Adv. Geom. **4** (2004), 105–117

Advances in Geometry

A four-class association scheme derived from a hyperbolic quadric in PG(3, q)

Tatsuya Fujisaki

(Communicated by E. Bannai)

Abstract. We prove the existence of a four-class association scheme on the set of external lines with respect to a hyperbolic quadric of $\mathrm{PG}(3,q)$ where $q \geqslant 4$ is a power of 2. This result is an analogue of the one by Ebert, Egner, Hollmann and Xiang. Taking a quotient of this association scheme yields a strongly regular graph of Latin square type. We show that this strongly regular graph can also be obtained by a generalization of the construction given by Mathon.

1 Introduction

In the paper [5], Ebert, Egner, Hollmann and Xiang constructed a four-class symmetric association scheme by using the set of secant lines with respect to an ovoid $\mathcal{O}$ of $\mathrm{PG}(3,q)$ for $q \geqslant 4$ a power of 2. We can regard this association scheme as defined on the set of external lines by taking the null polarity with respect to $\mathcal{O}$. In this paper, we consider an analogous construction by a hyperbolic quadric. We construct a four-class symmetric association scheme by using the set of external lines with respect to a hyperbolic quadric of $\mathrm{PG}(3,q)$. Each relation is invariant under the action of the orthogonal group $O^+(4,q)$ but the set of relations is not the set of orbitals on the set of external lines. Indeed, there are more orbitals than relations. Moreover, a quotient of this association scheme forms a strongly regular graph of Latin square type. We also prove that this strongly regular graph is isomorphic to the one constructed from a direct product of a pseudo-cyclic symmetric association scheme defined by the action of $\mathrm{SL}(2,q)$ on the right cosets $\mathrm{SL}(2,q)/O^-(2,q)$, which is a generalization of the construction given by Mathon [10]. This isomorphism is obtained by an isomorphism between $\mathrm{SL}(2,q)^2$ and $\Omega^+(4,q)$.

2 Association schemes, strongly regular graphs and projective spaces

Let X be a finite set and let $\{R_i\}_{0 \leqslant i \leqslant d}$ be relations on X, that is, subsets of $X \times X$. Then $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ is called a *d-class symmetric association scheme* if the following conditions are satisfied.

1. $\{R_i\}_{0 \leqslant i \leqslant d}$ is a partition of $X \times X$.
2. R_0 is diagonal, that is, $R_0 = \{(x, x) \mid x \in X\}$.
3. $\{(y, x) \mid (x, y) \in R_i\} = R_i$ for any i.
4. For any $i, j, k \in \{0, 1, \dots, d\}$, $p_{ij}^k := |\{z \in X \mid (x, z) \in R_i, (y, z) \in R_j\}|$ is independent of the choice of (x, y) in R_k.

For $i \in \{0, \dots, d\}$, let A_i be the adjacency matrix of the relation R_i, that is, A_i is indexed by X and

$$(A_i)_{xy} := \begin{cases} 1 & \text{if } (x, y) \in R_i, \\ 0 & \text{if } (x, y) \notin R_i. \end{cases}$$

Then we have

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k$$

for any $i, j \in \{0, \dots, d\}$. So $A_0, A_1, \dots, A_d$ form a basis of the commutative algebra generated by $A_0, A_1, \dots, A_d$ over the complex field (which is called the Bose–Mesner algebra of $\mathfrak{X}$). Moreover this algebra has a unique basis $E_0, E_1, \dots, E_d$ of primitive idempotents. One of the primitive idempotents is $|X|^{-1}J$ where J is the matrix whose entries are all 1. So we may assume $E_0 = |X|^{-1}J$. Let $P = (p_j(i))_{0 \leqslant i, j \leqslant d}$ be the matrix defined by

$$(A_0 \quad A_1 \dots A_d) = (E_0 \quad E_1 \dots E_d)P.$$

We call P the first eigenmatrix of $\mathfrak{X}$. Note that $\{p_j(i) \mid 0 \leqslant i \leqslant d\}$ is the set of eigenvalues of A_j. The first eigenmatrix satisfies the orthogonality relation:

$$\sum_{v=0}^{d} \frac{1}{k_v} p_v(i) p_v(j) = \frac{|X|}{m_i} \delta_{ij},$$

where $k_i = p_{ii}^0$ and $m_i = \operatorname{rank} E_i$. We say that $\mathfrak{X}$ is *pseudo-cyclic* if there exists an integer m such that $\operatorname{rank} E_i = m$ for all $i \in \{1, \dots, d\}$. Note that in this case, $|X| = dm + 1$ and $k_i = p_{ii}^0 = m$ for all $i \in \{1, \dots, d\}$ (see [1, p. 76]).

Let G be a finite group and K be a subgroup of G. Then G acts naturally on the set $G/K \times G/K$ with orbitals $R_0, R_1, \dots, R_d$, where we let $R_0 = \{(gK, gK) \mid gK \in G/K\}$. If all orbitals are self-paired, then $\mathfrak{X} = (G/K, \{R_i\}_{0 \leqslant i \leqslant d})$ forms a symmetric association scheme. We denote this association scheme by $\mathfrak{X}(G, K)$.

For a strongly regular graph with parameters (n, k, λ, μ), one of the eigenvalues of its adjacency matrix is k, and the others θ_1, θ_2 are the solutions of $x^2 + (\mu - \lambda)x + (\mu - k) = 0$. We can identify the pair of a strongly regular graph and its complement with a two-class symmetric association scheme whose first eigenmatrix is

$$\begin{bmatrix} 1 & k & n-k-1 \\ 1 & \theta_1 & -1-\theta_1 \\ 1 & \theta_2 & -1-\theta_2 \end{bmatrix} \tag{1}$$

In the paper [10], Mathon constructed a strongly regular graph from the pseudo-cyclic symmetric association scheme $\mathfrak{X}(\mathrm{SL}(2,8), O^-(2,8))$. The next lemma is a generalization of this construction, due to Brouwer and Mathon [2]. Godsil [7] remarks that it can also be proved by Koppinen's identity [9] (see also [6, Theorem 2.4.1]).

Lemma 2.1. *Let $\mathfrak{X} = (X, \{R_i\}_{0 \leqslant i \leqslant d})$ be a pseudo-cyclic symmetric association scheme on $dm + 1$ points. Then the graph $\Delta(\mathfrak{X})$ whose vertex set is $X \times X$, where two distinct vertices (x, y) and (x', y') are adjacent if and only if $(x, x'), (y, y') \in R_i$ for some $i \neq 0$, is a strongly regular graph of Latin square type with parameters*

$$(|X|^2, m(|X| - 1), |X| + m(m-3), m(m-1)).$$

Proof. The direct product of $\mathfrak{X}$ is $(X \times X, \{R_{ij}\}_{0 \leqslant i,j \leqslant d})$, where

$$R_{ij} := \{((x, y), (x', y')) \mid (x, x') \in R_i, (y, y') \in R_j\}.$$

If P is the first eigenmatrix of $\mathfrak{X}$, then $P \otimes P$ is the first eigenmatrix of $(X \times X, \{R_{ij}\}_{0 \leqslant i,j \leqslant d})$. The edge set of $\Delta(\mathfrak{X})$ is defined to be $\bigcup_{j=1}^{d} R_{jj}$. Then the eigenvalues of the adjacency matrix of $\Delta(\mathfrak{X})$ are

$$\left\{ \sum_{j=1}^{d} p_j(i) p_j(i') \mid 0 \leqslant i, i' \leqslant d \right\}.$$

Since $\mathfrak{X}$ is psuedo-cyclic, $k_0 = m_0 = 1$, $k_j = m_i = m$ for $i, j \neq 0$. Hence the orthogonality relation implies

$$\sum_{j=1}^{d} p_j(i) p_j(i') = \frac{m|X|}{m_i} \delta ii' - m = \begin{cases} m(|X| - 1) & \text{if } i = i' = 0, \\ |X| - m & \text{if } i = i' \neq 0, \\ -m & \text{if } i \neq i'. \end{cases}$$

Therefore $\Delta(\mathfrak{X})$ has three eigenvalues. This implies that $\Delta(\mathfrak{X})$ is strongly regular. The parameters of $\Delta(\mathfrak{X})$ can easily be calculated. □

In Lemma 2.1, if $\mathfrak{X} = \mathfrak{X}(G, K)$ for some finite group G and its subgroup K, then $\Delta(\mathfrak{X})$ has the following geometric interpretation.

Lemma 2.2. *Suppose that a finite group G and its subgroup K form a pseudo-cyclic symmetric association scheme $\mathfrak{X} = \mathfrak{X}(G, K)$. Then the graph $\Delta(\mathfrak{X})$ of Lemma 2.1 is isomorphic to the collinearity graph of the coset geometry $(G^2/K^2, G^2/D(G), *)$ where $D(G) := \{(x, x) \mid x \in G\}$ and for $x_1, x_2, y_1, y_2 \in G$, $(x_1, x_2)K^2 * (y_1, y_2)D(G)$ if and only if $(x_1, x_2)K^2 \cap (y_1, y_2)D(G) = \varnothing$.*

Proof. Since each relation of $\mathfrak{X}(G,K)$ is an orbital of the action of G on $G/K \times G/K$, two pairs (x_1K, y_1K), (x_2K, y_2K) are adjacent in the graph $\Delta(\mathfrak{X}(G,K))$ if and only if there exists $w \in G$ such that $y_1K = wx_1K$, $y_2K = wx_2K$. On the other hand, two pairs $(x_1, y_1)K^2$, $(x_2, y_2)K^2$ are adjacent in the collinearity graph of $(G^2/K^2, G^2/D(G), *)$ if and only if $(x_1^{-1}x_2, y_1^{-1}y_2)$ is in $K^2D(G)K^2$ (cf. [4, p. 15]).

For $x_1, x_2, y_1, y_2 \in G$,

$$\begin{aligned}(x_1^{-1}x_2, y_1^{-1}y_2) \in K^2D(G)K^2 &\Leftrightarrow x_1^{-1}x_2, y_1^{-1}y_2 \in KwK \text{ for some } w \in G,\\ &\Leftrightarrow x_1^{-1}x_2 \in Ky_1^{-1}y_2K\\ &\Leftrightarrow y_1kx_1^{-1} = y_2k'x_2^{-1} \text{ for some } k, k' \in K,\\ &\Leftrightarrow y_1 \in wx_1K, y_2 \in wx_2K \text{ for some } w \in G,\end{aligned}$$

Hence the mapping $G/K \times G/K \ni (xK, yK) \mapsto (x, y)K^2 \in G^2/K^2$ is an isomorphism between the above two graphs. □

For the rest of this section, we recall some terminology on finite projective spaces. In this paper, let q be a power of 2 and let $\mathrm{PG}(3, q)$ be the three-dimensional projective space over $\mathrm{GF}(q)$. For a non-degenerate quadratic form Q on $\mathrm{GF}(q)^4$, we say that a point $p = \langle v \rangle$ is *singular* if $Q(v) = 0$, and we call the set of singular points a *quadric*. For a set of points X, we say that a line l is *external* (respectively *secant*) to the set X if the number of points in $l \cap X$ is 0 (respectively 2).

It is well known that there are two types of non-degenerate quadratic forms on $\mathrm{GF}(q)^4$, which are called elliptic type or hyperbolic type. For a point p, denote by $p^\perp$ the orthogonal complement of p with respect to the symmetric bilinear form obtained from Q. Define for a line l or a plane π, $l^\perp := \bigcap_{p \in l} p^\perp$, $\pi^\perp := \bigcap_{p \in \pi} p^\perp$.

For a hyperbolic quadric in $\mathrm{PG}(3, q)$, since q is even, the polarity $\perp$ is a null polarity, that is, if p is a point, then $p \in p^\perp$. More precisely, if p is on the hyperbolic quadric, then $p^\perp$ is the plane determined by the two generators of the quadric through p. If p is not on the quadric, then through p there are $q+1$ tangent lines to the quadric and these $q+1$ lines are coplanar. The plane determined by these $q+1$ tangent lines is $p^\perp$. For a line l, we have $l^\perp = \{\pi^\perp \mid l \subseteq \pi\}$. If l is external, then since every plane π containing l satisfies the point $\pi^\perp$ is nonsingular and not on l, the line $l^\perp$ is also external to the quadric and skew to l. On the other hand, for an external line l to an ovoid, the line $l^\perp$ is skew to l and secant to the ovoid (see [8, pp. 24–26]).

A canonical form of the quadratic form of hyperbolic type is

$$Q(x_1, x_2, x_3, x_4) = x_1x_4 + x_2x_3.$$

Denote by $\Omega^+(4, q)$ the commutator group of the orthogonal group defined from the above Q.

3 Main results

A four-class symmetric association scheme on the set of secant lines with respect to any ovoid was constructed:

Theorem 3.1 ([5]). *Let $q = 2^f \geqslant 4$. Then the following relations on the set of secant lines of* $\mathrm{PG}(3,q)$ *with respect to an ovoid*

$$R_1 = \{(l,m) \mid l \cap m \textit{ is a singular point}\}$$

$$R_2 = \{(l,m) \mid l \cap m \textit{ is a nonsingular point}\}$$

$$R_3 = \{(l,m) \mid l^{\perp} \cap m \neq \varnothing\}$$

$$R_4 = \{(l,m) \mid l \cap m = \varnothing, l^{\perp} \cap m = \varnothing\}$$

and the diagonal relation R_0 define a four-class symmetric association scheme.

We can regard the above association scheme as defined on the set of external lines. The relations R_1, R_2, R_3 and R_4 correspond to the following relations on the set of external lines

$$\{(l,m) \mid \langle l,m\rangle^{\perp} \text{ is a singular point}\},$$

$$\{(l,m) \mid \langle l,m\rangle^{\perp} \text{ is a nonsingular point}\},$$

$$\{(l,m) \mid l^{\perp} \cap m \neq \varnothing\},$$

$$\{(l,m) \mid l \cap m = \varnothing, l^{\perp} \cap m = \varnothing\},$$

respectively.

In the paper [5], a plane π is called tangent if its orthogonal complement is a singular point.

For a hyperbolic quadric, we can construct a four-class symmetric association scheme similar to the above one. Let $\mathbf{L}$ be the set of external lines with respect to a hyperbolic quadric in $\mathrm{PG}(3,q)$.

Theorem 3.2. *Let $q = 2^f \geqslant 4$. Then the following relations on the set* $\mathbf{L}$ *of external lines of* $\mathrm{PG}(3,q)$ *with respect to a hyperbolic quadric*

$$R_1 = \{(l,m) \mid l \cap m \textit{ is a point}\}$$

$$R_2 = \{(l,m) \mid m = l^{\perp}\}$$

$$R_3 = \{(l,m) \mid l^{\perp} \cap m \textit{ is a point}\}$$

$$R_4 = \{(l,m) \mid l \cap m = \varnothing, l^{\perp} \cap m = \varnothing\}$$

and the diagonal relation R_0 define a four-class symmetric association scheme.

Moreover we can construct a strongly regular graph from this symmetric association scheme by taking a quotient.

Theorem 3.3. *Let $\Gamma = \Gamma_q$ $(q = 2^f \geqslant 4)$ be the graph with vertex set $\{\{l, l^\perp\} \mid l \in \mathbf{L}\}$, where two distinct vertices of Γ, $\{l, l^\perp\}$, $\{m, m^\perp\}$ are adjacent if and only if $l \cap m \neq \emptyset$ or $l \cap m^\perp \neq \emptyset$. Then Γ is a strongly regular graph of Latin square type with parameters*

$$v = \frac{1}{4}q^2(q-1)^2, \quad k = \frac{1}{2}(q-2)(q+1)^2, \quad \lambda = \frac{1}{2}(3q^2 - 3q - 4), \quad \mu = q(q+1).$$

Note that $l \cap m \neq \emptyset$ is equivalent to $l^\perp \cap m^\perp \neq \emptyset$, and $l \cap m^\perp \neq \emptyset$ is equivalent to $l^\perp \cap m \neq \emptyset$. So the adjacency in Γ is well-defined.

4 Proof of Theorem 3.2

To prove Theorem 3.2, we recall some facts about $\mathrm{PG}(3, q)$ with a hyperbolic quadric from Hirschfeld's book [8, §15–III]. From now on, put $q = 2^f \geqslant 4$. Let Π be the set of planes whose orthogonal complement is a nonsingular point.

Proposition 4.1. *For a hyperbolic quadric in* $\mathrm{PG}(3, q)$, *the following statements hold.*

(i) *A plane containing an external line is in* Π.

(ii) *The number of external lines is $q^2(q-1)^2/2$ and there are $q+1$ planes of Π containing a given external line.*

(iii) *The number of planes in Π is $q(q^2-1)$ and there are $q(q-1)/2$ external lines in a given plane of Π.*

(iv) *For $\pi \in \Pi$, there is no external line through $\pi^\perp$ on π. For a nonsingular point p of π distinct from $\pi^\perp$, there are $q/2$ external lines through p on π.*

(v) *There are $q(q-1)/2$ external lines through a given nonsingular point.*

(Remark: when q is an odd prime power, (i), (ii), (iii) and (v) also hold. For a plane π of Π, $\pi^\perp$ is not in π.)

First we show that the relations $R_0, \dots, R_4$ form a partition of $\mathbf{L} \times \mathbf{L}$. It is clear that any pair (l, m) of $\mathbf{L} \times \mathbf{L}$ is in one of $\{R_i\}_{0 \leqslant i \leqslant 4}$. Since any external line l is skew to $l^\perp$, R_1 and R_2 have no intersection. Suppose that $l, m \in \mathbf{L}$ satisfy that l meets m. Then the point $\langle l, m\rangle^\perp$ is on $l^\perp$, so m is skew to $l^\perp$ by Proposition 4.1 (iv). Hence R_1 and R_3 have no intersection. Therefore $\{R_i\}_{0 \leqslant i \leqslant 4}$ is a partition of $\mathbf{L} \times \mathbf{L}$.

Next we show that each relation is symmetric. It is clear that R_1, R_2 and R_4 are symmetric. If $(l, m) \in R_3$, then $\langle l^\perp, m\rangle$ forms a plane and $\langle l^\perp, m\rangle^\perp = l \cap m^\perp$, hence $(m, l) \in R_3$. Therefore R_3 is also symmetric.

Finally we show that for any $i, j, k \in \{0, \dots, 4\}$,

$$p_{ij}^k = |\{n \in \mathbf{L} \mid (l, n) \in R_i, (n, m) \in R_j\}|$$

is independent of the choice of $(l, m) \in R_k$. The assertion is clear when $k = 0$. For the moment, we put $p_{ij}(l,m) = |\{n \in \mathbf{L} \mid (l,n) \in R_i, (n,m) \in R_j\}|$. We can easily see that when $(l,m) \in R_k$, $p_{0j}(l,m) = \delta_{jk}$. Since each relation is symmetric, $p_{ji}(l,m) = p_{ij}(m,l)$. Since $R_0, \dots, R_4$ form a partition of $\mathbf{L} \times \mathbf{L}$, we have

$$\sum_{i=0}^{4} p_{ii}^0 = |\mathbf{L}| = \frac{1}{2} q^2 (q-1)^2,$$

and

$$\sum_{j=0}^{4} p_{ij}(l,m) = p_{ii}^0$$

for any $i \in \{0, \dots, 4\}$ and for any pair (l,m). Let σ be the permutation $(0,2)(1,3)$ on $\{0, \dots, 4\}$. Then since $(l,m) \in R_i$ if and only if $(l, m^\perp) \in R_{\sigma(i)}$,

$$p_{ij}(l,m) = p_{i\sigma(j)}(l, m^\perp) = p_{\sigma(i)\sigma(j)}(l,m). \tag{2}$$

Hence we only need to show that p_{11}^k $(1 \leqslant k \leqslant 4)$ are independent of the choice of $(l,m) \in R_k$.

Lemma 4.2. *For* $1 \leqslant k \leqslant 4$, p_{11}^k *is independent of the choice of* $(l,m) \in R_k$ *and*

$$p_{11}^0 = \frac{1}{2}(q-2)(q+1)^2, \quad p_{11}^1 = q^2 - \frac{3}{2}q - 2,$$

$$p_{11}^2 = 0, \quad p_{11}^3 = \frac{1}{2}q^2, \quad p_{11}^4 = \frac{1}{2}q(q+1).$$

Proof. Fix $l \in \mathbf{L}$. Any line which meets l in a point is in a plane through l, and conversely any line in a plane through l meets l in a point. Hence by Proposition 4.1 (ii) and (iii),

$$\begin{aligned} p_{11}^0 &= |\{n \in \mathbf{L} \mid (l,n) \in R_1\}| \\ &= \sum_{\pi \in \Pi_l} |\{n \in \mathbf{L} \mid n \subset \pi, n \neq l\}| \\ &= (q+1) \times \left(\frac{1}{2} q(q-1) - 1\right) \\ &= \frac{1}{2}(q-2)(q+1)^2 \end{aligned}$$

where $\Pi_l := \{\pi \in \Pi \mid l \subset \pi\}$.

For $(l,m) \in R_1$, if $n \in \mathbf{L}$ meets both l and m, then n has a point $l \cap m$ or n is in the plane $\langle l,m\rangle$. Hence by Proposition 4.1 (iii)–(v),

$$\begin{aligned} p_{11}^1 &= |\{n \in \mathbf{L} \mid (l,n),(n,m) \in R_1\}| \\ &= |\{n \in \mathbf{L} \mid n \subseteq \langle l,m\rangle, n \neq l,m\}| + |\{n \in \mathbf{L} \mid l \cap m \in n \not\subseteq \langle l,m\rangle\}| \\ &= \left(\frac{1}{2}q(q-1)-2\right) + \left(\frac{1}{2}q(q-1)-\frac{1}{2}q\right) \\ &= q^2 - \frac{3}{2}q - 2. \end{aligned}$$

From (2), we have $p_{11}^2 = 0$. For $(l,m) \in R_3 \cup R_4$, we have

$$|\{n \in \mathbf{L} \mid (l,n),(n,m) \in R_1\}| = \sum_{\pi \in \Pi_l} |\{n \in \mathbf{L} \mid m \cap \pi \in n \subseteq \pi\}|.$$

If $(l,m) \in R_3$, then there is just one plane $\pi_0 = \langle l, l^\perp \cap m\rangle \in \Pi_l$ such that $\pi_0^\perp = m \cap \pi_0$. By Proposition 4.1 (iv), there is no line of $\mathbf{L}$ through $m \cap \pi_0$ and in π_0, and for other plane π, there are $q/2$ lines of $\mathbf{L}$ through $m \cap \pi$ and in π. Hence

$$\begin{aligned} p_{11}^3 &= |\{n \in \mathbf{L} \mid (l,n),(n,m) \in R_1\}| \\ &= \sum_{\pi \in \Pi_l \setminus \{\pi_0\}} |\{n \in \mathbf{L} \mid m \cap \pi \in n \subseteq \pi\}| \\ &= q \times \frac{1}{2}q. \end{aligned}$$

For $(l,m) \in R_4$, any plane π of Π_l has $q/2$ lines of $\mathbf{L}$ through $m \cap \pi$. So

$$p_{11}^4 = |\{n \in \mathbf{L} \mid (l,n),(n,m) \in R_1\}| = (q+1) \times \frac{1}{2}q. \qquad \square$$

Therefore $(\mathbf{L}, \{R_i\}_{0 \leqslant i \leqslant 4})$ becomes a symmetric association scheme. For $i \in \{0,\dots,d\}$, let $B_i := (p_{ij}^k)_{0 \leqslant j,k \leqslant 4}$. Then B_0 is the identity matrix,

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ p_{11}^0 & q^2 - 3/2q - 2 & 0 & q^2/2 & q(q+1)/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & q^2/2 & p_{11}^0 & q^2 - 3/2q - 2 & q(q+1)/2 \\ 0 & q^2(q-3)/2 & 0 & q^2(q-3)/2 & s \end{pmatrix},$$

where $s = (q+1)(q^2 - 3q - 2)/2$,

$$B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & q^2/2 & p_{11}^0 & q^2 - 3/2q - 2 & q(q+1)/2 \\ 0 & 1 & 0 & 0 & 0 \\ p_{11}^0 & q^2 - 3/2q - 2 & 0 & q^2/2 & q(q+1)/2 \\ 0 & q^2(q-3)/2 & 0 & q^2(q-3)/2 & s \end{pmatrix},$$

and

$$B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & q^2(q-3)/2 & 0 & q^2(q-3)/2 & s \\ 0 & 0 & 0 & 0 & 1 \\ 0 & q^2(q-3)/2 & 0 & q^2(q-3)/2 & s \\ p_{44}^0 & q(q-3)(q^2-3q-2)/2 & p_{44}^0 & s & t \end{pmatrix},$$

where $p_{44}^0 = q(q-2)(q-3)(q+1)/2$ and $t = q(q-3)(q^2-3q-2)/2$. The first eigenmatrix of this association scheme is given by

$$P = \begin{pmatrix} 1 & (q-2)(q+1)^2/2 & 1 & (q-2)(q+1)^2/2 & p_{44}^0 \\ 1 & (q-2)(q+1)/2 & -1 & -(q-2)(q+1)/2 & 0 \\ 1 & -(q+1) & -1 & q+1 & 0 \\ 1 & -(q+1) & 1 & -(q+1) & 2q \\ 1 & (q^2-3q-2)/2 & 1 & (q^2-3q-2)/2 & -q(q-3) \end{pmatrix}.$$

5 Proof of Theorem 3.3

In this section, we prove Theorem 3.3 by using Theorem 3.2. The number of vertices of the graph Γ is $|\mathbf{L}|/2 = q^2(q-1)^2/4$. For a pair $\{l, l^\perp\} \in V\Gamma$,

$$\begin{aligned} &\{\{m, m^\perp\} \in V\Gamma \mid \{m, m^\perp\} \text{ is adjacent to } \{l, l^\perp\}\} \\ &= \{\{m, m^\perp\} \in V\Gamma \mid m \text{ meets } l \text{ in a point}\} \\ &= \{\{m, m^\perp\} \in V\Gamma \mid (l, m) \in R_1\}. \end{aligned}$$

So, the size of this set is $p_{11}^0 = (q-2)(q+1)^2/2$, which is just k in the definition of strongly regular graph. Next choose $\{l, l^\perp\}, \{m, m^\perp\} \in V\Gamma$ which are adjacent in Γ. We may suppose that l meets m in a point. Then

$$\begin{aligned}
&\{\{n, n^\perp\} \in V\Gamma \mid \{n, n^\perp\} \text{ is adjacent to both } \{l, l^\perp\} \text{ and } \{m, m^\perp\}\} \\
&\quad = \{\{n, n^\perp\} \in V\Gamma \mid n \text{ meets both } l \text{ and } m \text{ in a point}\} \\
&\qquad \cup \{\{n, n^\perp\} \in V\Gamma \mid n \text{ meets both } l \text{ and } m^\perp \text{ in a point}\} \\
&\quad = \{\{n, n^\perp\} \in V\Gamma \mid (l, n) \in R_1, (m, n) \in R_1 \cup R_3\}.
\end{aligned}$$

Hence the size of this set is $p_{11}^1 + p_{13}^1 = (3q^2 - 3q - 4)/2$. This is just λ in the definition of strongly regular graph.

Similarly, for $\{l, l^\perp\}, \{m, m^\perp\} \in V\Gamma$ which are not adjacent in Γ, since $(l, m) \in R_4$,

$$\begin{aligned}
&|\{\{n, n^\perp\} \in V\Gamma \mid \{n, n^\perp\} \text{ is adjacent to both } \{l, l^\perp\} \text{ and } \{m, m^\perp\}\}| \\
&\quad = p_{11}^4 + p_{13}^4 = q(q+1).
\end{aligned}$$

This is just μ in the definition of strongly regular graph.

Alternatively, we can prove Theorem 3.3 by using the quotient association scheme (cf. [1, p. 139, Theorem 9.4]). In the association scheme of Theorem 3.2, $R_0 \cup R_2$ is an equivalence relation on $\mathbf{L}$. So we can define a quotient association scheme on the set of equivalence classes $\{\{l, l^\perp\} \mid l \in \mathbf{L}\}$ whose relations are

$$\begin{aligned}
&\{(\{l, l^\perp\}, \{m, m^\perp\}) \mid (l, m) \in R_1 \cup R_3\} = \text{the edge set of } \Gamma, \\
&\{(\{l, l^\perp\}, \{m, m^\perp\}) \mid (l, m) \in R_4\},
\end{aligned}$$

and the diagonal relation. The first eigenmatrix of this association scheme can be computed from P (cf. [1, p. 148]):

$$\begin{pmatrix}
1 & (q-2)(q+1)^2/2 & q(q-2)(q-3)(q+1)/4 \\
1 & -(q+1) & q \\
1 & (q^2 - 3q - 2)/2 & -q(q-3)/2
\end{pmatrix}.$$

The first relation forms a strongly regular graph whose parameters are calculated from the second column of the above first eigenmatrix.

6 Another construction of Γ_q

In this section, we will give another construction of the strongly regular graph Γ_q. This construction uses a method which generalizes a construction of Mathon ([10, p. 137], see also [3, pp. 96–97]).

Let $G = \mathrm{SL}(2, q)$, $K = O^-(2, q)$. Then $\mathfrak{X}(G, K)$ is a $(q-2)/2$-class pseudo-cyclic symmetric association scheme (cf. [3, p. 96]). By Lemma 2.1, we can construct a strongly regular graph $\Delta(\mathfrak{X}(G, K))$ with parameters

$$\left(\frac{1}{4}q^2(q-1)^2, \frac{1}{2}(q-2)(q+1)^2, \frac{1}{2}(3q^2 - 3q - 4), q(q+1)\right)$$

which are the same as those of Γ_q. We shall prove that these graphs are isomorphic.

To show this, we use the isomorphism $G^2 \simeq \Omega^+(4,q)$ which maps (X, Y) to $X \otimes Y$ (see [11, p. 199]). Let l_0 be the external line generated by $v_1 = {}^t(0,1,1,0)$, $v_2 = {}^t(1,1,0,\alpha)$, where α is an element of $\mathbb{F}_q$ such that the polynomial $x^2 + x + \alpha$ is irreducible over $\mathbb{F}_q$. For an external line l, there are $2(q+1)$ bases (u_1, u_2) of l such that $Q(xu_1 + yu_2) = x^2 + xy + \alpha y^2$ for any $x, y \in \mathbb{F}_q$. Indeed, by Witt's Theorem, K acts regularly on the set of bases (u_1, u_2) of l with the above condition. It follows that the size of this set is equal to $|K| = 2(q+1)$. Let $\mathscr{P}$ be the set of nonsingular points in PG$(3, q)$ and let $\mathscr{L} = \{l \cup l^\perp \mid l \in \mathbf{L}\}$. Then the following lemma holds.

Lemma 6.1. *The group $\Omega^+(4,q) = \{X \otimes Y \mid X, Y \in G\}$ is flag-transitive on the incidence structure $(\mathscr{P}, \mathscr{L}, \in)$. Under the isomorphism $G^2 \simeq \Omega^+(4,q)$, the groups $D(G), K^2$ are the stabilizers of an element of $\mathscr{P}, \mathscr{L}$, respectively.*

Proof. Let $X = (x_{ij})_{1 \leqslant i,j \leqslant 2}$, $Y = (y_{ij})_{1 \leqslant i,j \leqslant 2} \in G$. Since

$$X \otimes Y = \begin{pmatrix} x_{11}y_{11} & x_{11}y_{12} & x_{12}y_{11} & x_{12}y_{12} \\ x_{11}y_{21} & x_{11}y_{22} & x_{12}y_{21} & x_{12}y_{22} \\ x_{21}y_{11} & x_{21}y_{12} & x_{22}y_{11} & x_{22}y_{12} \\ x_{21}y_{21} & x_{21}y_{22} & x_{22}y_{21} & x_{22}y_{22} \end{pmatrix},$$

$X \otimes Y$ fixes v_1 if and only if

$$\begin{aligned} x_{11}y_{12} + x_{12}y_{11} = x_{21}y_{22} + x_{22}y_{21} = 0, \\ x_{11}y_{22} + x_{12}y_{21} = x_{21}y_{12} + x_{22}y_{11} = 1. \end{aligned}$$

This implies

$$\Omega^+(4,q)_{v_1} = \{X \otimes X \mid X \in G\} \simeq D(G). \tag{3}$$

For $X \in G$, $X \otimes X$ fixes v_2 if and only if

$$\begin{aligned} x_{11}^2 + x_{11}x_{12} + \alpha x_{12}^2 &= 1, \\ x_{11}x_{21} + x_{12}x_{21} + \alpha x_{12}x_{22} &= 0, \\ x_{21}^2 + x_{21}x_{22} + \alpha x_{22}^2 &= \alpha. \end{aligned}$$

From these, we have

$$\Omega^+(4,q)_{v_1,v_2} = \left\{ X \otimes X \mid X = \begin{pmatrix} a & b \\ \alpha b & a+b \end{pmatrix} \in G \right\}$$

which is of order $q+1$. Hence

$$\begin{aligned} |\{(Mv_1, Mv_2) \mid M \in \Omega^+(4,q)\}| &= |\Omega^+(4,q)|/(q+1) \\ &= q^2(q-1)^2(q+1). \end{aligned}$$

Since $Q(xv_1 + yv_2) = x^2 + xy + \alpha y^2$ for any $x, y \in \mathbb{F}_q$,

$$\begin{aligned} &|\{(u_1, u_2) \mid Q(xu_1 + yu_2) = x^2 + xy + \alpha y^2 \text{ for all } x, y \in \mathbb{F}_q\}| \\ &\quad = |\mathbf{L}| \times 2(q+1) \\ &\quad = q^2(q-1)^2(q+1). \end{aligned}$$

Hence $\Omega^+(4,q)$ acts transitively on the set of pairs (u_1, u_2) such that $Q(xu_1 + yu_2) = x^2 + xy + \alpha y^2$ for any $x, y \in \mathbb{F}_q$. In particular, $\Omega^+(4,q)$ is flag-transitive on $(\mathscr{P}, \mathscr{L}, \in)$.

The equality (3) means that the stabilizer of $\langle v_1 \rangle \in \mathscr{P}$ is isomorphic to $D(G)$. Let

$$A := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} a_0 & b_0 \\ \alpha b_0 & a_0 + b_0 \end{pmatrix} \in G$$

such that B is of order $q+1$. Then the group $\langle A, B \rangle$ is isomorphic to K. $A \otimes I$, $I \otimes A$ interchange l_0 and $l_0^{\perp}$, while $B \otimes I$, $I \otimes B$ fix l_0 and $l_0^{\perp}$. So $\{X \otimes Y \mid X, Y \in \langle A, B \rangle\}$ is a subgroup of $\Omega^+(4,q)_{l_0 \cup l_0^{\perp}}$. Since $\Omega^+(4,q)_{l_0 \cup l_0^{\perp}}$ has order $4(q+1)^2 = |K|^2$, we have that $\Omega^+(4,q)_{l_0 \cup l_0^{\perp}}$ is isomorphic to K^2. □

Theorem 6.2. *The graph Γ_q is isomorphic to $\Delta(\mathfrak{X}(\mathrm{SL}(2,q), O^-(2,q)))$.*

Proof. The graph Γ_q is isomorphic to the collinearity graph of the dual of the incidence structure $(\mathscr{P}, \mathscr{L}, \in)$. From Lemma 6.1, the dual of $(\mathscr{P}, \mathscr{L}, \in)$ is isomorphic to the coset geometry $(G^2/K^2, G^2/D(G), *)$ defined in Lemma 2.2. From Lemma 2.2, the collinearity graph of $(G^2/K^2, G^2/D(G), *)$ is isomorphic to $\Delta(\mathfrak{X}(G,K))$. Therefore Γ_q is isomorphic to $\Delta(\mathfrak{X}(G,K))$. □

References

[1] E. Bannai, T. Ito, *Algebraic combinatorics. I.* Benjamin/Cummings Publishing Co. 1984. MR 87m:05001 Zbl 0555.05019

[2] A. E. Brouwer, R. Mathon, unpublished.

[3] A. E. Brouwer, J. H. van Lint, Strongly regular graphs and partial geometries. In: *Enumeration and design* (*Waterloo, Ont.*, 1982), 85–122, Academic Press 1984. MR 87c:05033 Zbl 0555.05016

[4] P. Dembowski, *Finite geometries*. Springer 1968. MR 38 #1597 Zbl 0159.50001

[5] G. L. Ebert, S. Egner, H. D. L. Hollmann, Q. Xiang, A four-class association scheme. *J. Combin. Theory Ser. A* **96** (2001), 180–191. MR 2002j:05152 Zbl 1010.05083

[6] C. D. Godsil, Association schemes. Preprint.

[7] C. D. Godsil, private communication.

[8] J. W. P. Hirschfeld, *Finite projective spaces of three dimensions*. Oxford Univ. Press 1985. MR 87j:51013 Zbl 0574.51001

[9] M. Koppinen, On algebras with two multiplications, including Hopf algebras and Bose-Mesner algebras. *J. Algebra* **182** (1996), 256–273. MR 97e:16092 Zbl 0897.16022

[10] R. Mathon, 3-class association schemes. In: *Proceedings of the Conference on Algebraic Aspects of Combinatorics* (*Univ. Toronto, Toronto, Ont.*, 1975), 123–155. Congressus Numerantium, No. XIII, Utilitas Math., Winnipeg, Man. 1975. MR 54 #2503 Zbl 0326.05023

[11] D. E. Taylor, *The geometry of the classical groups*. Heldermann 1992. MR 94d:20028 Zbl 0767.20001

Received 3 June, 2002

T. Fujisaki, Graduate school of Mathematics, Kyushu University, 6-10-1 Hakozaki, Fukuoka 812-8581, Japan
Email: fujisaki@math.kyushu-u.ac.jp