Adv. Geom. 3 (2003), 469–485

Advances in Geometry

Finite line-transitive linear spaces: parameters and normal point-partitions

Anne Delandtsheer, Alice C. Niemeyer and Cheryl E. Praeger*

(Communicated by T. Penttila)

Abstract. Until the 1990's the only known finite linear spaces admitting line-transitive, point-imprimitive groups of automorphisms were Desarguesian projective planes and two linear spaces with 91 points and line size 6. In 1992 a new family of 467 such spaces was constructed, all having 729 points and line size 8. These were shown to be the only linear spaces attaining an upper bound of Delandtsheer and Doyen on the number of points. Projective planes, and the linear spaces just mentioned on 91 or 729 points, are the only known examples of such spaces, and in all cases the line-transitive group has a non-trivial normal subgroup intransitive on points. The orbits of this normal subgroup form a partition of the point set called a normal point-partition. We give a systematic analysis of finite line-transitive linear spaces with normal point-partitions. As well as the usual parameters of linear spaces there are extra parameters connected with the normal point-partition that affect the structure of the linear space. Using this analysis we characterise the line-transitive linear spaces for which the values of various of these parameters are small. In particular we obtain a classification of all imprimitive line-transitive linear spaces that 'nearly attain' the Delandtsheer–Doyen upper bound.

Key words. Finite linear space, line-transitive automorphism group, imprimitive permutation group.

2000 Mathematics Subject Classification. 20C05

1 Introduction

A *finite linear space* $\mathscr{D} = (\mathscr{P}, \mathscr{L})$ consists of a finite set $\mathscr{P}$ of points, together with a set $\mathscr{L}$ of distinguished subsets of $\mathscr{P}$, called *lines*, such that any two points lie on exactly one line, and each line contains at least two points. The *automorphism group* $\operatorname{Aut}\mathscr{D}$ of $\mathscr{D}$ is the subgroup of all permutations of $\mathscr{P}$ which leave $\mathscr{L}$ invariant. We shall be concerned with finite linear spaces $\mathscr{D}$ for which $\operatorname{Aut}\mathscr{D}$ acts transitively on $\mathscr{L}$, that is, $\mathscr{D}$ is *line-transitive*. In particular, for such linear spaces, the lines have a con-

* This paper forms part of an Australian Research Council funded project of the second and third authors.

stant size, k say, and so the linear space is a 2-$(v, k, 1)$ block design, where $v = |\mathscr{P}|$. We assume throughout the paper that $\mathscr{D}$ has more than one line, that is $k < v$.

It is well-known that in a 2-$(v, k, 1)$ design line-transitivity forces point-transitivity and that the two transitivities imply each other in finite projective planes. Over the last 30 years, various sufficient conditions for point-primitivity of linear spaces have been studied, with the hope that "most" line-transitive linear spaces would prove to be point-primitive. Dembowski, in his book "Finite Geometries" [11], asked whether a line-primitive collineation group of a finite projective plane is necessarily point-primitive, and it took almost 20 years until this question was answered in the affirmative by Kantor [14], using the classification of primitive permutation groups of odd degree (and hence relying on the finite simple group classification). However the question of whether line-primitivity implies point-primitivity for finite linear spaces is still open. Similar problems have been raised in more general contexts, for example, for incidence structures whose incidence matrices have maximal rank (see [19]). Higman and McLaughlin [13] proved that (point, line)-flag transitivity also implies point-primitivity in a 2-$(v, k, 1)$ design.

However line-transitivity alone does not imply point-primitivity for 2-$(v, k, 1)$ designs, and in this paper we shall study *imprimitive pairs* $(\mathscr{D}, G)$, where $\mathscr{D}$ is a 2-$(v, k, 1)$ design admitting a line-transitive but point-imprimitive subgroup G of $\mathrm{Aut}\,\mathscr{D}$. Thus G leaves invariant a partition $\mathscr{C}$ of the point set with classes of size c, where $1 < c < v$, in the sense that for $g \in G$ and a class $C \in \mathscr{C}$ the image $C^g = \{\alpha^g \mid \alpha \in C\}$ is also a class of $\mathscr{C}$. Examples of imprimitive pairs are provided by the Desarguesian projective planes on a non-prime number of points, taking the group G to be a cyclic Singer group. The only other known imprimitive pairs involve two 2-$(91, 6, 1)$ designs found by Mills [16] and Colbourn & Colbourn [8], and 467 examples (up to isomorphism) which are 2-$(729, 8, 1)$ designs (see [17]). The line-transitive 2-$(91, 6, 1)$ designs were studied in [3]; the second one was named after McCalla by its discoverer Colbourn (Charlie Colbourn, private communication). In all these examples v and k are coprime, and we are tempted to conjecture that this may be true in general.

Delandtsheer and Doyen [10] proved that the number of imprimitive pairs is bounded above by a function of the line size k, by showing that $v \leqslant \left(\binom{k}{2} - 1\right)^2$. The 467 examples constructed in [17] were proved in [2] and [18] to be the only imprimitive pairs for which v attains this upper bound, and they were found in the course of investigating this extreme case. The methodology involved a detailed group theoretic analysis to identify the possibilities for $\mathrm{Aut}\,\mathscr{D}$, followed by a sophisticated computer search for the designs. The success of this classification demonstrated the importance of understanding the structure of line-transitive automorphism groups of such designs.

The aims of this paper are two-fold. Firstly we investigate some combinatorial consequences of having two structures on the point set left invariant by the automorphism group, namely the set of lines and the point-partition. Secondly we study the structure of the line-transitive, point-imprimitive automorphism group. This enables us to characterise line-transitive linear spaces for which various of these parameters are small, see Theorem 1.2. Throughout the paper we will assume that the following hypothesis and notation hold.

Hypothesis 1. $(\mathscr{D}, G)$ is an imprimitive pair, that is, $\mathscr{D} = (\mathscr{P}, \mathscr{L})$ is a 2-$(v, k, 1)$ design with point-set $\mathscr{P}$ and line-set $\mathscr{L}$, and $G \leqslant \mathrm{Aut}\,\mathscr{D}$ is transitive on $\mathscr{L}$ and is imprimitive on $\mathscr{P}$, leaving invariant a non-trivial point-partition $\mathscr{C}$, where $\mathscr{C}$ non-trivial means that both $d = |\mathscr{C}| > 1$ and the class size $c = |C| > 1$, for $C \in \mathscr{C}$. We assume that $\mathscr{D}$ is non-trivial in the sense that $3 \leqslant k \leqslant v - 3$. By [10], the parameters c, d have the following form

$$c = \frac{\binom{k}{2} - x}{y} \quad \text{and} \quad d = \frac{\binom{k}{2} - y}{x}, \tag{1}$$

where x, y are positive integers called the *Delandtsheer–Doyen parameters*. The integer x is the number of *inner pairs* on a line $L \in \mathscr{L}$, that is, the number of unordered pairs of L which lie in the same class of $\mathscr{C}$. We denote the class of $\mathscr{C}$ containing a point α by $C(\alpha)$.

Our first aim is to investigate some of the rich combinatorial structure on the point set $\mathscr{P}$ generated by interactions between the lines of $\mathscr{L}$ and the classes of $\mathscr{C}$. We define in Section 2 several parameters which help to measure these line-class interactions. The generic name we give to these parameters is *intersection parameters*, and we prove two technical propositions about them. These results are then used in Section 3 to obtain upper bounds on the sizes of the line-class intersections in terms of c and x. For small c, or small x, it is possible to determine all possibilities for the set I_0 of non-zero line-class intersection sizes. In Tables 2 and 3 we give this information for $c \leqslant 12$ and $x \leqslant 6$ respectively. The values taken by the intersection parameters in the known examples of line-transitive, point-imprimitive linear spaces, apart from projective planes, are a small subset of these possible values and are recorded in Table 4. In particular, in all the known examples that are not projective planes, the Delandtsheer–Doyen parameter x is at most 2, and these tables of possibilities raise the still open question as to whether there may be further imprimitive pairs $(\mathscr{D}, G)$ corresponding to some of the other entries.

Question 1. Are there imprimitive pairs $(\mathscr{D}, G)$ corresponding to any of the parameter values in Table 3 other than the parameter values in Table 4?

The elementary arguments in Sections 2 and 3 show that $\min\{c, d\} \geqslant 3$ and that if either $c = 3$ or $d = 3$, then $\mathscr{D}$ must be a projective plane (see Corollary 2.2 and Corollary 2.5). However we do not even know if there exists an imprimitive pair with class size $c = 4$ or with $d = 4$ classes. For such an example one of the Delandtsheer–Doyen parameters must equal 1, and it follows from [4, Lemma 8] that if $c = 4$ then k, v must satisfy $k = 8h + 2$, $v = 4(24h^2 + 9h + 1)$ for some positive integer h.

Question 2. Is there an imprimitive pair $(\mathscr{D}, G)$ for which $\min\{c, d\} = 4$?

Our inability to answer this question using elementary methods is rather disappointing, but more group-theoretic methods can provide extra information. A recent result of Camina and the third author [7] implies that, for an imprimitive pair $(\mathcal{D}, G)$, if every non-trivial normal subgroup of G is transitive on the points of $\mathcal{D}$, that is, if G is *quasiprimitive* on $\mathcal{P}$, then G is *almost simple*, that is $T \leqslant G \leqslant \mathrm{Aut}(T)$ for some non-abelian simple group T. This suggests that it may be useful to consider separately those imprimitive pairs $(\mathcal{D}, G)$ for which G is quasiprimitive on points, from those for which G is not quasiprimitive on points. Apart possibly from projective planes, there are no known imprimitive pairs $(\mathcal{D}, G)$ for which G is point-quasiprimitive, and we have chosen to focus in this paper on the case where G is not point-quasiprimitive. Nevertheless, the question of existence of imprimitive pairs with a point-quasiprimitive group is of great interest and importance. It has recently been shown in [5] that no imprimitive pairs exist in the case where G is quasiprimitive and is a finite alternating or symmetric group.

Question 3. Does there exist an imprimitive pair $(\mathcal{D}, G)$, with $\mathcal{D}$ not a projective plane, for which G is point-quasiprimitive (and hence is almost simple)?

For any transitive group action, the set of orbits of a normal subgroup is an invariant partition of the point set. Such invariant partitions for the action of a group G are called *G-normal partitions*, and it follows from the definition of quasiprimitivity that a permutation group G on a set $\mathcal{P}$ is quasiprimitive if and only if the only G-normal partitions are the trivial partitions (that is, the partition consisting of singletons, and the partition consisting of the single class $\mathcal{P}$). If $(\mathcal{D}, G)$ is an imprimitive pair with G not quasiprimitive on $\mathcal{P}$, then we may take $\mathcal{C}$ to be a non-trivial G-normal partition of $\mathcal{P}$. It turns out that much stronger information can be obtained in this case than in the general case. In Section 4 we begin an investigation of the structure of line-transitive groups which preserve a non-trivial normal point-partition. If the partition $\mathcal{C}$ in Hypothesis 1 is the set of point orbits of a normal subgroup K of G, then by [6, Theorem 1] we know that K acts faithfully on each of the classes of $\mathcal{C}$. Since K is normal in G and G is line-transitive, all K-orbits in $\mathcal{L}$ have the same length, say b_K. In Proposition 4.1 we make a few observations about b_K, depending on both the form of the set I_0 of non-zero line-class intersection numbers (as defined in Section 2), and the relationships between the permutation groups K^C induced on the classes $C \in \mathcal{C}$. Our results are strongest in the special case where $I_0 = \{1, 2\}$. These are given in Theorem 1.1 which is proved after Proposition 4.1. We note that, for all the known line-transitive, point-imprimitive linear spaces apart from projective planes, the set I_0 has this form (see Table 4). Also, I_0 must be $\{1, 2\}$ if $x \leqslant 2$ (see Table 3).

Theorem 1.1. *Let $(\mathcal{D}, G)$ be an imprimitive pair satisfying Hypothesis* 1 *with $\mathcal{C}$ the set of K-orbits and K the kernel of G on $\mathcal{C}$. Suppose also that $I_0 = \{1, 2\}$. Then c is odd, all K-orbits on lines have length c, and K_α $(\alpha \in \mathcal{P})$ is an elementary abelian* 2*-group with all point orbits of length at most* 2. *Also, every minimal normal subgroup of G contained in K is elementary abelian of odd order dividing c.*

Most of the arguments used in the proof have an elementary combinatorial and

Table 1. Results for Theorems 1.2 and 1.4

x	y	c	d	k	Comments
1	3	3	7	5	$\mathscr{D} = \mathrm{PG}(2,4)$, $K = S_3$, $G = K \times H$, where $H = Z_7$ or $Z_7 \cdot Z_3$
2	1	13	7	6	Colbourn design, $K = D_{26}$, $G = Z_{91} \cdot Z_e$, where $e = 6$ or 12.
1	1	27	27	8	N^2OP^2 designs [17]
1	2	7	13	6	Colbourn and Mills designs [8], [16]

group theoretic basis. However, for the last assertion we need to use the classification by Walter [20] of the finite simple groups with abelian Sylow 2-subgroups. We derive several consequences of this result in the cases where either c or x is small in Theorem 1.2 below, the proof of which follows from Lemma 4.2.

Theorem 1.2. *Let* $(\mathscr{D}, G)$ *be an imprimitive pair satisfying Hypothesis* 1 *with* $\mathscr{C}$ *the set of K-orbits and K the kernel of G on* $\mathscr{C}$. *Then the following all hold.*

(a) *If* $c \leqslant 6$ *then either* $c = 5$, $K = Z_5$ *and* $I_0 = \{1,2\}$, *or* $c = 3$ *and* $\mathscr{D}$ *is a projective plane.*

(b) *If* $x \leqslant 2$, *then either K is elementary abelian of order* $c = p^a$, *for some odd prime p and* $a \geqslant 1$, *or* $\mathscr{D}$ *satisfies line* 1 *or* 2 *of Table* 1.

(c) *If* $x = 1$ *then* $c = p^a$ *for some odd prime p and integer* $a \geqslant 1$, *and* $y \geqslant (c-3)/2$ *if* $p > 3$, *or* $y \geqslant (c-9)/18$ *if* $p = 3$.

Remark 1.3. (1) In Theorem 1.2(a), if $\mathscr{D}$ is not a projective plane then it follows from Proposition 2.4(vi) that $\mathscr{D}$ has $2v = 10d$ lines with d odd, and $x = 1$, $y = (d-1)/4$. No examples are known.

(2) The case $x = 1$ in Part (b) was proved in [18, Theorem 1.1].

Theorem 1.2 enables us to obtain a complete classification in the case where $x = 1$, $y \leqslant 2$. This is a generalisation of the classification in [18] for the case $x = y = 1$ in the case where there is a normal point partition. The proof given below involved an exhaustive computer search and we are grateful to Anton Betten for carrying out this search.

Theorem 1.4. *Let* $(\mathscr{D}, G)$ *be an imprimitive pair satisfying Hypothesis* 1 *with* $\mathscr{C}$ *the set of K-orbits and K the kernel of G on* $\mathscr{C}$. *If* $x = 1$ *and* $y \leqslant 2$ *then* $\mathscr{D}$ *satisfies line* 3 *or* 4 *of Table* 1.

Proof of Theorem 1.4. By Lemma 4.2, either $\mathscr{D}$ satisfies line 3 or 4 of Table 1 as claimed, or there is one further possibility, namely $(x, y, c, d, k) = (1, 2, 27, 53, 11)$, the group G has a point-regular normal subgroup $R = Z_{53} \times Z_3^3$, and G has a line-regular normal subgroup $R \cdot Z_{13}$ of index dividing 4. (A permutation group is called *regular* if it is transitive and the only element fixing a point is the identity.) Moreover a subgroup of order 13 acts non-trivially on each of the Sylow subgroups of

R. An exhaustive computer search similar to that described in [1] was conducted for line-transitive 2-$(1431, 11, 1)$ designs admitting a line-regular group of the form $(Z_{53} \times Z_3^3) \cdot Z_{13}$. The search showed that no such designs exist.

2 Intersection parameters

In this section we introduce several parameters to describe the interaction between the lines of $\mathscr{D}$ and the partition $\mathscr{C}$. We refer to these parameters generically as the *intersection parameters* for $\mathscr{D}, G, \mathscr{C}$. We denote the number of lines of $\mathscr{D}$ by b. For a given point α, the remaining points $\mathscr{P}\backslash\{\alpha\}$ are partitioned into disjoint sets of size $k-1$ by the lines through α. Thus the number r of lines through each point is $(v-1)/(k-1)$. Counting incident point-line pairs we have $bk = vr$. These equations, together with the well-known Fisher Inequality, namely $b \geqslant v$, are the basic relations between the parameters for any 2-$(v, k, 1)$ design.

We derive further equations related to the G-invariant partition $\mathscr{C}$. Note that since r divides $v-1$, we have $\gcd(r, v) = 1$. For an integer n we call $n^{(r)} = \gcd(n, r)$ and $n^{(v)} = \gcd(n, v)$ the r-part and v-part of n, respectively. For a line L and a class C we call $k_{L,C} := |L \cap C|$ the *line-class intersection number* for C and L and, for $0 \leqslant i \leqslant k$, we say that C and L are *i-incident* if $k_{L,C} = i$. The number of classes that are i-incident with some line L is

$$d_{L,i} = d_i = |\{C \in \mathscr{C} \mid k_{L,C} = i\}|.$$

As G is line-transitive this number is independent of L, and hence is denoted d_i. Similarly the number of lines that are i-incident to some class C is

$$b_{C,i} = b_i = |\{L \in \mathscr{L} \mid k_{L,C} = i\}|.$$

As G is transitive on $\mathscr{C}$ this number also is independent of C. Let I be the set of line-class intersection numbers, that is,

$$I := \{k_{L,C} \mid L \in \mathscr{L}, C \in \mathscr{C}\} \tag{2}$$

and let $I_0 = I\backslash\{0\}$ be the set of all non-zero line-class intersection numbers. Our starting point is the following result which follows implicitly from the proof of Higman and McLaughlin in [13] (see also [9]).

Proposition 2.1 (Higman and McLaughlin). *For an imprimitive pair $(\mathscr{D}, G)$, there are at least two distinct non-zero line-class intersection numbers, that is $|I_0| \geqslant 2$.*

This proposition has several useful corollaries.

Corollary 2.2. *If $(\mathscr{D}, G)$ is an imprimitive pair, then*

(a) *a class does not contain a line*;

(b) *a line does not contain a class*;

(c) *the class size $c \geqslant 3$.*

Proof. If a line is contained in a class, then $I_0 = \{k\}$, which contradicts Proposition 2.1. Hence (a) holds. If some line contains a class of $\mathscr{C}$, then every line contains a class of $\mathscr{C}$ since G is line-transitive. However each class of $\mathscr{C}$ is contained in at most one line as $\mathscr{D}$ is a linear space. Hence $b \leqslant |\mathscr{C}| = d < v$, which is a contradiction. In particular, this means that $c \neq 2$.

We extend the meaning of i-incidence for lines and classes, by defining a point α and a line L to be *i-incident* if $\alpha \in L$ and $k_{L,C(\alpha)} = i$, where $C(\alpha)$ is the class of $\mathscr{C}$ containing α. The number of points that are i-incident to some line L can be computed as

$$i \cdot d_i = |\{\alpha \in \mathscr{P} \mid \alpha \in L, k_{L,C(\alpha)} = i\}|.$$

The number of lines that are i-incident to the point α is called the *i-degree* of α and is denoted by $r_{\alpha,i}$. As G is transitive on $\mathscr{P}$ and preserves i-incidence, the number $r_{\alpha,i}$ is independent of α and is usually denoted r_i. Thus

$$r_{\alpha,i} = r_i = |\{L \in \mathscr{L} \mid \alpha \in L, k_{L,C(\alpha)} = i\}|,$$

and by convention, $r_0 = 0$.

The line-size k and the total number of lines $b = vr/k$ can be factorised into their v and r parts as follows:

$$k^{(r)} = \gcd(k, r) = \gcd(k, v-1), \quad k^{(v)} = \gcd(k, v),$$
$$b^{(r)} = \gcd(b, r) = \gcd(b, v-1), \quad b^{(v)} = \gcd(b, v).$$

Then, since $bk = vr$ and $\gcd(v, r) = 1$,

$$k = k^{(v)}k^{(r)} \quad \text{and} \quad b = b^{(v)}b^{(r)}. \tag{3}$$

Using again $\gcd(v, r) = 1$, we deduce from $vr = bk = b^{(v)}b^{(r)}k^{(v)}k^{(r)}$ that

$$v = k^{(v)}b^{(v)} \quad \text{and} \quad r = k^{(r)}b^{(r)}. \tag{4}$$

Next we record some information about the configuration induced on a class C by i-incidence, where $i \in I$, $i \geqslant 2$. Recall that such an i exists by Proposition 2.1.

Proposition 2.3. *Let $(\mathscr{D}, G)$ be an imprimitive pair. Let $C \in \mathscr{C}$, and for $0 \leqslant i \leqslant k$, let $\mathscr{S}(i) = \{C \cap L \mid L \in \mathscr{L}, |C \cap L| = i\}$ ($\mathscr{S}(i)$ may be empty). Then*

(i) *$b \cdot d_i = d \cdot b_i$ and $i \cdot b_i = c \cdot r_i$, and hence $bd_i i = vr_i$.*

(ii) *$b_i' := \frac{cd_i}{k^{(v)}} = \frac{b_i}{b^{(r)}}$ and $r_i' := \frac{id_i}{k^{(v)}} = \frac{r_i}{b^{(r)}}$ are integers; moreover, $cr_i' = ib_i'$.*

(iii) *If $i \in I$ and $i \geqslant 2$, then $(C, \mathscr{S}(i))$ is a 1-design admitting G_C acting point-transitively, with b_i blocks, c points, block size i, and r_i blocks on each point. Moreover,*

(a) $r_i \geqslant 2$, *and if* $r_i = 2$ *then* $i = 2$ *and* $\mathscr{D}$ *is a projective plane*;
(b) $c \geqslant r_i(i-1)+1$, *with equality if and only if* $I_0 = \{1, i\}$, *which in turn holds if and only if* $(C, \mathscr{S}(i))$ *is a* 2-$(c, i, 1)$-*design.*

Proof. Counting the number of i-incident class-line pairs (C, L) in two ways yields $b \cdot d_i = d \cdot b_i$. For a fixed class C counting the number of i-incident point-line pairs (α, L) with $\alpha \in C$ yields the second equality of (i). Then using the first equality we have $bd_i i = db_i i = dcr_i = vr_i$.

Using Part (i), we have $b^{(r)}d_i = (b/b^{(v)}) \cdot (db_i/b) = (db_i)/b^{(v)} = (vb_i)/(b^{(v)}c) = (k^{(v)}b_i)/c$. Hence $b^{(r)}cd_i = k^{(v)}b_i$, and as $\gcd(k^{(v)}, b^{(r)}) = 1$, it follows that $b_i' := cd_i/k^{(v)} = b_i/b^{(r)}$ is an integer. By Part (i), $b^{(v)}b^{(r)}d_i i = b^{(v)}k^{(v)}r_i$ and so $b^{(r)}d_i i = k^{(v)}r_i$. Since $\gcd(k^{(v)}, b^{(r)}) = 1$, it follows that $r_i' := id_i/k^{(v)} = r_i/b^{(r)}$ is an integer. By definition, $cr_i' = ib_i'$. Thus (ii) is proved.

Now suppose that $i \in I$ and $i \geqslant 2$. The fact that $(C, \mathscr{S}(i))$ is a 1-design follows from the discussion preceding the proposition. Let $L \in \mathscr{L}$ be such that $S := C \cap L \in \mathscr{S}(i)$. If $r_i = 1$ then S is a block of imprimitivity for G_C in C, and hence S is also a block of imprimitivity for G in $\mathscr{P}$, contradicting Corollary 2.2(b). Hence $r_i \geqslant 2$. Suppose next that $r_i = 2$. Then by Part (i), $b_i = 2c/i \leqslant c$. Since L is the unique line containing S, G_S fixes L setwise, and hence $G_S \leqslant G_L$ so $b = |G : G_L| \leqslant |G : G_S| = d|G_C : G_S| \leqslant db_i \leqslant dc = v \leqslant b$. Hence $b = v$, and so $\mathscr{D}$ is a projective plane. Also we must have $b_i = c$, so $i = 2$ by (i). Thus (iii)(a) is proved.

Let $\alpha \in C$. We count incident point-line pairs (β, L) for which $\beta \in C \backslash \{\alpha\}$. As two points determine a line there are $c - 1$ choices for β and 1 choice for L. On the other hand, for each $h \geqslant 1$, there are r_h choices for lines that contain α and meet C in $h - 1$ other points. Hence $c - 1 = \sum_h r_h(h-1)$. It follows that $c \geqslant r_i(i-1)+1$ with equality if and only if $I \subseteq \{0, 1, i\}$. This holds if and only if any line intersecting a class C in at least two points has exactly i points in C, or equivalently $\mathscr{D}$ induces the structure of a 2-$(c, i, 1)$-design on C.

In the course of the proof above, we proved the equality $c - 1 = \sum_h r_h(h-1)$. We state this formally below, together with other equalities obtained by averaging over the set I. Recall that by convention $r_0 = 0$.

Proposition 2.4. *Let* $(\mathscr{D}, G)$ *be an imprimitive pair satisfying Hypothesis* 1.

(i) $k = \sum_{i=0}^{k} id_i = \sum_{C \in \mathscr{C}} k_{L,C}$, *for* $L \in \mathscr{L}$, *and* $d = \sum_{i=0}^{k} d_i$;
(ii) $r = \sum_{i=0}^{k} r_i$ *and* $b = \sum_{i=0}^{k} b_i$;
(iii) $r \cdot c = \sum_{i \in I} b_i i = \sum_{L \in \mathscr{L}} k_{L,C}$, *for* $C \in \mathscr{C}$;
(iv) $c - 1 = \sum_{i \in I} r_i(i-1)$;
(v) $x = \sum_{i \in I} \binom{i}{2} d_i = \sum_{i \in I} \frac{(i-1)}{2} r_i' k^{(v)} = \frac{(c-1)k^{(v)}}{2b^{(r)}}$, *with* r_i' *as in Proposition* 2.3.
(vi) $(d-1)x = (c-1)y$, $c - 1 = 2xb/v$ *and* $d - 1 = 2yb/v$.

Proof. The equalities in (i) and (ii) follow from the definitions of the d_i, $k_{L,C}$, r_i and

b_i. We obtain (iii) by counting incident point-line pairs (α, L) for $\alpha \in C$. The equality in (iv) was proved in the proof of Proposition 2.3. For (v), the number x of inner pairs on a line is $\sum_{i\in I}\binom{i}{2}d_i$, which, by Proposition 2.3(ii) is $\sum_{i\in I} r_i' k^{(v)}(i-1)/2 = \left(\sum_{i\in I}(i-1)r_i\right)k^{(v)}/2b^{(r)}$, and by (iv) this equals $(c-1)k^{(v)}/2b^{(r)}$. From Hypothesis 1 we have

$$(d-1)x = \left(\frac{\binom{k}{2}-y}{x} - 1\right)x = \binom{k}{2} - y - x = (c-1)y \geqslant c-1,$$

which is the first equality in (vi). Since $k^{(v)}/b^{(r)} = v/b$, the second part of (vi) follows from (v). The third part is then immediate.

We can use these relationships between the parameters to obtain a lower bound on the number d of classes and on their size c, and in particular to prove that both c and d are at least 3.

Corollary 2.5. *Let $(\mathscr{D}, G)$ be an imprimitive pair satisfying Hypothesis* 1. *Then c and d are both at least* $1 + 2b/v$. *In particular*, $c \geqslant 3$, $d \geqslant 3$, *and if $c = 3$ or $d = 3$ then $\mathscr{D}$ is a projective plane.*

Proof. The first inequalities follow from Proposition 2.4(vi) and the fact that x and y are positive integers. The other assertions are then consequences of Fisher's Inequality, namely $b \geqslant v$ with equality if and only if $\mathscr{D}$ is a projective plane.

Refinement of the intersection parameters to orbit parameters. Let $(\mathscr{D}, G)$ be an imprimitive pair as in Hypothesis 1. Then there is an equivalence relation on $\mathscr{L} \times \mathscr{C}$ such that two pairs $(L, C), (L', C') \in \mathscr{L} \times \mathscr{C}$ are equivalent whenever $|L \cap C| = |L' \cap C'|$. The partition of $\mathscr{L} \times \mathscr{C}$ determined by this equivalence relation has equivalence classes indexed by the set I of line-class intersection numbers defined in (2). Moreover the group G fixes each equivalence class setwise, and so this partition is refined by the partition of $\mathscr{L} \times \mathscr{C}$ into G-orbits. For each $i \in I$ let $O_{(i,1)}, O_{(i,2)}, \ldots$ denote the G-orbits on i-incident line-class pairs, and let $\mathscr{I} \subset I \times \mathbb{N}_0$ denote the set of indices (i, j) indexing all G-orbits in $\mathscr{L} \times \mathscr{C}$. For a line L and a class C we let $h_{L,C} := (i, j)$ if and only if $(L, C) \in O_{(i,j)}$ and we say that C and L are (i, j)-incident if $h_{L,C} = (i, j)$. Hence $\mathscr{I} = \{h_{L,C} \mid L \in \mathscr{L}, C \in \mathscr{C}\}$ and we can generalize the above definitions of d_i, b_i and r_i to $d_{(i,j)}$, $b_{(i,j)}$ and $r_{(i,j)}$, so that

$$\sum_j d_{(i,j)} = d_i, \quad \sum_j b_{(i,j)} = b_i, \quad \sum_j r_{(i,j)} = r_i.$$

It is then straightforward to refine Propositions 2.3 and 2.4 using these orbit parameters.

We finish this section by examining the orbit lengths of a point stabiliser in its actions on lines and on the partition $\mathscr{C}$. The results below strengthen Proposition 2.3.

We denote by $G^{\mathscr{C}}$ the permutation group induced by G on the partition $\mathscr{C}$. We note that part of this result can be found in [15, Proposition 3.3 and Corollary 3.2].

Proposition 2.6. *Let* $(\mathscr{D}, G)$ *be an imprimitive pair satisfying Hypothesis* 1, *and let* $\alpha \in \mathscr{P}$. *Then the parameter* $b^{(r)}$ *divides the length of*

(a) *every* G_α*-orbit on the set of lines through* α;

(b) *every* G_α*-orbit on* $\mathscr{P}\backslash\{\alpha\}$, *and in particular on* $C(\alpha)\backslash\{\alpha\}$; *and*

(c) *every* G_α*-orbit on* $\mathscr{C}\backslash\{C(\alpha)\}$, *and hence also every* G_C*-orbit on* $\mathscr{C}\backslash\{C\}$.

Proof. Suppose that α lies on the line L, and let a be the length of the G_α-orbit containing L in $\mathscr{L}$. Then $va = |G : G_\alpha| \cdot |G_\alpha : G_{\alpha,L}| = |G : G_{\alpha,L}|$ is divisible by $|G : G_L| = b$. Since $b^{(r)}$ divides b and is coprime to v, it follows that $b^{(r)}$ divides a. Thus Part (a) is proved.

Let Γ be a G_α-orbit in $\mathscr{P}\backslash\{\alpha\}$. The lines through α induce a partition of Γ. Any two lines in the same G_α-orbit intersect Γ in the same number of points, so $|\Gamma|$ is divisible by the length of the G_α-orbit of any line through α intersecting Γ. Thus by Part (a), $b^{(r)}$ divides $|\Gamma|$. This holds in particular for the G_α-orbits on $C(\alpha)\backslash\{\alpha\}$, so Part (b) is proved.

Let Δ be a G_α-invariant subset of $\mathscr{C}\backslash\{C(\alpha)\}$, let $e = |\Delta|$, and let $\mathscr{P}_\Delta$ be the union of the classes of $\mathscr{C}$ that are contained in Δ. We claim that $b^{(r)}$ divides e. Note that Part (c) follows from this claim. Let $\mathscr{L}_\Delta$ be the subset of $\mathscr{L}$ consisting of lines containing α and intersecting some class in Δ non-trivially. Since Δ is G_α-invariant, so also $\mathscr{L}_\Delta$ is G_α-invariant. Let $\mathscr{L}_1, \ldots, \mathscr{L}_t$ be the G_α-orbits in $\mathscr{L}_\Delta$. By Part (a), each $|\mathscr{L}_i|$ is divisible by $b^{(r)}$. For each i, choose $L_i \in \mathscr{L}_i$. Note that, for each $\beta \in \mathscr{P}_\Delta$, the unique line containing α and β lies in $\mathscr{L}_\Delta$. Counting the number of pairs (β, L) where $\beta \in \mathscr{P}_\Delta$, $L \in \mathscr{L}_\Delta$, and $\beta \in L$, we find

$$ec = \sum_{i=1}^{t} |\mathscr{L}_i| \cdot |L_i \cap \mathscr{P}_\Delta|.$$

Since $b^{(r)}$ divides each $|\mathscr{L}_i|$, it follows that $b^{(r)}$ divides ec, and since $v = cd$ is coprime to $b^{(r)}$ we conclude that $b^{(r)}$ divides e.

3 Bounds on line-class intersection numbers

Let $\mathscr{D}, G, \mathscr{C}$ be as in Hypothesis 1, and let I be the set of line-class intersection numbers. Set

$$i_{\max} = \max\{i \mid i \in I\}. \tag{5}$$

By Corollary 2.2, $i_{\max}$ is strictly less than the class size c. The equalities of Proposition 2.4 allow us to reduce this upper bound to approximately $\sqrt{c}$.

Lemma 3.1. $i_{\max} < \sqrt{c} + 1/2$, *and* $i_{\max} < \sqrt{2x} + 1$.

Proof. By Proposition 2.4(iv) and Proposition 2.3,

$$c - 1 = \sum_{i \in I} r_i(i-1) = \frac{b}{v} \sum_{i \in I} d_i i(i-1) \geqslant i_{\max}(i_{\max} - 1),$$

and the first inequality follows. By Proposition 2.4(v), $2x \geqslant i_{\max}(i_{\max} - 1)$, and we obtain the second inequality.

This result suggests that it may be possible to obtain a limited number of feasible sets of intersection parameters if either or both of c and x are bounded. Such information may lead to the discovery of new block-transitive, point-imprimitive linear spaces, and thereby lead to a better understanding of such linear spaces. Accordingly we determine the possibilities for I and the parameters d_i for small values of c and x. We note that upper bounds on x alone do not imply upper bounds for c.

Lemma 3.2. *For* $c \leqslant 12$, *the possibilities for* I_0, c, x *and the* d_i *are given in Table* 2. *For* $x \leqslant 6$ *the possibilities for* I_0, x *and the* d_i *are given in Table* 3.

Proof. By Proposition 2.1, $|I_0| \geqslant 2$. By Propositions 2.3(i) and 2.4(iv) and (v),

$$c - 1 = \sum_{i \in I} r_i(i-1) = \frac{b}{v} \sum_{i \in I} d_i i(i-1) = \frac{2bx}{v}.$$

Also, by Lemma 3.1, $i_{\max} < \sqrt{c} + 1/2$ and $i_{\max} < \sqrt{2x} + 1$. These inequalities imply that, for $c \leqslant 12$, we have $i_{\max} \leqslant 3$, and also for $x \leqslant 6$ we have $i_{\max} \leqslant 4$.

Table 2. Intersection parameters for $3 \leqslant c \leqslant 12$

I_0	c	d_i	b/v	x
$\{1,2\}$	$3 \leqslant c \leqslant 12$	$1 \leqslant d_2 \leqslant (c-1)/2$	$(c-1)/2d_2$	d_2
$\{1,3\}$	$7 \leqslant c \leqslant 12$	$d_3 = 1$	$(c-1)/6$	3
$\{2,3\}$ or $\{1,2,3\}$	$9 \leqslant c \leqslant 12$	$1 \leqslant d_2 \leqslant (c-7)/2$, $d_3 = 1$	$(c-1)/(2d_2+6)$	$d_2 + 3$

Table 3. Intersection parameters for $1 \leqslant x \leqslant 6$

I_0	d_i	b/v	x
$\{1,2\}$	$1 \leqslant d_2 \leqslant 6$	$(c-1)/2d_2$	d_2
$\{1,3\}$	$d_3 = 1$ or 2	$(c-1)/6d_3$	$3d_3$
$\{1,4\}$	$d_4 = 1$	$(c-1)/12$	6
$\{2,3\}$ or $\{1,2,3\}$	$1 \leqslant d_2 \leqslant 3$, $d_3 = 1$	$(c-1)/2(d_2+3)$	$d_2 + 3$

Table 4. Intersection parameters for known examples

I_0	c	d	$x = d_2$	y	b/v	k	Comments
$\{1,2\}$	7	13	1	2	3	6	Colbourn and Mills designs [8], [16]
$\{1,2\}$	13	7	2	1	3	6	Colbourn and Mills designs [8], [16]
$\{1,2\}$	27	27	1	1	13	8	N^2OP^2 designs [17]

If $I_0 = \{1,2\}$, then the number x of inner pairs is d_2, and the displayed equations above give $c - 1 = (b/v) \cdot 2d_2 \geqslant 2d_2$, and hence the first line of Table 2 holds if $c \leqslant 12$, and the first line of Table 3 holds if $x \leqslant 6$. If $3 \leqslant c \leqslant 6$, then $i_{\max} = 2$, and hence $I_0 = \{1,2\}$, and we have the required result.

Thus we may assume that $c \geqslant 7$ and that $i_{\max} \geqslant 3$, that is, $I_0 \neq \{1,2\}$. If $I_0 = \{1,3\}$, then $x = 3d_3$, and $c - 1 = (b/v) \cdot 6d_3 \geqslant 6d_3$. Hence the second line of Table 3 holds if $x \leqslant 6$. If $7 \leqslant c \leqslant 12$, then $i_{\max} = 3$, and $c - 1 = (b/v)(2d_2 + 6d_3) \geqslant 2d_2 + 6d_3 \geqslant 2d_2 + 6$, so either $d_2 = 0$, $d_3 = 1$ and $I_0 = \{1,3\}$, or we have $1 \leqslant d_2 \leqslant (c-7)/2$, $d_3 = 1$ and $c \geqslant 9$. Hence if $c = 7$ or 8 then the second line of Table 2 holds. In summary, if either $7 \leqslant c \leqslant 8$ or $I_0 = \{1,3\}$, then we have required result. If $9 \leqslant c \leqslant 12$ and $I_0 \neq \{1,2\}$ and $\{1,3\}$, then $I_0 = \{2,3\}$ or $\{1,2,3\}$, and the third line of Table 2 holds.

Thus we may assume that $x \leqslant 6$ and $I_0 \neq \{1,2\}$ and $\{1,3\}$. Since $i_{\max} < \sqrt{2x} + 1$, we have $3 \leqslant i_{\max} \leqslant 4$ and $2x = \sum d_i i(i-1)$. If $i_{\max} = 4$, this implies that $I_0 = \{1,4\}$ and $d_4 = 1$, so $x = 6$; also $c - 1 = (b/v) \sum d_i i(i-1) = 12b/v$, so the third line of Table 3 holds. Thus we may assume that $i_{\max} = 3$. Then $12 \geqslant 2x = 2d_2 + 6d_3$ and both d_2 and d_3 are non-zero, so $d_3 = 1$ and $d_2 = x - 3$ is 1, 2 or 3. Finally $c - 1 = (b/v) \sum d_i i(i-1) = (b/v)(2d_2 + 6)$, and hence the fourth line of Table 3 holds.

Comparable information about all the known line-transitive, point imprimitive linear spaces, apart from projective planes, is contained in Table 4. In both of the 2-$(91,6,1)$ designs there are two non-trivial invariant partitions, as in lines 1 and 2 of Table 4. For most of the N^2OP^2 designs there is a unique non-trivial invariant partition, but for a few of these designs there are two such partitions, and for two of these designs there are 28 such partitions corresponding to the 28 parallel classes of lines in the affine plane $\mathrm{AG}(2,27)$. For all of the partitions corresponding to the N^2OP^2 designs, the intersection parameters are as in line 3 of Table 4.

4 Normal partitions

Now we assume that $(\mathscr{D}, G)$ is an imprimitive pair satisfying Hypothesis 1 such that G has a non-trivial normal subgroup K which is intransitive on points. We further assume that $\mathscr{C}$ is the set of K-orbits in $\mathscr{P}$, and that K is the subgroup consisting of all elements of G which fix every class of $\mathscr{C}$ setwise, that is, K is the kernel of the action of G on $\mathscr{C}$. By [6, Theorem 1], K acts faithfully on each class of $\mathscr{C}$, that is, for each $C \in \mathscr{C}$, the permutation group K^C induced by K on C is isomorphic to K. Since K is normal in G and G is line-transitive, it follows that all K-orbits in $\mathscr{L}$ have the same length, say b_K. We begin by making a few simple observations about b_K, depending on both the form of the set I of line-class intersection numbers, and the relationships between the various permutation groups K^C, $C \in \mathscr{C}$.

Proposition 4.1. *Let* $(\mathscr{D}, G)$ *be an imprimitive pair satisfying Hypothesis* 1 *with* $\mathscr{C}$ *the set of* K*-orbits and* K *the kernel of* G *on* $\mathscr{C}$. *Let* I_0 *be the set of non-zero line-class intersection numbers, and let* b_K *be the length of the* K*-orbits on lines.*

(a) *Then* b_K *divides* c.

(b) *If* $1 \in I_0$, *then* $b_K = c$, *where* c *is odd, and* $K_L = K_\alpha$ *for any point* α *and any line* L *such that* $L \cap C(\alpha) = \{\alpha\}$.

Proof. Let α, β be distinct points in $\mathscr{P}$, let L denote the unique line containing α and β, and let $C' \in \mathscr{C}\backslash\{C(\alpha)\}$. Then $K_{\alpha\beta} \leqslant K_L$ and hence $b_K = |K : K_L|$ divides $|K : K_{\alpha\beta}| = |K : K_\alpha|\,|K_\alpha : K_{\alpha\beta}| = c|K_\alpha : K_{\alpha\beta}|$. Thus b_K divides c times the length of every K_α-orbit in $\mathscr{P}\backslash\{\alpha\}$. Since K_α leaves $C(\alpha)\backslash\{\alpha\}$ invariant it follows that b_K divides $c|C(\alpha)\backslash\{\alpha\}| = c(c-1)$, and since K_α leaves C' invariant it follows that b_K divides $c|C'| = c^2$. Hence b_K divides $\gcd(c(c-1), c^2) = c$.

Suppose that $1 \in I_0$. Let $\alpha \in \mathscr{P}$ and $L \in \mathscr{L}$ be such that $L \cap C(\alpha) = \{\alpha\}$. Then $K_L \leqslant K_\alpha$, and hence $c = |K : K_\alpha|$ divides $b_K = |K : K_L|$. By Part (a) it follows that $b_K = c$ and hence $K_L = K_\alpha$. Now let $\beta \in C(\alpha)\backslash\{\alpha\}$ and let L' be the line containing α and β. Then $K_{\{\alpha,\beta\}} \leqslant K_{L'}$, and hence $c = b_K = |K : K_{L'}|$ divides $|K : K_{\{\alpha,\beta\}}|$. Thus c divides the length of every K-orbit on unordered pairs from a class $C \in \mathscr{C}$, and so c divides $c(c-1)/2$. It follows that c is odd.

Our next task is to prove Theorem 1.1. We use the concept of permutational equivalence defined as follows. Suppose that a group H acts on sets X, Y. These actions, and also the corresponding permutation groups H^X, H^Y, are said to be *permutationally isomorphic* if there is a bijection $\psi : X \to Y$ and an automorphism $\varphi \in \mathrm{Aut}(H)$ such that, for all $\alpha \in X$, $h \in H$, we have $((\alpha)\psi)^{(h)\varphi} = (\alpha^h)\psi$; if $\varphi = 1$ then the actions and permutation groups are said to be *permutationally equivalent* or simply *equivalent*.

Proof of Theorem 1.1. Let $(\mathscr{D}, G)$ be an imprimitive pair satisfying Hypothesis 1 with $\mathscr{C}$ the set of K-orbits and K the kernel of G on $\mathscr{C}$, and suppose that $I_0 = \{1, 2\}$. Let $p = \{\alpha, \beta\} \subset C$, and let $L(p)$ be the line containing p. Then $L(p) \cap C = p$, and hence $K_p = K_{L(p)}$. Then the K_α-orbit containing β has length $|K_\alpha : K_{\alpha\beta}| = |K : K_{\alpha\beta}|/c = |K_p : K_{\alpha\beta}| \leqslant 2$. Thus all K_α-orbits in C have length 1 or 2 and hence $K_\alpha \cong K_\alpha^C$ is an elementary abelian 2-group. Since c is odd, K_α must therefore fix a point in each class of $\mathscr{C}$, and hence the K-actions on the classes of $\mathscr{C}$ are all equivalent. Since for each $C' \in \mathscr{C}$ there exists $\alpha' \in \mathscr{C}'$ such that $K_\alpha = K_{\alpha'}$, it follows that all K_α-orbits in C' have length at most 2 also.

Suppose that N is a minimal normal subgroup of G contained in K. Then $N = T^t$ for some simple group T and some $t \geqslant 1$. Suppose that T is a non-abelian simple group. Then $N_\alpha = N \cap K_\alpha$ is an elementary abelian 2-group and all N_α-orbits in

$\mathscr{P}$ have length at most 2. Also, $|N : N_\alpha| = |NK_\alpha : K_\alpha|$ divides $|K : K_\alpha| = c$, and hence $|N : N_\alpha|$ is odd. Thus N_α is a Sylow 2-subgroup of N, and hence the Sylow 2-subgroups of T are elementary abelian. By a result of Walter [20], T is one of $\mathrm{PSL}(2,q)$, $q = 2^n \geqslant 4$ or $q \equiv 3, 5 \pmod 8$, Janko's first group J_1, or $\mathrm{Ree}(q)'$, $q = 3^{2a+1} \geqslant 3$. Since all N_α-orbits in $\mathscr{P}$ have length at most 2, it follows that distinct Sylow 2-subgroups of T intersect in a subgroup of index 2 of each. In the case of $\mathrm{PSL}(2,q)$, $q = 2^n \geqslant 4$, distinct Sylow 2-subgroups of T intersect trivially, which is a contradiction. Similarly, in the other cases, a Sylow 2-subgroup S of T is contained in a subgroup A_4, $2 \times A_5$, or $2 \times \mathrm{PSL}(2,q)$ (and hence in $2 \times A_4$), respectively, and it is easy to verify that S intersects one of its conjugates in a subgroup of index 4. These contradictions imply that $T = Z_p$ for some prime p. Since N^C is normal in the transitive group K^C it follows that all N-orbits in C have the same size. In particular p divides $|C| = c$ so p is odd and hence $|N|$ is odd. Also, since K_α is a 2-group and K is faithful on C, it follows that $N \cap K_\alpha = 1$ so $|N|$ divides c.

We complete this section by proving Theorem 1.2. This is achieved by proving the following lemma which also deduces the extra information needed for Theorem 1.4.

Lemma 4.2. *Let $(\mathscr{D}, G)$ be an imprimitive pair satisfying Hypothesis* 1 *with $\mathscr{C}$ the set of K-orbits and K the kernel of G on $\mathscr{C}$. Then all the assertions of Theorem* 1.2 *hold. Moreover, if $x = 1$ and $y \leqslant 2$, then either $\mathscr{D}$ satisfies line* 3 *or* 4 *of Table* 1, *or* $(x, y, c, d, k) = (1, 2, 27, 53, 11)$, *the group G has a point-regular normal subgroup $R = Z_{53} \times Z_3^3$, G has a line-regular normal subgroup $R \cdot Z_{13}$ of index dividing* 4, *and a subgroup of G of order* 13 *acts non-trivially on each of the Sylow subgroups of R.*

Proof. First we prove Theorem 1.2. Suppose that either $c \leqslant 6$ or $x \leqslant 2$. Then (see Tables 2 and 3) $I_0 = \{1, 2\}$, and hence, by Theorem 1.1, c is odd, and K_α ($\alpha \in \mathscr{P}$) is an elementary abelian 2-group which by [6, Theorem 1] acts faithfully on $C(\alpha)$. First we show that Theorem 1.2(a) follows from Theorem 1.2(b). Suppose that $c \leqslant 6$ and that Theorem 1.2(b) is true. Then by Table 2, $x \leqslant (c-1)/2 < 3$ so $x \leqslant 2$. Since $c \leqslant 6$, it follows from Theorem 1.2(b) that either $\mathscr{D}$ is $\mathrm{PG}(2,4)$, $K = S_3$ and $c = 3$, or else $K = Z_c$ and $c = 3$ or 5. If $c = 3$ then $\mathscr{D}$ is a projective plane by Corollary 2.5. Thus Theorem 1.2(a) follows.

Our next step is to prove Theorem 1.2(b). If $x = 1$ then by [18, Theorem 1.1], $c = p^a$ for some odd prime p and integer $a \geqslant 1$ and either $K = Z_p^a$ or line 1 of Table 1 holds. Thus Theorem 1.2(b) holds if $x = 1$. Suppose now that $x = 2$, and let $C \in \mathscr{C}$ and $\alpha \in C$. First we prove that C is a minimal block of imprimitivity for G. If this is not the case then there exists a proper subset $B \subset C$ containing α such that B is a block of imprimitivity for G and $c_0 = |B| \geqslant 2$. By (1), $c_0 = \left(\binom{k}{2} - x_0\right)/y_0$, for some positive integers x_0, y_0. Since B is a block of imprimitivity for G_C in C it follows that c_0 divides $c = \left(\binom{k}{2} - 2\right)/y$, and hence

$$\binom{k}{2} - x_0 \text{ divides } cyy_0 = y_0\left(\binom{k}{2} - 2\right). \tag{6}$$

Let $\mathscr{C}_0 = B^G$ denote the point-partition generated by B. Then $\mathscr{C}_0$ is a refinement of $\mathscr{C}$,

Table 5. Computations for the divisibility condition

k	9	8	7	6	5
$\binom{k}{2}-2$	34	26	19	13	8
$\binom{k-4}{2}$	10	6	3	1	0

so every pair of points lying in the same block of $\mathscr{C}_0$ is an inner pair for $\mathscr{C}$. However, since G is line-transitive, every line contains at least one inner pair of points which do not lie in the same block of $\mathscr{C}_0$. Hence $1 \leqslant x_0 < x = 2$, so $x_0 = 1$, and hence $\binom{k}{2} - x_0$ is relatively prime to $\binom{k}{2} - 2$. Therefore, by (6), $\binom{k}{2} - x_0$ divides y_0, but this means that $c_0 \leqslant 1$, which is a contradiction. Hence C is a minimal block of imprimitivity.

Thus the stabiliser G_C of $C \in \mathscr{C}$ induces a primitive group G_C^C on C. It follows that a minimal normal subgroup M of G contained in K must be transitive and faithful on C. Then by Theorem 1.1 we deduce that $c = |M| = p^a$ for some odd prime p and integer $a \geqslant 1$, $M = Z_p^a$, $K = M \cdot K_\alpha$ $(\alpha \in \mathscr{P})$, and K_α is an elementary abelian 2-group acting faithfully on C. If $K_\alpha = 1$ then $K = Z_p^a$ and Theorem 1.2(b) holds, so we may assume that $K_\alpha \neq 1$. Arguing as in the proof of Theorem 1.1, the actions of K on the blocks of $\mathscr{C}$ are permutationally equivalent. Thus K_α fixes the same number of points in each block of $\mathscr{C}$. Since K^C is a normal subgroup of G_C^C, the set of fixed points of K_α in C is a block of imprimitivity for G_C^C, and since G_C^C is primitive it follows that K_α fixes exactly one point of C. Thus K_α fixes exactly one point of each block of $\mathscr{C}$. The set F of fixed points of K_α is therefore a block of imprimitivity for G of size d, and the G-invariant partition $\mathscr{F} = F^G$ it generates contains c blocks. The integer y is the number of $\mathscr{F}$-inner pairs of points on each line. Let $\{\beta, \gamma\}$ be an orbit of K_α of length 2, and let L be the unique line containing $\{\beta, \gamma\}$. Since K_α fixes $\{\beta, \gamma\}$, we have $K_\alpha \leqslant K_L$. By Theorem 1.1, $|K : K_L| = c$ and hence $K_\alpha = K_L$. Since K_α has a unique fixed point in each block of $\mathscr{C}$, it follows that L consists of $x = 2$ orbits of K_α of length 2, and $k - 4$ points of F, and by Proposition 2.1, $k - 4 \geqslant 1$. These latter $k - 4$ points all lie in the single block $F \in \mathscr{F}$, while each $\mathscr{C}$-inner pair on L consists of one point from each of two different $\mathscr{F}$-blocks which are interchanged by K_α. Hence $y = \binom{k-4}{2} + \delta$, where $\delta = 0$ or 2. Now by (1) and Corollary 2.5,

$$\binom{k}{2} - 2 = cy \geqslant 3y = 3\left(\binom{k-4}{2} + \delta\right) \geqslant 3\binom{k-4}{2}$$

so $k \leqslant 9$. Thus $5 \leqslant k \leqslant 9$. Since $y = \binom{k-4}{2} + \delta$ divides $cy = \binom{k}{2} - 2$, and $\delta = 0$ or 2, we see from Table 5 that (k, c, d, y) is either $(6, 13, 7, 1)$ or $(5, 4, 4, 2)$. It follows from [4] that the latter case cannot arise, and from [3] and [4] that in the former case $\mathscr{D}$ is the design discovered by Colbourn and Colbourn, and for this design there is a group G for which the subgroup K_α is cyclic of order 2. Thus line 2 of Table 1 holds, and Theorem 1.2(b) is proved.

Now we deal with Theorem 1.2(c), so suppose that $x = 1$. By Theorem 1.2(b), $c = p^a$ for some odd prime p and integer $a \geqslant 1$. We have $c = p^a = \left(\binom{k}{2} - 1\right)/y$, so $2yp^a = k^2 - k - 2 = (k-2)(k+1)$. If $p > 3$, then, as $\gcd(k-2, k+1)$ divides 3, p^a

must divide either $k-2$ or $k+1$. Hence $p^a \leqslant k+1 = (k-2)+3 = 2yp^a/(k+1) + 3 \leqslant 2y+3$, and so $y \geqslant (c-3)/2$. Similarly, if $p = 3$ then 3^{a-1} divides one of $k-2$ or $k+1$, and we obtain the required lower bound for y of Theorem 1.2(c). This completes the proof of Theorem 1.2.

To finish the proof of Lemma 4.2, we suppose that $x = 1$ and $y \leqslant 2$. Then the inequalities of Theorem 1.2(c) alone show that $c \in \{3, 5, 7, 9, 27\}$, and if $c = 7$ then $y = 2$. Next the fact that $\binom{k}{2} = 1 + cy$ implies that (y, c, k) is one of $(1, 5, 4)$, $(1, 9, 5)$, $(1, 27, 8)$, $(2, 7, 6)$, $(2, 27, 11)$. It was shown in [17] and [18] that the first two triples do not arise, and that in the case of the third triple, the examples are precisely the 467 examples constructed in [17] so line 3 of Table 1 holds. In the case of the fourth triple, it was shown in [4, Theorem 1] that the only designs which arise here are the ones of Mills and Colbourn so line 4 of Table 1 holds. So suppose we are in the last case. By Theorem 1.2(b), K is elementary abelian of order $c = 27$. Also $d = \binom{k}{2} - y = 53$, and G/K acts faithfully and transitively on $\mathscr{C}$ of degree 53. Since 53 is prime, this action of G/K is primitive, and by [12, Table B.4], the only primitive groups of degree 53 are subgroups of $\mathrm{AGL}(1, 53)$ and the alternating and symmetric groups A_{53} and S_{53}. Since G is line-transitive and each line contains exactly $x = 1$ inner pair, it follows that the setwise stabiliser G_C of a class $C \in \mathscr{C}$ must be transitive on the pairs of points from C. If $G/K = A_{53}$ or S_{53}, then the group A_{53} would centralise K, making it impossible for G_C to induce a permutation group on C transitive on unordered pairs. Hence $G/K \leqslant \mathrm{AGL}(1, 53)$. Since $|\mathrm{Aut}\, K| = |\mathrm{GL}(3, 3)|$ is not divisible by 53, it follows that K is centralised by a subgroup of order 53, and hence G has a normal subgroup $R \cong Z_{53} \times K$, and $G/R \leqslant Z_{52}$. The group R is transitive on $\mathscr{C}$ and $R_C = K$ is transitive on C, and it follows that R is regular on points. Since G_C is transitive on the 27×13 unordered pairs from C it follows that G/R has order divisible by 13, and hence G has a normal subgroup $N = R \cdot Z_{13}$ of index dividing 4. Now N is transitive, and hence regular, on inner pairs, and hence N is regular on lines. We note that a subgroup of G_C of order 13 must act non-trivially on both Sylow subgroups of R. This completes the proof.

References

[1] A. Betten, G. Cresp, A. C. Niemeyer, C. E. Praeger, Searching for line-transitive linear spaces preserving a grid structure on points. Preprint, 2002.

[2] P. J. Cameron, C. E. Praeger, Block-transitive t-designs. I. Point-imprimitive designs. *Discrete Math.* **118** (1993), 33–43. MR 95f:05117 Zbl 0780.05006

[3] A. R. Camina, L. Di Martino, The group of automorphisms of a transitive 2-$(91, 6, 1)$ design. *Geom. Dedicata* **31** (1989), 151–164. MR 91a:20006 Zbl 0697.05013

[4] A. R. Camina, S. Mishcke, Line-transitive automorphism groups of linear spaces. *Electron. J. Combin.* **3** (1996), Research Paper 3, 16 pp. (electronic). http://www.combinatorics.org MR 96k:05020 Zbl 0853.51001

[5] A. R. Camina, P. M. Neumann, C. E. Praeger, Alternating groups acting on finite linear spaces. *Proc. London Math. Soc.* (3) **87** (2003), 29–53.

[6] A. R. Camina, C. E. Praeger, Line-transitive automorphism groups of linear spaces. *Bull. London Math. Soc.* **25** (1993), 309–315. MR 94f:20005 Zbl 0792.05018

[7] A. R. Camina, C. E. Praeger, Line-transitive, point quasiprimitive automorphism groups of finite linear spaces are affine or almost simple. *Aequationes Math.* **61** (2001), 221–232. MR 2002b:20002 Zbl 01642560

[8] M. J. Colbourn, C. J. Colbourn, Cyclic Steiner systems having multiplier automorphisms. *Utilitas Math.* **17** (1980), 127–149. MR 81h:05022 Zbl 0455.05020

[9] A. Delandtsheer, Line-primitive automorphism groups of finite linear spaces. *European J. Combin.* **10** (1989), 161–169. MR 90c:05021 Zbl 0679.51013

[10] A. Delandtsheer, J. Doyen, Most block-transitive t-designs are point-primitive. *Geom. Dedicata* **29** (1989), 307–310. MR 90d:05034 Zbl 0673.05010

[11] P. Dembowski, *Finite geometries*. Springer 1968. MR 38 #1597 Zbl 0159.50001

[12] J. D. Dixon, B. Mortimer, *Permutation groups*. Springer 1996. MR 98m:20003 Zbl 0951.20001

[13] D. G. Higman, J. E. McLaughlin, Geometric ABA-groups. *Illinois J. Math.* **5** (1961), 382–397. MR 24 #A1069 Zbl 0104.14702

[14] W. M. Kantor, Primitive permutation groups of odd degree, and an application to finite projective planes. *J. Algebra* **106** (1987), 15–45. MR 88b:20007 Zbl 0606.20003

[15] H. Li, W. Liu, Line-primitive 2-$(v,k,1)$ designs with $k/(k,v) \leqslant 10$. *J. Combin. Theory Ser. A* **93** (2001), 153–167. MR 2002f:05031 Zbl 0972.05007

[16] W. H. Mills, Two new block designs. *Utilitas Math.* **7** (1975), 73–75. MR 51 #10123 Zbl 0312.05009

[17] W. Nickel, A. C. Niemeyer, C. M. O'Keefe, T. Penttila, C. E. Praeger, The block-transitive, point-imprimitive 2-$(729,8,1)$ designs. *Appl. Algebra Engrg. Comm. Comput.* **3** (1992), 47–61. MR 96j:05024 Zbl 0766.05010

[18] C. M. O'Keefe, T. Penttila, C. E. Praeger, Block-transitive, point-imprimitive designs with $\lambda = 1$. *Discrete Math.* **115** (1993), 231–244. MR 94e:05070 Zbl 0777.05015

[19] J. Siemons, B. Webb, On a problem of Wielandt and a question by Dembowski. In: *Advances in finite geometries and designs* (*Chelwood Gate*, 1990), 353–358, Oxford Univ. Press 1991. MR 93d:20006 Zbl 0731.20004

[20] J. H. Walter, The characterization of finite groups with abelian Sylow 2-subgroups. *Ann. of Math.* (2) **89** (1969), 405–514. MR 40 #2749 Zbl 0184.04605

Received 19 December, 2000; revised 6 August, 2002

A. Delandtsheer, Faculty of Applied Sciences, CP165/11 Mathematics, Université Libre de Bruxelles, Avenue F. D. Roosevelt 50, B 1050 Bruxelles, Belgium.
Email: adelandt@ulb.ac.be

A. C. Niemeyer, Ch. E. Praeger, Department of Mathematics & Statistics, University of Western Australia, 35 Stirling Highway, Crawley, Western Australia 6009, Australia.
Email: {alice, praeger}@maths.uwa.edu.au