Delta(t, s)$$

and therefore equality holds at each $t, s \in \Delta$. Equation (4.1) implies that (4.2) holds for any $t, s \in \Delta$ and this concludes the proof. $\square$

It is well known that there exist no complex geodesics in the annuli $\Omega(r)$: this statement can be generalized to any couple of points belonging to $\Omega(r) \times \{0\} \subset \Omega$.

Proposition 4.4. *For any $\xi_1, \zeta_1 \in \Omega(r)$ with $\xi_1 \neq \zeta_1$ there are no complex geodesics in Ω through $\xi = (\xi_1, 0, \dots, 0)$ and $\zeta = (\zeta_1, 0, \dots, 0)$. For any $\xi_1 \in \Omega(r)$ and $v_1 \in \mathbb{C}$ there are no infinitesimal complex geodesics in Ω through $\xi = (\xi_1, 0, \dots, 0)$ with tangent vector $v = (v_1, 0, \dots, 0)$.*

Proof. We perform the proof in the case of complex geodesics, the case of infinitesimal complex geodesics is analogous and is left to the reader.

Suppose φ is a complex geodesic through ξ and ζ and let $\tilde{\varphi}$ be a lifting of φ to $\mathbb{B}^n$; by Proposition 4.3 the map $\tilde{\varphi}$ is a complex geodesic in $\mathbb{B}^n$. The form of χ implies that $\chi^{-1}(\Omega(r) \times \{0\}) = \Delta \times \{0\}$, and hence $\tilde{\varphi}(\Delta)$ intersects $\Delta \times \{0\}$ in two distinct points $z = (z_1, 0, \dots, 0) \in \chi^{-1}(\xi)$ and $w = (w_1, 0, \dots, 0) \in \chi^{-1}(\zeta)$. Since the image of a complex geodesic in $\mathbb{B}^n$ is an affine subset of $\mathbb{B}^n$, i.e. the intersection of $\mathbb{B}^n$ with an affine line, we have $\tilde{\varphi}(\Delta) = \Delta \times \{0\}$, and therefore $\varphi(\Delta) = \chi(\tilde{\varphi}(\Delta)) = \Omega(r) \times \{0\}$. As $\Omega(r) \times \{0\}$ is a holomorphic retract of Ω, we obtain that the map

$$\hat{\varphi} : \Delta \ni t \mapsto \varphi_1(t) \in \Omega(r)$$

is a complex geodesic in $\Omega(r)$ and this is a contradiction. $\square$

As we already noticed, complex geodesics in Ω are projections on Ω of complex geodesics in $\mathbb{B}^n$ for which (4.2) holds for any $t, s \in \Delta$. To simplify computations, which are very long in general, we will focus our attention on complex geodesics in Ω passing through the point $P_0 = (\sqrt{r}, 0, \dots, 0)$, i.e. complex geodesics in $\mathbb{B}^n$ passing through the origin. Up to holomorphic automorphisms of Δ we can therefore suppose that $\varphi(0) = 0$; then, since complex geodesics in $\mathbb{B}^n$ passing through the origin of $\mathbb{B}^n$ are given by maps of the form $\Delta \ni t \mapsto tp$ for any $p \in \partial\mathbb{B}^n$, we are led to investigate the following question, which is equivalent to the existence of a complex geodesic in Ω passing through the point P_0 and with tangent vector $d\chi_0(p)$ at P_0:

Does the equality

$$k_\Delta(s, t) = \inf\{k_{\mathbb{B}^n}(\gamma_0^j(sp), tp) : j \in \mathbb{Z}\} \tag{4.3}$$

hold for any $t, s \in \Delta$?

Denote by $\langle \cdot, \cdot \rangle$ the standard Hermitian product in $\mathbb{C}^n$ and for any $a \in \mathbb{B}^n \setminus \{0\}$ define $P_a, Q_a : \mathbb{C}^n \to \mathbb{C}^n$ and $s_a \in \mathbb{R}$ by

$$P_a(z) = \frac{\langle z, a \rangle}{\langle a, a \rangle} a, \quad Q_a(z) = z - P_a(z), \quad s_a = (1 - \|a\|^2)^{1/2}$$

and consider $\gamma_a : \mathbb{B}^n \to \mathbb{C}^n$ given by

$$\gamma_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle}.$$

Then γ_a is an involution in $\operatorname{Aut}\mathbb{B}^n$ which maps a to the origin and

$$1 - \|\gamma_a(z)\|^2 = (1 - \|a\|^2)(1 - \|z\|^2)|1 - \langle z, a \rangle|^{-2} \tag{4.4}$$

holds for any $z \in \mathbb{B}^n$ (for a proof see [1] p. 152–153).

Let γ_{tp} be the involution defined above which maps tp to the origin; since tanh is increasing, by developing computations and by (4.4) we obtain that (4.3) is equivalent to

$$\frac{1 - |s|^2}{|1 - \bar{t}s|^2} \geqslant \frac{1 - \|\gamma_0^j(sp)\|^2}{|1 - \langle tp, \gamma_0^j(sp) \rangle|^2} = \frac{1 - \|\gamma_0^j(sp)\|^2}{|1 - \langle \gamma_0^j(sp), tp \rangle|^2}$$

for any $t, s \in \Delta$ and any $j \in \mathbb{Z}$.

Setting $c_j = \cosh(jT)$, $s_j = \sinh(jT)$, $p' = (p_2, \dots, p_n)$, $W = \operatorname{diag}[e^{i\theta_2}, \dots, e^{i\theta_n}]$ and developing computations, we obtain the following question which is again equivalent to the existence of a complex geodesic in Ω passing through the point P_0 and with tangent vector $d\chi_0(p)$ at P_0:

Does

$$|1 - \bar{t}s|^2 \leqslant |c_j + s_j s p_1 - \langle (c_j s p_1 + s_j, W^j(sp')), tp \rangle|^2 \tag{4.5}$$

hold for any $t, s \in \Delta$ and any $j \in \mathbb{Z}$?

A first, very simple algebraic remark again stresses the fact that p' cannot be equal to zero. In fact, if $p' = 0$, taking $s = -\overline{p_1} \in \partial\Delta$ and $t = -s$, by continuity (4.5) implies $2 \leqslant 2(c_j - s_j)$ for all $j \in \mathbb{Z}$, which is impossible since $T > 0$.

Remark 4.5. Inequality (4.5) holds for any $t, s \in \Delta$ and any $j \in \mathbb{Z}$ if and only if it holds for any $t, s \in \partial\Delta$ and any $j \in \mathbb{Z}$.

Proof. If $j = 0$, it is obvious that (4.5) is an equality for any $t, s \in \bar{\Delta}$ and there is nothing to prove. Now suppose that (4.5) holds for any $t, s \in \partial\Delta$ and any $j \in \mathbb{Z}$. First of all we prove that $c_j + s_j s p_1 - \langle (c_j s p_1 + s_j, W^j(sp')), tp \rangle \neq 0$ for all $t, s \in \bar{\Delta}$ and for all $j \neq 0$. In fact, if $c_j + s_j s p_1 - \langle (c_j s p_1 + s_j, W^j(sp')), tp \rangle = 0$ for some $t, s \in \bar{\Delta}$ and some $j \in \mathbb{Z} \backslash \{0\}$, then we obtain that $\langle \gamma_0^j(sp), tp \rangle = 1$, and therefore $tp = \gamma_0^j(sp)$ and $|t| = \|\gamma_0^j(sp)\| = 1$ which implies $t, s \in \partial\Delta$. Then the fact that (4.5) holds for any $t, s \in \partial\Delta$ gives $1 - \bar{t}s = 0$, that is $t = s$. Since $j \neq 0$, the unique fixed points of γ_0^j are $\pm e_1$, and hence $sp = \pm e_1$, that is $p' = 0$, which is a contradiction to the previous remark. So, for any $j \in \mathbb{Z} \backslash \{0\}$, the holomorphic maps

$$h_j : \Delta \times \Delta \ni (t, s) \mapsto \frac{1 - ts}{c_j + s_j s p_1 - \langle (c_j s p_1 + s_j, W^j(sp')), \bar{t}p \rangle} \in \mathbb{C}$$

extend continuously to the boundary; if $|h_j(t, s)| \leqslant 1$ on the Shĭlov boundary of the bidisk, then $|h_j(t, s)| \leqslant 1$ for any $t, s \in \Delta$ and any $j \in \mathbb{Z} \backslash \{0\}$; this implies (4.5) for any $t, s \in \Delta$ and any $j \in \mathbb{Z}$. The other implication is trivial by continuity. □

Then we are led to investigate on the following question: for which $p \in \partial\mathbb{B}^n$ does

$$|1 - \bar{t}s|^2 \leqslant |c_j + s_j s p_1 - \langle (c_j s p_1 + s_j, W^j(sp')), tp \rangle|^2 \tag{4.6}$$

hold for any $t, s \in \partial\Delta$ and any $j \in \mathbb{Z}$?

To simplify notation, we denote by q_j the quantity $\langle p', W^j p' \rangle$ and obtain

$$|1 - \bar{t}s|^2 \leqslant |c_j - \bar{t}s(c_j |p_1|^2 + q_j) + s_j(sp_1 - \overline{tp_1})|^2;$$

setting $\zeta = \bar{t}s$ (which belongs to $\partial\Delta$ if both t and s do) we get

$$|1 - \zeta|^2 \leqslant |c_j + s_j s p_1 - \zeta(c_j |p_1|^2 + q_j + s_j \overline{sp_1})|^2$$

for all $\zeta, s \in \partial\Delta$ and $j \in \mathbb{Z}$. A simple computation proves that the above inequality holds for any $\zeta, s \in \partial\Delta$ and $j \in \mathbb{Z}$ if and only if

$$2 + 2|(c_j |p_1|^2 + q_j + s_j \overline{sp_1})(c_j + s_j \overline{sp_1}) - 1| \leqslant |c_j + s_j s p_1|^2 + |c_j |p_1|^2 + q_j + s_j \overline{sp_1}|^2$$

for all $s \in \partial\Delta$ and $j \in \mathbb{Z}$. Setting $G_j = c_j |p_1|^2 + q_j$ we get

$$2 + 2|c_j G_j - 1 + s_j(c_j + G_j)\overline{sp_1} + s_j^2 (\overline{sp_1})^2| \leqslant |c_j + s_j s p_1|^2 + |G_j + s_j \overline{sp_1}|^2$$

for all $s \in \partial\Delta$ and $j \in \mathbb{Z}$. Setting $s = e^{-i(x+\arg p_1)}$ for $x \in \mathbb{R}$ and developing computations we get

$$2 + 2|c_j G_j - 1 + s_j(c_j + G_j)|p_1|e^{ix} + s_j^2|p_1|^2 e^{2ix}| \leqslant c_j^2 + 2s_j^2|p_1|^2 + |G_j|^2 + 2s_j|p_1|\Re(e^{-ix}(c_j + G_j)) \tag{4.7}$$

for all $x \in \mathbb{R}$ and $j \in \mathbb{Z}$.

To simplify computations, which are very heavy in the general case, we focus our attention on two cases: when $p_1 = 0$ and when $W = I_{n-1}$ (in this last case $q_j = \langle p', W^j p' \rangle = 1 - |p_1|^2$ is a real positive number which does not depend on j).

Case $\boldsymbol{p_1 = 0}$. In this Case (4.7) becomes

$$2 + 2|c_j q_j - 1| \leqslant c_j^2 + |q_j|^2 \tag{4.8}$$

for all $j \in \mathbb{Z}$ (and it does not depend on x any more). The next two remarks show that if the radius r is large enough there always exist "vertical" complex geodesics, while if it is small in some cases there exist no "vertical" complex geodesics.

Remark 4.6. If $r \geqslant \exp(\pi^2/\ln(3 - \sqrt{8}))$, then for any $\theta_2, \ldots, \theta_n \in \mathbb{R}$ and any $p' \in \partial\mathbb{B}_{n-1}$, there exists a complex geodesic in $\Omega(r, \theta_2, \ldots, \theta_n)$ passing through P_0 with tangent vector $d\chi_0((0, p'))$ in P_0.

Proof. First of all notice that the relation between r and T entails $c_1 \geqslant 3$. The above reasoning implies that a complex geodesic through P_0 with tangent vector $d\chi_0((0, p'))$ in P_0 exists iff (4.8) is satisfied for all $j \in \mathbb{Z}$. This inequality is surely satisfied if

$$2 + 2(c_j|q_j| + 1) \leqslant c_j^2 + |q_j|^2$$

for any $j \in \mathbb{Z}\backslash\{0\}$; developing computations we obtain $4 \leqslant (c_j - |q_j|)^2$, that is $2 \leqslant c_j - |q_j|$. Then $3 \leqslant c_1 \leqslant c_j$ for any $j \in \mathbb{Z}\backslash\{0\}$ and $|q_j| \leqslant \|p'\|^2 \leqslant 1$ yield the conclusion. $\square$

Remark 4.7. If $r < \exp(\pi^2/\ln(3 - \sqrt{8}))$, then for any $p' \in \partial\mathbb{B}_{n-1}$, there exists no complex geodesic in $\Omega(r, \pi, \ldots, \pi)$ passing through P_0 with tangent vector $d\chi_0((0, p'))$ in P_0.

Proof. As above the relation between r and T entails $c_1 < 3$; moreover $W = -I_{n-1}$ and hence $q_1 = \langle p', Wp' \rangle = -\|p'\|^2 = -1$. A complex geodesic through P_0 with tangent vector $d\chi_0((0, p'))$ in P_0 exists iff (4.8) is satisfied for all $j \in \mathbb{Z}$; in particular for $j = 1$ it becomes $c_1^2 - 2c_1 - 3 \geqslant 0$. As $c_1 < 3$, the last inequality is not satisfied and this yields the conclusion. $\square$

At last we bring forward a result which concerns "vertical" geodesics in the case when $W = I_{n-1}$. In this case $q_j = \langle p', W^j p' \rangle = 1$ and hence we have the following

Remark 4.8. For any $r \in (0,1)$ and $p' \in \partial\mathbb{B}_{n-1}$, there exists a complex geodesic in $\Omega(r,0,\dots,0)$ passing through P_0 with tangent vector $d\chi_0((0,p'))$ in P_0.

Proof. In this Case (4.8) becomes $2 + 2(c_j - 1) \leqslant c_j^2 + 1$, that is $2c_j \leqslant c_j^2 + 1$ which is obviously satisfied for any $j \in \mathbb{Z}$. □

This remark can also be seen as a consequence of the fact that $\Omega(r,0,\dots,0)$ retracts holomorphically (but not by deformation) on $\{\xi \in \Omega(r,0,\dots,0) \mid \xi_1 = \sqrt{r}\}$ which is biholomorphic to the unit ball in $\mathbb{C}^{n-1}$ and this ensures the existence of "vertical" complex geodesics through P_0.

Case $W = I_{n-1}$. As we already noticed, in this case $q_j = \langle p', W^j p'\rangle = 1 - |p_1|^2$ is a real positive number which does not depend on j and therefore $G_j = c_j|p_1|^2 + 1 - |p_1|^2$ is also a real positive number. Using this property and setting

$$A_j = c_j G_j - 1, \quad B_j = s_j|p_1|(c_j + G_j), \quad C_j = s_j^2|p_1|^2, \quad D_j = c_j^2 + 2s_j^2|p_1|^2 + G_j^2,$$

(4.7) becomes

$$2|A_j e^{-ix} + B_j + C_j e^{ix}| \leqslant D_j + 2B_j \cos x - 2 \tag{4.9}$$

for all $x \in \mathbb{R}$ and $j \in \mathbb{Z}$. Developing computations, setting $L_j = 4(B_j^2 - 4A_jC_j)$, $M_j = 4B_j(D_j - 2 - 2A_j - 2C_j)$, $N_j = (D_j - 2)^2 - 4B_j^2 - 4(C_j - A_j)^2$ and $t = \cos x$, we obtain that the above inequality is equivalent to the system of equations

$$D_j - 2|B_j| - 2 \geqslant 0, \quad L_j t^2 + M_j t + N_j \geqslant 0 \tag{4.10}$$

for all $t \in [-1,1]$ and for all $j \in \mathbb{Z}$. A simple though long computation gives $L_j = 4|p_1|^2 s_j^2((c_j - 1)^2(1 - |p_1|^2)^2 + 4) \geqslant 0$ for all $j \in \mathbb{Z}$; then (4.10) splits into two parts (according to whether the vertex of the parabola $t \mapsto L_j t^2 + M_j t + N_j$ belongs to the interval $[-1,1]$ or not) and we can state the following

Theorem 4.9. *Let $G_j, A_j, B_j, C_j, D_j, L_j, M_j, N_j$ be as above and let $p \in \partial\mathbb{B}_n$. There exists a complex geodesic in $\Omega(r,0,\dots,0)$ passing through P_0 with tangent vector $d\chi_0(p)$ in P_0 if and only if $D_j - 2|B_j| - 2 \geqslant 0$ holds for any $j \in \mathbb{Z}$ and for the $j \in \mathbb{Z}$ such that $|M_j| \leqslant 2L_j$ we have $M_j^2 \leqslant 4L_jN_j$ and for the $j \in \mathbb{Z}$ such that $|M_j| \geqslant 2L_j$ we have $|M_j| \leqslant L_j + N_j$.*

This characterization could seem useless since it involves very complicated inequalities but it can be easily handled by a program dealing with symbolic computation like Mathematica or Maple.

References

[1] M. Abate, *Iteration theory of holomorphic maps on taut manifolds*. Mediterranean Press, Rende 1989. MR 92i:32032 Zbl 0747.32002

[2] D. E. Barrett, Holomorphic equivalence and proper mapping of bounded Reinhardt domains not containing the origin. *Comment. Math. Helv.* **59** (1984), 550–564. MR 86j:32052 Zbl 0601.32030
[3] C. de Fabritiis, Commuting holomorphic functions and hyperbolic automorphisms. *Proc. Amer. Math. Soc.* **124** (1996), 3027–3037. MR 97b:32023 Zbl 0866.32004
[4] C. de Fabritiis, A family of complex manifolds covered by Δ_n. *Complex Variables Theory Appl.* **36** (1998), 233–252. MR 99k:32055 Zbl 01464104
[5] C. de Fabritiis, G. Gentili, On holomorphic maps which commute with hyperbolic automorphisms. *Adv. Math.* **144** (1999), 119–136. MR 2000c:32052 Zbl 0976.32014
[6] M. Jarnicki, P. Pflug, *Invariant distances and metrics in complex analysis.* de Gruyter 1993. MR 94k:32039 Zbl 0789.32001
[7] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$. Springer 1980. MR 82i:32002 Zbl 0495.32001
[8] M. Schiffer, On the modulus of doubly-connected domains. *Quart. J. Math., Oxford Ser.* **17** (1946), 197–213. MR 8,325b Zbl 0060.23707
[9] S. Shimizu, Automorphisms and equivalence of bounded Reinhardt domains not containing the origin. *Tohoku Math. J.* (2) **40** (1988), 119–152. MR 89d:32003 Zbl 0646.32003

Received 15 October, 2001

C. de Fabritiis, Dipartimento di Matematica "Vito Volterra", Università degli Studi di Ancona, Via Brecce Bianche, I-60131 Ancona, Italia
Email: fabritii@dipmat.unian.it