Adv. Geom. **4** (2004), 9–17

Advances in Geometry

A geometrical construction of the oval(s) associated with an α-flock

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(Communicated by T. Penttila)

Abstract. It is known, via algebraic methods, that a flock of a quadratic cone in $\mathrm{PG}(3,q)$ gives rise to a family of $q+1$ ovals of $\mathrm{PG}(2,q)$ and similarly that a flock of a cone over a translation oval that is not a conic gives rise to an oval of $\mathrm{PG}(2,q)$. In this paper we give a geometrical construction of these ovals and provide an elementary geometrical proof of the construction. Further we also give a geometrical construction of a spread of the GQ $T_2(\mathcal{O})$ for $\mathcal{O}$ an oval corresponding to a flock of a translation oval cone in $\mathrm{PG}(3,q)$, previously constructed algebraically.

1 Introduction and definitions

The essence of this paper is a geometrical construction of an oval $\mathcal{O}$ of $\mathrm{PG}(2,q)$, q even, from a flock of a translation oval cone in $\mathrm{PG}(3,q)$ and a spread of the corresponding GQ $T_2(\mathcal{O})$. This construction, along with a geometrical proof that it does indeed give an oval $\mathcal{O}$ and a spread of $T_2(\mathcal{O})$, can be found in Section 3 and preliminary results required can be found in Section 2. Much of this introduction gives the known algebraic constructions of these objects while in Section 4 it is shown that the geometrical construction we present here is the same as the algebraic one.

An *oval* $\mathcal{O}$ of $\mathrm{PG}(2,q)$ is a set of $q+1$ points no three of which are collinear. A line of $\mathrm{PG}(2,q)$ is called an *external* line, a *tangent* line or a *secant* line of $\mathcal{O}$ depending on whether it is incident with zero, one or two points of $\mathcal{O}$, respectively. From this point we assume that q is even. In the case where q is even the tangents to $\mathcal{O}$ are concurrent in a point N called the *nucleus* of $\mathcal{O}$. A *hyperoval* of $\mathrm{PG}(2,q)$ is a set of $q+2$ points no three collinear. An oval together with its nucleus forms a hyperoval of $\mathrm{PG}(2,q)$. If an oval $\mathcal{O}$ has a tangent line ℓ such that there exists a group of q elations of $\mathrm{PG}(2,q)$ each element of which has axis ℓ and fixes $\mathcal{O}$, then $\mathcal{O}$ is called a *translation oval*. The line ℓ is called an *axis* of $\mathcal{O}$. It was proved by Payne in [5] that each translation oval is of the form $\mathcal{D}(\alpha) = \{(1,t,t^\alpha) : t \in \mathrm{GF}(q)\} \cup \{(0,0,1)\}$, for some generator α of $\mathrm{Aut}(\mathrm{GF}(q))$. Note that in the case where $\alpha : x \mapsto x^2$, or abusing notation $\alpha = 2$, that the translation oval is the classical oval, the non-degenerate conic.

Let $\mathcal{K}$ be a quadratic cone in $\mathrm{PG}(3,q)$ with vertex V. A *flock* $\mathcal{F}$ of $\mathcal{K}$ is a set of q

planes of $\mathrm{PG}(3,q)$ partitioning the points of $\mathscr{K}\backslash\{V\}$. If we suppose that $\mathscr{K}$ is defined by the equation $x_0x_2 = x_1^2$, then following Thas in [7] we may write the flock in the form $\mathscr{F} = \{\pi_t : t \in \mathrm{GF}(q)\}$ where

$$\pi_t : a_t x_0 + b_t x_1 + c_t x_2 + x_3 = 0.$$

It follows that $t \mapsto a_t$, $t \mapsto b_t$ and $t \mapsto c_t$ are permutations of $\mathrm{GF}(q)$. Without loss of generality the elements of the flock may be normalised to

$$\pi_t : f(t)x_0 + t^{1/2}x_1 + ag(t)x_2 + x_3 = 0,$$

for permutations f and g of $\mathrm{GF}(q)$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$ and $\mathrm{trace}(a) = 1$. In [3] the authors prove the following theorem concerning flocks of the above form.

Theorem 1.1. *Each of the sets*

$$\left\{\left(1, t, \frac{f(x) + asg(x) + s^{1/2}x^{1/2}}{1 + as + s^{1/2}}\right) : x \in \mathrm{GF}(q)\right\} \cup \{(0,1,0),(0,0,1)\}$$

for $s \in \mathrm{GF}(q)$ *and*

$$\{(1, t, g(t)) : t \in \mathrm{GF}(q)\} \cup \{(0,1,0),(0,0,1)\}$$

is a hyperoval of $\mathrm{PG}(2,q)$.

In [3] the set of $q+1$ functions defining the hyperovals as above is called a *herd*. In [8] Thas gave a geometrical construction of these hyperovals from the flock (although not a geometrical proof of the construction).

Let α be a generator of $\mathrm{Aut}(\mathrm{GF}(q))$, $q = 2^e$. Following Cherowitzo in [2] define an *α-cone* $\mathscr{K}_\alpha$ of $\mathrm{PG}(3,q)$ to be a cone with point vertex V and base an oval equivalent to $\mathscr{D}(\alpha)$. If X is a point of the base oval on an axis, then the line $\langle X, V\rangle$ is called an *axial line* of $\mathscr{K}_\alpha$. A *flock* of $\mathscr{K}_\alpha$, also known as an *α-flock*, is a set of q planes of $\mathrm{PG}(3,q)$ partitioning the points of $\mathscr{K}_\alpha\backslash\{V\}$. If $\mathscr{K}_\alpha$ is defined by the equation $x_1^\alpha = x_0x_2^{\alpha-1}$ and $\mathscr{F}_\alpha$ a flock of $\mathscr{K}_\alpha$, then similarly to the case of a flock of a quadratic cone we may write the elements of $\mathscr{F}_\alpha$ as

$$\pi_t : f(t)x_0 + t^{1/\alpha}x_1 + ag(t)x_2 + x_3 = 0 \quad \text{for } t \in \mathrm{GF}(q),$$

where f and g are permutations of $\mathrm{GF}(q)$ with $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$ and $\mathrm{trace}(a) = 1$. Then Cherowitzo ([2]) proves the following result concerning α-flocks.

Theorem 1.2. *The set* $\{(1, t, f(t)) : t \in \mathrm{GF}(q)\} \cup \{(0,1,0),(0,0,1)\}$ *is a hyperoval of* $\mathrm{PG}(2,q)$.

In Section 3 we give a generalisation of a construction in [8] that each pair (axial line of $\mathscr{K}_\alpha$, flock of $\mathscr{K}_\alpha$) gives rise to an oval of $\mathrm{PG}(2,q)$. In the case where $\mathscr{K}_\alpha$ is a quadratic cone it was shown in [8] that in this way a flock gives rise to the $q+1$ hyperovals of the corresponding herd, while in Section 4 we show that for a general α-flock the oval completes to the hyperoval of Theorem 1.2. In this way we have a geometric proof of Theorem 1.1 and Theorem 1.2.

We now consider the Generalized Quadrangle (GQ) $T_2(\mathcal{O})$ of Tits; see [4]. Let $\mathcal{O}$ be an oval in $\mathrm{PG}(2,q)$ and embed $\mathrm{PG}(2,q)$ in $\mathrm{PG}(3,q)$, then $T_2(\mathcal{O})$ is a GQ of order q and is constructed in the following manner. *Points* are (i) the points of $\mathrm{PG}(3,q)\setminus \mathrm{PG}(2,q)$, (ii) the planes of $\mathrm{PG}(3,q)$ which meet $\mathrm{PG}(2,q)$ in a single point of $\mathcal{O}$ and (iii) a symbol (∞); *lines* are (a) the lines of $\mathrm{PG}(3,q)$, not in $\mathrm{PG}(2,q)$, which meet $\mathrm{PG}(2,q)$ in a single point of $\mathcal{O}$, and (b) the points of $\mathcal{O}$; with *incidence* inherited from $\mathrm{PG}(3,q)$ plus (∞) is incident with all lines of type (b). Note that $T_2(\mathcal{O})$ is the classical GQ $Q(4,q)$ if and only if $\mathcal{O}$ is a conic; see [6, 3.2.2]. A *spread* $\mathscr{S}$ of $T_2(\mathcal{O})$ is a set of lines such that each point of $T_2(\mathcal{O})$ is incident with a unique element of $\mathscr{S}$. It follows that $\mathscr{S}$ has size q^2+1. In [1] the authors show that $\mathscr{S}$ must consist of a point P of $\mathcal{O}$ and the q^2 lines not in $\mathrm{PG}(2,q)$ of q oval cones, $\mathscr{K}_X$, $X \in \mathcal{O}\setminus\{P\}$; where $\mathscr{K}_X$ has vertex X, contains P and has nuclear line $\langle X, N\rangle$, with N the nucleus of the oval $\mathcal{O}$. The following theorem, in an equivalent form, also appears in [1].

Theorem 1.3. *Let* $\mathcal{O} = \{(t,1,f(t)) : t \in \mathrm{GF}(q)\} \cup \{(0,0,1)\}$, *with* $f(0)=0$ *and* $f(1)=1$, *be an oval of* $\mathrm{PG}(2,q)$, *q even. Embed* $\mathrm{PG}(2,q)$ *in* $\mathrm{PG}(3,q)$ *as* $x_2=0$ *and let* α *be a generator of* $\mathrm{Aut}(\mathrm{GF}(q))$. *Let* $\mathscr{K}_t$ *be the cone with vertex* $(t,1,0,f(t))$ *and base* $\{(s^\alpha + a^\alpha g(t)^\alpha, 0, 1, s) : s \in \mathrm{GF}(q)\} \cup \{(0,0,0,1)\}$, with $\mathrm{trace}(a)=1$. *Then* $(0,0,0,1)$ *plus the* q^2 *lines not in* $\mathrm{PG}(2,q)$ *of the cones* $\mathscr{K}_t$ *form a spread of* $T_2(\mathcal{O})$ *if and only if* $\{f(t)x_0 + t^{1/\alpha}x_1 + ag(t)x_2 + x_3 = 0 : t \in \mathrm{GF}(q)\}$ *is an* α-*flock of* $\mathscr{K}_\alpha : x_1^\alpha = x_0 x_2^{\alpha-1}$, *with* $g(0)=0$ *and* $g(1)=1$.

In this way the ovals corresponding to an α-flock, as in Theorem 1.2, are characterised as those for which the corresponding Tits GQ admits a spread of the form above. Our geometrical construction in Section 3 characterises these ovals in the same way and by attaching coordinates in Section 4 we see that it gives a non-algebraic proof of Theorem 1.3.

Now we state our main theorem.

Theorem 1.4. *For* α *a generator of* $\mathrm{GF}(q)$, *q even, let* $\mathscr{K}_\alpha$ *be a cone in* $\mathrm{PG}(3,q)$ *over a translation oval equivalent to* $\mathscr{D}(\alpha) = \{(1,t,t^\alpha) : t \in \mathrm{GF}(q)\} \cup \{(0,0,1)\}$. *If* $\mathscr{F}_\alpha$ *is a flock of* $\mathscr{K}_\alpha$, *then to each pair* (F_α, a), *where a is an axial line of* $\mathscr{K}_\alpha$, *there corresponds an oval* $\mathcal{O}$ *of* $\mathrm{PG}(2,q)$. *Further, there also corresponds a spread* $\mathscr{S}$ *of the generalized quadrangle* $T_2(\mathcal{O})$ *which consists of one point* Y *of* $\mathcal{O}$ *and the* q^2 *lines not in* $\mathrm{PG}(2,q)$ *of q* α-*cones* $\mathscr{K}_X$, *where* $\mathscr{K}_X$ *has vertex* $X \in \mathcal{O}\setminus\{Y\}$, *base oval equivalent to* $\mathscr{D}(\alpha)$ *and is tangent to* $\mathrm{PG}(2,q)$ *at the axial line* $\langle Y, X\rangle$.

Conversely, if a GQ $T_2(\mathcal{O})$ *has such a spread* $\mathscr{S}$, *then there corresponds an* α-*flock giving rise to the oval* $\mathcal{O}$.

In Section 2 we shall state some basic properties of translation ovals and flocks of translation oval cones which shall be used in the proof of Theorem 1.4 in Section 3. In Section 4 we apply coordinates to the construction in the proof of Theorem 1.4 to show that it gives both Theorem 1.2 and Theorem 1.3.

2 Preliminaries

In this section we give some basic results on translation ovals and α-flocks to be used in the proof of our main theorem.

By Payne ([5]) we know that any translation oval of $PG(2, 2^h)$ is equivalent to an oval of the form $\mathscr{D}(\alpha) = \{(1, t, t^\alpha) : t \in GF(q)\} \cup \{(0, 0, 1)\}$ with nucleus $(0, 1, 0)$, where α is a generator of $\mathrm{Aut}(GF(q))$. From this form it is clear that $\mathscr{D}(\alpha)$ is a conic if and only if $\alpha = 2$. In the case where $\mathscr{D}(\alpha)$ is a conic each tangent to $\mathscr{D}(\alpha)$ is an axis of $\mathscr{D}(\alpha)$ and the group of the conic is transitive on the axes. In the case where $\mathscr{D}(\alpha)$ is not a conic then $\mathscr{D}(\alpha)$ has a unique axis $[1, 0, 0]$. From the canonical form of a translation oval it is also straight-forward to see that for a given line ℓ of $PG(2, q)$ and distinct points P, N incident with ℓ that there are exactly $q(q-1)$ ovals equivalent to $\mathscr{D}(\alpha)$ containing P and with nucleus N, such that ℓ is an axis of the oval. If R is a fixed point of $PG(2, q) \backslash \ell$, then there are $q(q-1)$ ovals equivalent to $\mathscr{D}(\alpha)$ with axis ℓ and containing the points P and R.

Another notion that we shall need is that of compatibility of ovals. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two ovals of $PG(2, q)$ and let P be a point of $PG(2, q)$ not on either of the ovals and distinct from their nuclei. Then $\mathcal{O}_1$ and $\mathcal{O}_2$ are *compatible* at P if they have the same nucleus, they have a point Q in common, the line $\langle P, Q\rangle$ is a tangent to both $\mathcal{O}_1$ and $\mathcal{O}_2$ and every secant line to $\mathcal{O}_1$ on P is external to $\mathcal{O}_2$. As a consequence every external line to $\mathcal{O}_1$ on P is a secant line to $\mathcal{O}_2$. In particular we will need information regarding points of compatibility in the case where $\mathcal{O}_1$ and $\mathcal{O}_2$ are both ovals equivalent to $\mathscr{D}(\alpha)$ with a common axis ℓ, common nucleus N, $\ell \cap \mathcal{O}_1 = \ell \cap \mathcal{O}_2 = \{Q\}$ and such that $\mathcal{O}_2$ is the image of $\mathcal{O}_1$ under an elation with axis ℓ and centre Q. Without loss of generality we may assume that $\mathcal{O}_1 = \{(1, u, u^\alpha) : u \in GF(q)\} \cup \{(0, 0, 1)\}$ and that $\mathcal{O}_2 = \{(1, t, t^\alpha + B) : t \in GF(q)\} \cup \{(0, 0, 1)\}$ for $B \in GF(q)$. A point $(0, 1, s)$, $s \neq 0$, on the common axis of $\mathcal{O}_1$ and $\mathcal{O}_2$ is a point of compatibility of $\mathcal{O}_1$ and $\mathcal{O}_2$ if and only if $\mathrm{trace}(B/s^{\alpha/(\alpha-1)}) = 1$, which has $q/2$ solutions for $s \in GF(q)$. Hence $\mathcal{O}_1$ and $\mathcal{O}_2$ have $q/2$ points of compatibility on the common axis.

Now consider a cone $\mathscr{K}_\alpha$ in $PG(3, q)$ with vertex V and base an oval equivalent to $\mathscr{D}(\alpha)$. Let ℓ be an axial line of the cone and let P be any point incident with ℓ distinct from V and let π be any plane of $PG(3, q)$ not containing P. If we project the $q^3 - q^2$ oval sections of $\mathscr{K}_\alpha$ not containing P, from P onto π, then we obtain a one-to-one correspondence between this set and the $q^2(q-1)$ ovals of π equivalent to $\mathscr{D}(\alpha)$ that contain the point $Y = \ell \cap \pi$ and have axis $n = \pi_\ell \cap \pi$, where π_ℓ is the plane tangent to $\mathscr{K}_\alpha$ at ℓ. Similarly, the q^2 oval sections of $\mathscr{K}_\alpha$ containing P are in one-to-one correspondence with the q^2 lines of π not incident with Y. This correspondence is the *planar representation* of $\mathscr{K}_\alpha$. If we consider a set of q oval sections of $\mathscr{K}_\alpha$ that are mutually tangent at a point of $\langle P, V\rangle \backslash \{P, V\}$, then in the planar representation this set of q ovals is called an *axial linear pencil* of ovals. Equivalently, such a set of ovals may be described as the images of an oval equivalent to $\mathscr{D}(\alpha)$ under the group of elations with axis the axis of the oval and centre the point of the oval on the axis.

Now consider a flock $\mathscr{F} = \{\pi_1, \dots, \pi_q\}$ of $\mathscr{K}_\alpha$. Without loss of generality suppose that $P \in \pi_q$. For $i = 1, \dots, q-1$ let the projection of the oval $\pi_i \cap \mathscr{K}_\alpha$ from P onto π be $\mathscr{O}_i$ and let w denote the line $\pi_q \cap \pi$. Then it follows that in the planar representation of $\mathscr{K}_\alpha$ that $\mathscr{F}$ is the set $\{\mathscr{O}_1, \dots, \mathscr{O}_{q-1}, w\}$. Thus $\mathscr{O}_1, \dots, \mathscr{O}_{q-1}, w$ partition the points of $\pi \backslash n$, and it also follows that the nuclei of the $\mathscr{O}_i$ are distinct points of $n \backslash \{Y\}$ and that the line w intersects n in the remaining point of $n \backslash \{Y\}$. Conversely, any such set $\{\mathscr{O}_1, \dots, \mathscr{O}_{q-1}, w\}$ partitioning the points of $\pi \backslash \{n\}$ corresponds to a flock of $\mathscr{K}_\alpha$.

3 Proof of Theorem 1.4

Suppose $\mathscr{F}_\alpha$ is a flock of $\mathscr{K}_\alpha$ and a an axis of the base oval of $\mathscr{K}_\alpha$. If V is the vertex of $\mathscr{K}_\alpha$, then $\langle V, a \rangle$ contains the axial line ℓ of $\mathscr{K}_\alpha$. Then, as in Section 2, if we project the elements of $\mathscr{F}_\alpha$ from a non-vertex point P of ℓ onto a plane π, not containing P, we obtain a planar representation $\{\mathscr{O}_1, \mathscr{O}_2, \dots, \mathscr{O}_{q-1}, w\}$ of $\mathscr{F}_\alpha$. Let the common point of the ovals $\mathscr{O}_1, \dots, \mathscr{O}_{q-1}$ be Y, the common axis of the ovals be n and $n \cap w = X'$.

Now consider two other planes $\mathrm{PG}(2,q)$ and ξ, such that $\mathrm{PG}(2,q) \cap \pi = n$, $\pi \cap \mathrm{PG}(2,q) \cap \xi = \{Y\}$, $\mathrm{PG}(2,q) \cap \xi = m$ and $\pi \cap \xi = u$. In ξ we consider an oval $\mathscr{O}'_1$ equivalent to $\mathscr{D}(\alpha)$ such that $\mathscr{O}'_1$ has axis m, contains the point Y on m, and has nucleus N. Let $\{\mathscr{O}'_1, \mathscr{O}'_2, \dots, \mathscr{O}'_q\}$ be the axial linear pencil containing $\mathscr{O}'_1$ with axis m. The ovals $\mathscr{O}'_1, \mathscr{O}'_2, \dots, \mathscr{O}'_q$ partition $\xi \backslash m$, and in particular the points of $u \backslash \{Y\}$. Consequently we may choose indices such that $\mathscr{O}_i \cap \mathscr{O}'_i = \{Y, W_i\}$, with $W_i \in u$ and $i = 1, 2, \dots, q-1$.

We now show that for each $i = 1, 2, \dots, q-1$ there is a unique cone containing $\mathscr{O}_i$ and $\mathscr{O}'_i$. Since n and m are tangents to $\mathscr{O}_i$ and $\mathscr{O}'_i$ at Y, respectively, it follows that the vertex of any cone containing the two ovals must be in the plane $\langle n, m \rangle = \mathrm{PG}(2,q)$. Now there are $q(q-1)$ cones containing $\mathscr{O}'_1$ and with vertex in $\mathrm{PG}(2,q) \backslash (n \cup m)$, and also $q(q-1)$ ovals of π equivalent to $\mathscr{D}(\alpha)$ with axis n and containing the points Y and W_1. Thus, if we can find a group fixing $\mathscr{O}'_1$, Y and W_1 as well as acting regularly on both the set of points of $\mathrm{PG}(2,q) \backslash (n \cup m)$ and the set of ovals of π equivalent to $\mathscr{D}(\alpha)$ with axis n and containing the points Y and W_1, then it follows there must be exactly one cone containing $\mathscr{O}'_1$ and such an oval. To show the existence of such a group we (briefly) apply coordinates. Let $\mathrm{PG}(2,q) : x_2 = 0$, $\pi : x_3 = 0$, $\xi : x_1 = 0$. We may assume that $\mathscr{O}'_1$ has the form $\{(t^\alpha, 0, 1, t) : t \in \mathrm{GF}(q)\} \cup \{(1,0,0,0)\}$. The required group has elements of the form

$$\begin{pmatrix} \lambda^\alpha & \rho & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \quad \text{for } \rho \in \mathrm{GF}(q) \text{ and } \lambda \in \mathrm{GF}(q) \backslash \{0\}.$$

By the above there is a unique cone $\mathscr{K}_1$ containing $\mathscr{O}_1$ and $\mathscr{O}'_1$ which has vertex X_1, say, in $\mathrm{PG}(2,q)$. Similarly we have cones $\mathscr{K}_i$, $i = 1,2,\ldots,q-1$, where $\mathscr{K}_i$ contains $\mathscr{O}_i$ and $\mathscr{O}_i'$ and has vertex X_i. We also define $\mathscr{K}_q$ to be the cone containing $\mathscr{O}_q'$ and with vertex X'. For convenience we will relabel the point X' as X_q.

We now show that X_i, X_j, Y, $i \neq j$, are not collinear and that the two cones $\mathscr{K}_i$ and $\mathscr{K}_j$, $i \neq j$, intersect in exactly Y. Without loss of generality we will consider $\mathscr{K}_1$ and $\mathscr{K}_2$. First suppose that Y, X_1, X_2 are collinear on a line o (which is necessarily a generator of both $\mathscr{K}_1$ and $\mathscr{K}_2$). Of the $q+1$ planes of $\mathrm{PG}(3,q)$ on o, $\mathrm{PG}(2,q)$ is a tangent plane to both $\mathscr{K}_1$ and $\mathscr{K}_2$ while each of the other q planes contains a second generator of both $\mathscr{K}_1$ and $\mathscr{K}_2$ and so a second point of $\mathscr{K}_1 \cap \mathscr{K}_2$. Hence $|\mathscr{K}_1 \cap \mathscr{K}_2| = q+1$. Now we consider the planes on the line m. The plane $\mathrm{PG}(2,q)$ is tangent to both $\mathscr{K}_1$ and $\mathscr{K}_2$ while each of the other q planes intersects both $\mathscr{K}_1$ and $\mathscr{K}_2$ in an oval equivalent to $\mathscr{D}(\alpha)$ with axis m, containing the point Y and with nucleus N. Two such ovals may intersect in either $0,1,2$ or $q+1$ points. Suppose that there exists a plane η distinct from $\mathrm{PG}(2,q)$ on m for which $\eta \cap \mathscr{K}_1 = \eta \cap \mathscr{K}_2 = \bar{O}$, $\bar{O}$ an oval. Now since $\bar{O}$ is the set of common points of $\mathscr{K}_1$ and $\mathscr{K}_2$ it follows that $\pi \cap \bar{O} = \mathscr{O}_1 \cap \mathscr{O}_2 = \{Y\}$. Hence the line $\pi \cap \eta$ is tangent to both $\mathscr{K}_1$ and $\mathscr{K}_2$ at Y and so must be n. This implies that $m, n \subset \eta$ and so $\eta = \mathrm{PG}(2,q)$, a contradiction. It follows that each plane on m distinct from $\mathrm{PG}(2,q)$ contains exactly two points of $\mathscr{K}_1 \cap \mathscr{K}_2$, Y and one other. However this must also hold for ξ, a contradiction. Therefore Y, X_1, X_2 are not collinear.

If $N \in \langle X_1, X_2\rangle$, then $\mathscr{O}_1$ and $\mathscr{O}_2$ have a common nucleus and so it follows that N, X_1, X_2 are not collinear. So the line $\langle X_1, X_2\rangle$ contains a point P of $m \backslash \{Y, N\}$. If $\mathscr{K}_1$ and $\mathscr{K}_2$ are to meet in exactly Y, then no line of ξ distinct from m and incident with P can contain a point of both $\mathscr{O}_1'$ and $\mathscr{O}_2'$. Hence $\mathscr{O}_1'$ and $\mathscr{O}_2'$ are compatible at P. From this we see that the number of cones containing $\mathscr{O}_2'$, with vertex in $\mathrm{PG}(2,q)$, that meet $\mathscr{K}_1$ in exactly Y is the number of points on m at which $\mathscr{O}_1'$ and $\mathscr{O}_2'$ are compatible, multiplied by $q-2$ for the possible vertices in $\langle X_1, P\rangle \backslash \{X_1, P\}$ not on n, for each such point of compatibility P. By Section 2 this is $q(q-2)/2$. In the planar representation of $\mathscr{K}_\alpha$ in π, this is the same as the number of ovals equivalent to $\mathscr{D}(\alpha)$ meeting $\mathscr{O}_1$ in exactly Y, containing W_2, and with nucleus distinct from that of $\mathscr{O}_1$. It follows that the cones $\mathscr{K}_1$ and $\mathscr{K}_2$ meet in exactly Y.

We now show that the cone $\mathscr{K}_q$ and any cone $\mathscr{K}_i$, $i \in \{1,2,\ldots,q-1\}$ intersect in exactly Y. If we consider a plane π' such that $u \subset \pi'$, but $m, n \not\subset \pi'$, then we have the same situation as above except that $\mathscr{K}_q \cap \pi'$ is an oval and not a line. By choosing π' appropriately we see that $\mathscr{K}_q \cap \mathscr{K}_i = \{Y\}$ for $i = 1,2,\ldots,q-1$.

Since the cones $\mathscr{K}_i$ intersect pairwise in exactly Y it follows that they partition the points of $\mathrm{PG}(3,q) \backslash \mathrm{PG}(2,q)$.

We now show that the set $\mathscr{O} = \{Y, X_1, X_2, \ldots, X_q\}$ is an oval with nucleus N. Consider the three points X_i, X_j, X_k for distinct i,j,k in $\{1,2,\ldots,q-1\}$. Suppose that X_i, X_j, X_k are collinear on the line ℓ_{ijk}. There are q planes on ℓ_{ijk} distinct from $\mathrm{PG}(2,q)$, and the q lines of $\mathscr{K}_i \backslash \langle Y, X_i\rangle$ lie on these planes with at most two per plane; and similarly for X_j and X_k. It follows that there is a plane on ℓ_{ijk} which contains a line from at least two of the cones $\mathscr{K}_i, \mathscr{K}_j, \mathscr{K}_k$, which implies two cones intersecting in a point other than Y, a contradiction. Hence X_i, X_j, X_k cannot be collinear. Similarly, X_q, X_i, X_j are not collinear for distinct i,j in $\{1,2,\ldots,q-1\}$ and $\mathscr{O} = \{Y, X_1, X_2, \ldots, X_q\}$ is an oval. Since the lines $\langle N, X_i\rangle$, $i = 1,2,\ldots,q-1$ and $\langle N, X_q\rangle$ are the nuclear lines of the cones $\mathscr{K}_i$, $i = 1, 2, \dots, q$, respectively, and these lines intersect n in distinct points it follows that N is the nucleus of the oval $\{Y, X_1, X_2, \dots, X_q\}$.

If we construct the GQ $T_2(\mathcal{O})$ in $\mathrm{PG}(3,q)$, then the set $\mathscr{S} = \{Y\} \cup \{\mathscr{K}_i \backslash \langle Y, X_i \rangle : i = 1, 2, \dots, q\}$ is a spread of $T_2(\mathcal{O})$, and the cone $\mathscr{K}_i$ has base oval equivalent to $\mathscr{D}(\alpha)$ and axial line $\langle Y, X_i \rangle$.

Conversely, suppose that we have such a spread $\mathscr{S}$ of $T_2(\mathcal{O})$. If we take any plane π on Y, distinct from the plane $\mathrm{PG}(2,q)$ of $\mathcal{O}$, that intersects $\mathcal{O}$ in a secant, then the intersection of the cones of $\mathscr{S}$ with π yields an α-flock in the planar representation; if we take any plane ξ on Y and N, distinct from $\mathrm{PG}(2,q)$, then the intersection of the cones of $\mathscr{S}$ with ξ yields ovals $\mathcal{O}_1', \mathcal{O}_2', \dots, \mathcal{O}_q'$. It is clear that the above construction gives us the oval $\mathcal{O}$.

Note that this result characterises the ovals $\mathcal{O}$ that may be constructed from an α-flock by the existence of the corresponding spread of $T_2(\mathcal{O})$. (This result was first proved algebraically in [1].)

4 Algebraic description of $\mathcal{O}$ and $\mathscr{S}$

In this section we add coordinates to the construction of Theorem 1.4 to show that the hyperoval completion of $\mathcal{O}$ is the same as the hyperoval constructed from an α-flock by Cherowitzo and that the spread $\mathscr{S}$ of $T_2(\mathcal{O})$ is the same as that constructed by Brown, O'Keefe, Payne, Penttila and Royle. Note that in [8] Thas showed that in the case of a flock of a quadratic cone that the $q+1$ (flock, axis to base oval of cone) pairs gave rise to the $q+1$ herd hyperovals constructed from a flock as formalised in Theorem 1.1.

Adding coordinates as in the proof of Theorem 1.4, let $\mathrm{PG}(2,q) : x_2 = 0$, $\pi : x_3 = 0$, $\xi : x_1 = 0$. Thus $n : x_2 = x_3 = 0$, $m : x_1 = x_2 = 0$ and $u : x_3 = x_1 = 0$ with $Y(1,0,0,0)$.

Let $\mathscr{K}_\alpha : x_1^\alpha = x_0 x_2^{\alpha-1}$ and let $\mathscr{F}_\alpha$ be a flock of $\mathscr{K}_\alpha$. From Section 1 we may assume that $\mathscr{F}_\alpha$ has elements $\pi_t : f(t)x_0 + t^{1/\alpha}x_1 + ag(t)x_2 + x_3 = 0$, $t \in \mathrm{GF}(q)$, where f and g are permutations such that $f(0) = g(0) = 0$ and $f(1) = g(1) = 1$ and $\mathrm{trace}(a) = 1$. Let $\mathcal{O}_t''$ denote the oval $\mathscr{K}_\alpha \cap \pi_t$, and so

$$\mathcal{O}_t'' = \{(s^\alpha, s, 1, f(t)s^\alpha + t^{1/\alpha}s + g(t)) : s \in \mathrm{GF}(q)\} \cup \{(1, 0, 0, f(t))\}$$

with nucleus $(0, 1, 0, t^{1/\alpha})$. We now choose to project these $\mathcal{O}_t''$ from the point $U = (1, 0, 0, 1)$ on the axial line $x_1 = x_2 = 0$ of $\mathscr{K}_\alpha$, onto the plane π. As $f(1) = 1$ the point U is contained in π_1 and so

$$\mathcal{O}_1'' \mapsto w : x_3 = x_0 + x_1 + x_2 = 0.$$

For $t \neq 1$

$$\mathcal{O}_t'' \mapsto \mathcal{O}_t = \{((1 + f(t))s^\alpha + t^{1/\alpha}s + ag(t), s, 1, 0) : s \in \mathrm{GF}(q)\} \cup \{Y\}$$

with nucleus $(t^{1/\alpha}, 1, 0, 0)$. Thus the planar representation of the α-flock is $\{\mathcal{O}_t : t \in \mathrm{GF}(q) \backslash \{1\}\} \cup \{w\}$. For $t \neq 1$ define W_t to be the second point (other than Y) of

$\mathcal{O}_t$ on u, that is, $W_t = (ag(t), 0, 1, 0)$ and define W_1 to be the intersection of w and u, that is $W_1 = (1, 0, 1, 0)$.

Next, in the plane ξ we choose the axial linear pencil of ovals equivalent to $\mathcal{D}(\alpha)$ to be

$$\mathcal{O}'_B = \{(r^\alpha + aB, 0, 1, r) : r \in \mathrm{GF}(q)\} \cup \{Y\}, \quad B \in \mathrm{GF}(q), \quad \text{with nucleus } (0, 0, 0, 1).$$

The second point (other than Y) of the oval O'_B on u is $(aB, 0, 1, 0) = W_{g^{-1}(B)}$.

For $t \neq 1$ the unique cone on $\mathcal{O}_t$ and $\mathcal{O}'_{g(t)}$ has vertex $(t^{1/\alpha}, 1, 0, (1 + f(t))^{1/\alpha})$. Thus by Theorem 1.4 we have that

$$\{(t^{1/\alpha}, 1, 0, (1 + f(t))^{1/\alpha}) : t \in \mathrm{GF}(q)\} \cup \{(1, 0, 0, 0)\}$$

is an oval of $\mathrm{PG}(2, q)$ with nucleus $(0, 0, 0, 1)$.

Applying the collineation $x'_3 = x_3 + x_1$ and then the automorphic collineation induced by α, the oval is equivalent to

$$\mathcal{O} = \{(t, 1, 0, f(t)) : t \in \mathrm{GF}(q)\} \cup \{(1, 0, 0, 0)\} \quad \text{with nucleus } (0, 0, 0, 1).$$

This implies that the hyperoval completion of the oval is indeed the same hyperoval as that in Theorem 1.2.

Now considering the corresponding spread of $T_2(\mathcal{O})$, we see that the cone with vertex $(t, 1, 0, f(t))$ intersects the plane ξ in the oval

$$\{(r^\alpha + a^\alpha g(t)^\alpha, 0, 1, r) : r \in \mathrm{GF}(q)\} \cup \{(1, 0, 0, 0)\} \quad \text{with nucleus } (0, 0, 0, 1).$$

This is the same as the spread given in Theorem 1.3.

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Received 20 April, 2001; revised 15 October, 2002

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