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On the second sectional geometric genus of quasi-polarized manifolds

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Abstract. Let (X, L) be a quasi-polarized manifold of $\dim X = n$. In a previous paper we gave a new invariant (the i-th sectional geometric genus) of (X, L), which is a generalization of the degree and the sectional genus of (X, L). In this paper we study some properties of the second sectional geometric genus.

Key words. Polarized manifolds, sectional genus, sectional geometric genus, Chern class.

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0 Introduction

Let X be a projective variety of $\dim X = n$ over the complex number field $\mathbb{C}$, and let L be a nef and big (resp. an ample) line bundle on X. Then we call the pair (X, L) a *quasi-polarized* (resp. *polarized*) *variety*, and (X, L) is called a quasi-polarized (resp. polarized) *manifold* if X is smooth. In [6], we gave a new invariant of (X, L) which is called the *i-th sectional geometric genus* $g_i(X, L)$ of (X, L) for $0 \leqslant i \leqslant n$. We note that $g_i(X, L)$ is a generalization of the degree L^n and the sectional genus $g(L)$. (Namely $g_0(X, L) = L^n$ and $g_1(X, L) = g(L)$.) Here we recall the reason why we call this invariant the sectional geometric genus. Let (X, L) be a quasi-polarized manifold of dimension $n \geqslant 2$ with $\mathrm{Bs}|L| = \emptyset$, where $\mathrm{Bs}|L|$ is the base locus of $|L|$. Let i be an integer with $1 \leqslant i \leqslant n$, and let Y be the transversal intersection of general $n - i$ elements of $|L|$. In this case Y is a smooth projective variety of dimension i. Then we can prove that $g_i(X, L) = h^i(\mathcal{O}_Y)$, that is, $g_i(X, L)$ is the geometric genus of Y.

In [6] we study some fundamental properties of the i-th sectional geometric genus. We find that we can generalize some problems about the sectional genus to the case of the sectional geometric genus. For example, in [6] we proposed the following conjecture:

Conjecture 0.1. *Let* (X, L) *be a quasi-polarized manifold of* $\dim X = n$ *and let* i *be an integer with* $0 \leqslant i \leqslant n$. *Then* $g_i(X, L) \geqslant h^i(\mathcal{O}_X)$.

Here we note that if $i = 0$, then this is true because $g_0(X, L) = L^n \geqslant 1 = h^0(\mathcal{O}_X)$. If $i = 1$, then this is Fujita's conjecture. (See [3, (13.7)] or [1, Question 7.2.11].) Namely we can find that an inequality $g(L) \geqslant h^1(\mathcal{O}_X)$ is a generalization of an inequality $L^n \geqslant 1$. In [6] we proved that this conjecture is true if $\mathrm{Bs}|L| = \emptyset$. Moreover we classified polarized manifolds (X, L) which satisfy the following properties:

(A) $\dim X \geqslant 3$, $\mathrm{Bs}|L| = \emptyset$, and $g_2(X, L) = h^2(\mathcal{O}_X)$,

(B) $\dim X \geqslant 3$, L is very ample, and $g_2(X, L) = h^2(\mathcal{O}_X) + 1$.

In a future paper, we will classify polarized manifolds (X, L) such that L is very ample and $g_2(X, L) - h^2(\mathcal{O}_X) \leqslant 5$. In [7] we study the conjecture for the case where $0 \leqslant \dim \mathrm{Bs}|L| \leqslant n - 1$.

Furthermore in [6] we proved the following which is analogous to a theorem of Sommese ([11, Theorem 4.1]):

Theorem 0.2 ([6, Corollary 3.5]). *Let (X, L) be a polarized manifold of* $\dim X = n \geqslant 3$. *Assume that L is spanned. Then the following are equivalent*:

(1) $g_2(X, L) = h^2(\mathcal{O}_X)$,

(2) $h^0(K_X + (n-2)L) = 0$,

(3) $\kappa(K_X + (n-2)L) = -\infty$,

(4) (X, L) *is one of the types from* (1) *to* (7-4) *in Theorem* 1.13 *below*.

In this way, it is interesting and very important to study the sectional geometric genus, and we hope that by using this invariant we can study polarized manifolds more deeply.

In this paper, we mainly study the second sectional geometric genus of (quasi-) polarized manifolds. The contents of this paper are the following: In Section 1, we prepare for some results which are used later. In Section 2, we give an explicit formula of the second sectional geometric genus of quasi-polarized manifolds. In Section 3, we study the second sectional geometric genus of polarized manifolds and we obtain the following:

(1) We give a lower bound of $g_2(X, L)$ for $\dim X \geqslant 4$ and $\kappa(X) \geqslant 0$. (Theorem 3.5 (1).) In particular we get that $g_2(X, L) \geqslant h^1(\mathcal{O}_X)$. (Corollary 3.5.2 (1).)

(2) We give some numerical conditions of (X, L) with $g_2(X, L) = 0$ if $\dim X \geqslant 4$ and $\kappa(X) \geqslant 0$. (Corollary 3.5.4.)

(3) We prove that $g_2(X, 2L) \geqslant 0$ for $\dim X = 3$. (Theorem 3.7 and Corollary 3.7.1.)

(4) We give a classification of (X, L) with $\dim X = 3$ and $g_2(X, 2L) = 0$. (Proposition 3.10 and Proposition 3.11.)

(5) We study the case where $\dim X \geqslant 3$, K_X is nef and $\kappa(X) \geqslant 0$. (Theorem 3.5 (2), Corollary 3.5.2 (2), and Proposition 3.9.)

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Notation and Conventions. In this paper, we shall study mainly a smooth projective variety X over the complex number field $\mathbb{C}$. The words "line bundles" and "Cartier divisors" are used interchangeably.

$\mathcal{O}(D)$: invertible sheaf associated with a Cartier divisor D on X.

$\mathcal{O}_X$: the structure sheaf of X.

$\chi(\mathcal{F})$: the Euler–Poincaré characteristic of a coherent sheaf $\mathcal{F}$.

$h^i(\mathcal{F}) = \dim H^i(X, \mathcal{F})$ for a coherent sheaf $\mathcal{F}$ on X.

$h^i(D) = h^i(\mathcal{O}(D))$ for a divisor D.

$|D|$: the complete linear system associated with a divisor D.

K_X: the canonical divisor of X.

$\kappa(D)$: Iitaka dimension of a Cartier divisor D on X.

$\kappa(X)$: Kodaira dimension of X.

$\mathbb{P}^n$: projective space of dimension n.

$\mathbb{Q}^n$: hyperquadric surface in $\mathbb{P}^{n+1}$.

$\sim$ (or $=$): linear equivalence.

$\equiv$: numerical equivalence.

1 Preliminaries

Definition 1.1. Let X be a normal projective variety of $\dim X = n$, and let $\mathcal{E}$ be a vector bundle on X. Let $\mathcal{U} = (h_1, \dots, h_{n-1})$ be an $(n-1)$-tuple of numerically effective $\mathbb{Q}$-divisors on X. Then $\mathcal{E}$ is said to be *$\mathcal{U}$-semistable* if

$$\delta_{\mathcal{U}}(\mathcal{F}) \leqslant \delta_{\mathcal{U}}(\mathcal{E})$$

for every nonzero subsheaf $\mathcal{F}$ of $\mathcal{E}$, where

$$\delta_{\mathcal{U}}(\mathcal{G}) := \frac{c_1(\mathcal{G}) h_1 \dots h_{n-1}}{\operatorname{rank} \mathcal{G}}$$

for any torsion free sheaf $\mathcal{G}$ on X.

Theorem 1.2 (Harder–Narashimhan filtration). *Let X be a normal projective variety of* $\dim X = n$ *and let $\mathcal{E}$ be a torsion free sheaf on X. Let $\mathcal{U} = (h_1, \dots, h_{n-1})$ be an $(n-1)$-tuple of numerically effective $\mathbb{Q}$-divisors on X. Then there exists a unique filtration*

$$\Sigma_{\mathscr{U}} : 0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \cdots \subsetneq \mathscr{E}_s = \mathscr{E}$$

that has the following properties: for any integer i with $1 \leqslant i \leqslant s$

(1) $\mathrm{Gr}_i(\Sigma_{\mathscr{U}}) := \mathscr{E}_i/\mathscr{E}_{i-1}$ *is a torsion free* $\mathscr{U}$*-semistable sheaf,*

(2) $\delta_{\mathscr{U}}(\mathrm{Gr}_i(\Sigma_{\mathscr{U}}))$ *is a strictly decreasing function on i.*

Proof. See [10, Theorem 2.1]. □

Remark 1.2.1. We say that the above filtration $\Sigma_{\mathscr{U}}$ of $\mathscr{E}$ is the *Harder–Narashimhan filtration of* $\mathscr{E}$ *with respect to* $\mathscr{U}$.

Definition 1.3. (1) The coherent subsheaf $\mathscr{E}_1$ in Theorem 1.2 is said to be the *maximal* $\mathscr{U}$*-destabilizing subsheaf of* $\mathscr{E}$.

(2) Let X be a normal projective variety of $\dim X = n$ and let $\mathscr{U} = (h_1, \dots, h_{n-2})$ be a $(n-2)$-tuple of numerically effective $\mathbb{Q}$-divisors on X. A torsion free sheaf $\mathscr{E}$ on X is said to be *generically* $\mathscr{U}$*-semipositive* if for every numerically effective $\mathbb{Q}$-divisor D on X, $\delta_{(\mathscr{U},D)}((\mathscr{E}^*)_1) \leqslant 0$, where $\mathscr{E}^*$ denotes the dual of $\mathscr{E}$, and $(\mathscr{E}^*)_1$ is the maximal $(\mathscr{U}, D)$-destabilizing subsheaf of $\mathscr{E}^*$. (See [10, Section 6].)

Theorem 1.4. *Let X be a normal projective variety of* $\dim X = n$ *such that X is smooth in codimension two. Let* $NA(X) \subset \{\mathrm{Pic}(X)/\textit{numerical equivalence}\} \otimes \mathbb{R}$ *be the ample cone. Let* $\mathscr{E}$ *be a torsion free sheaf on X, with its first Chern class being a numerically effective* $\mathbb{Q}$*-divisor. Assume that* $\mathscr{E}$ *is generically* $\mathscr{B}$*-semipositive, where* $\mathscr{B} = (h_1, \dots, h_{n-2})$ *and* $h_i \in NA(X)_{\mathbb{Q}}$ *for each i. Then*

$$c_2(\mathscr{E})h_1 \dots h_{n-2} \geqslant 0.$$

Proof. See [10, Theorem 6.1]. □

Theorem 1.5. *Let X be a smooth projective variety of* $\dim X = n$. *Let* $H_1, \dots, H_{n-2}$ *be ample Cartier divisors on X. Then* Ω_X^1 *is generically* $(H_1, \dots, H_{n-2})$*-semipositive unless X is uniruled.* (*For the definition that X is uniruled, see Definition* 1.15 *below.*)

Proof. See [10, Corollary 6.4]. □

Theorem 1.6 (Hirzebruch–Riemann–Roch). *Let X be a smooth complete variety and let* $\mathscr{E}$ *be a locally free sheaf on X. Then*

$$\chi(\mathscr{E}) = \int_X \mathrm{ch}(\mathscr{E})\,\mathrm{td}(\mathscr{T}_X),$$

where $\mathscr{T}_X$ *is the tangent bundle of X,* $\mathrm{ch}(\mathscr{E})$ (*resp.* $\mathrm{td}(\mathscr{T}_X)$) *is the Chern character of* $\mathscr{E}$ (*resp. the Todd class of* $\mathscr{T}_X$), *and* $\int_X$ *denotes the degree of the zero-dimensional component of* $(\mathrm{ch}(\mathscr{E})\,\mathrm{td}(\mathscr{T}_X)) \cap [X]$.

Proof. See [9, Chapter Four]. □

Notation 1.7. Let (X, L) be a quasi-polarized manifold of $\dim X = n \geqslant 3$ and $\mathrm{Bs}|L| = \emptyset$. (Here $\mathrm{Bs}|L|$ denotes the base locus of $|L|$.) We put $X_0 := X$ and $L_0 := L$. Let $X_i \in |L_{i-1}|$ be a smooth member of $|L_{i-1}|$ and $L_i = L_{i-1}|_{X_i}$ for $1 \leqslant i \leqslant n-1$.

Definition 1.8. Let (X, L) be a quasi-polarized variety of $\dim X = n$, and let $\chi(tL)$ be the Euler–Poincaré characteristic of tL. Here we put

$$\chi(tL) = \sum_{j=0}^{n} \chi_j(X, L) \frac{t^{[j]}}{j!},$$

where $t^{[j]} = t(t+1)\dots(t+j-1)$ for $j \geqslant 1$ and $t^{[0]} = 1$. Then the *sectional genus* $g(L)$ *of* (X, L) is defined by the following:

$$g(L) = 1 - \chi_{n-1}(X, L).$$

Remark 1.8.1. If X is smooth, then the sectional genus of (X, L) can be expressed by the following formula:

$$g(L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1},$$

where K_X is the canonical divisor of X.

Definition 1.9 (See [6, Definition 2.1]). Let (X, L) be a quasi-polarized variety of $\dim X = n$. Then for an integer $0 \leqslant i \leqslant n$ the *i-th sectional geometric genus* $g_i(X, L)$ *of* (X, L) is defined by the following formula:

$$g_i(X, L) = (-1)^i(\chi_{n-i}(X, L) - \chi(\mathcal{O}_X)) + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{O}_X).$$

(Here we use notation in Definition 1.8.)

Remark 1.9.1. (1) Since $\chi_{n-i}(X, L) \in \mathbb{Z}$, $g_i(X, L)$ is an integer by definition.

(2) If $i = 0$ (resp. $i = 1$), then $g_i(X, L)$ is equal to the degree (resp. the sectional genus) of (X, L).

(3) If $i = n$, then $g_n(X, L) = h^n(\mathcal{O}_X)$, and $g_n(X, L)$ is independent of L.

Theorem 1.10. *Let* (X, L) *be a quasi-polarized manifold of* $\dim X = n$. *Let i be an integer such that* $0 \leqslant i \leqslant n-1$. *Then*

$$g_i(X, L) = \sum_{j=0}^{n-i-1} (-1)^j \binom{n-i}{j} h^0(K_X + (n-i-j)L) + \sum_{k=0}^{n-i} (-1)^{n-i-k} h^{n-k}(\mathcal{O}_X).$$

Proof. See [6, Theorem 2.3]. □

Theorem 1.11. *Let X be a variety of* $\dim X = n$ *and let* $L_1, L_2, A_1, \dots, A_{n-2}$ *be nef* $\mathbb{Q}$*-bundles on X. Then*

$$(L_1 L_2 A_1 \dots A_{n-2})^2 \geqslant (L_1 L_1 A_1 \dots A_{n-2})(L_2 L_2 A_1 \dots A_{n-2}).$$

Proof. See [3, (0.4.6)], or [1, Proposition 2.5.1]. □

Definition 1.12. (1) Let X (resp. Y) be an n-dimensional projective manifold, and L (resp. A) an ample line bundle on X (resp. Y). Then (X, L) is called a *simple blowing up of* (Y, A) if there exists a birational morphism $\pi : X \to Y$ such that π is a blowing up at a point of Y and $L = \pi^*(A) - E$, where E is the π-exceptional reduced divisor.

(2) Let X (resp. Y) be an n-dimensional projective manifold, and L (resp. A) an ample line bundle on X (resp. Y). Here we put $(X_0, L_0) := (X, L)$. Then we say that (Y, A) is the *first reduction of* (X, L) if there exist polarized manifolds (X_j, L_j) for $1 \leqslant j \leqslant t+1$ and birational morphisms $\mu_j : X_j \to X_{j+1}$ for $0 \leqslant j \leqslant t$ such that $(X_{t+1}, L_{t+1}) = (Y, A)$, (X_j, L_j) is a simple blowing up of (X_{j+1}, L_{j+1}) for any j with $0 \leqslant j \leqslant t$, and (Y, A) is not obtained by a simple blowing up of any polarized manifold. The birational morphism $\mu := \mu_t \circ \cdots \circ \mu_0 : X \to Y$ is called the *first reduction map*.

Remark 1.12.1. If (X, L) is not obtained by a simple blowing up of any polarized manifold, then (X, L) is the first reduction of itself.

Theorem 1.13. *Let* (X, L) *be a polarized manifold of* $n = \dim X \geqslant 3$. *Then* (X, L) *is one of the following types*:

(1) $(\mathbb{P}^n, \mathcal{O}(1))$,

(2) $(\mathbb{Q}^n, \mathcal{O}(1))$,

(3) *a scroll over a smooth curve*,

(4) $K_X \sim -(n-1)L$, *that is*, (X, L) *is a Del Pezzo manifold*,

(5) *a hyperquadric fibration over a smooth curve*,

(6) *a scroll over a smooth surface*,

(7) *let* (X', L') *be the first reduction of* (X, L),
 (7-1) $n = 4$, $(X', L') = (\mathbb{P}^4, \mathcal{O}(2))$,
 (7-2) $n = 3$, $(X', L') = (\mathbb{Q}^3, \mathcal{O}(2))$,
 (7-3) $n = 3$, $(X', L') = (\mathbb{P}^3, \mathcal{O}(3))$,
 (7-4) $n = 3$, X' *is a* $\mathbb{P}^2$*-bundle over a smooth curve C with* $(F', L'|_{F'}) = (\mathbb{P}^2, \mathcal{O}(2))$ *for any fiber* F' *of it*,

(8) $K_{X'} + (n-2)L'$ *is nef*.

Proof. See [1, Proposition 7.2.2, Theorem 7.2.4, Theorem 7.3.2, and Theorem 7.3.4]. See also [4]. □

Remark 1.13.1. (X, L) is the type (1) (resp. the type (1), (2), or (3)) in Theorem 1.13 if and only if $K_X + nL$ (resp. $K_X + (n-1)L$) is not nef.

Proposition 1.14. *Let (X, L) be a polarized manifold of $\dim X = n$, and let i be an integer with $1 \leqslant i \leqslant n$. Let (M, A) be the first reduction of (X, L). Then $g_i(X, L) = g_i(M, A)$.*

Proof. See [6, Proposition 2.6]. □

Definition 1.15. A variety X of dimension n is said to be *uniruled* if there exist a variety Y of dimension $n-1$ and a dominant rational map $\mathbb{P}^1 \times Y \dashrightarrow X$. (Here we note that $\mathbb{P}^n$ is uniruled.)

Definition 1.16 (See [1, (13.1)]). Let X be a normal and 1-Gorenstein projective variety of $\dim X = n$ and let L be a line bundle on X. For an integer j with $0 \leqslant j \leqslant n$, the *$j$-th pluridegree $d_j(L)$ of the pair (X, L)* is defined as

$$d_j(L) := (K_X + (n-2)L)^j L^{n-j},$$

where K_X is the canonical sheaf of X.

Remark 1.16.1. Let (X, L) be as in Definition 1.16. Then by easy calculations, we obtain the following:

(1) $K_X L^{n-1} = d_1(L) - (n-2)d_0(L)$,

(2) $K_X^2 L^{n-2} = d_2(L) - 2(n-2)d_1(L) + (n-2)^2 d_0(L)$.

Lemma 1.17. *Let X be a smooth projective variety of $\dim X = n \geqslant 3$ and let L be an ample line bundle on X. Assume that $\kappa(X) \geqslant 0$. Let (M, A) be the first reduction of (X, L). Let $d_j(A)$ be the j-th pluridegree of (M, A). Then for $j = 1, \dots, n$,*

$$d_j(A) \geqslant (n-2)d_{j-1}(A).$$

Furthermore if $\kappa(X) \geqslant 1$, then the inequalities are strict.

Proof. See [1, Lemma 13.1.3]. □

2 An explicit formula for the second sectional geometric genus

In this section we will give an explicit formula for the second sectional geometric genus of quasi-polarized manifolds.

Proposition 2.1. *Let* (X, L) *be a quasi-polarized manifold of* $\dim X = 3$. *Then*

$$g_2(X, L) = -1 + h^1(\mathcal{O}_X) + \frac{1}{12}((K_X + 2L)(K_X + L) + c_2)L,$$

where c_2 *is the second Chern class of* X.

Proof. By the Hirzebruch–Riemann–Roch theorem (see Theorem 1.6), we get that

$$\chi(-L) = -\frac{1}{6}L^3 + \frac{1}{4}c_1L^2 - \frac{1}{12}(c_1^2 + c_2)L + \frac{1}{24}c_1c_2,$$

where $c_i = c_i(\mathcal{T}_X)$ for the tangent bundle $\mathcal{T}_X$ of X. By the Kawamata–Viehweg vanishing theorem and the Serre duality, we have

$$-h^0(K_X + L) = \chi(-L).$$

Hence

$$h^0(K_X + L) = \frac{1}{6}L^3 - \frac{1}{4}c_1L^2 + \frac{1}{12}(c_1^2 + c_2)L - \frac{1}{24}c_1c_2.$$

By the Hirzebruch–Riemann–Roch theorem, we obtain that

$$\chi(\mathcal{O}_X) = \frac{1}{24}c_1c_2.$$

Therefore since $c_1 = -K_X$, we get that

$$\begin{aligned}
h^0(K_X + L) &= \frac{1}{6}L^3 + \frac{1}{4}K_XL^2 + \frac{1}{12}(K_X^2 + c_2)L - \chi(\mathcal{O}_X) \\
&= \frac{1}{6}L^3 + \frac{1}{4}K_XL^2 + \frac{1}{12}(K_X^2 + c_2)L - (1 - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) - h^3(\mathcal{O}_X)) \\
&= \frac{1}{12}(2L^3 + 3K_XL^2 + K_X^2L) + \frac{1}{12}c_2L - (1 - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) - h^3(\mathcal{O}_X)) \\
&= \frac{1}{12}((K_X + 2L)(K_X + L) + c_2)L - 1 + h^1(\mathcal{O}_X) - h^2(\mathcal{O}_X) + h^3(\mathcal{O}_X).
\end{aligned}$$

So by Theorem 1.10 we get the assertion. □

Next we consider the case in which $\dim X \geqslant 4$.

Proposition 2.2. *Let* (X, L) *be a quasi-polarized manifold of* $\dim X = n \geqslant 4$. *Assume that* L *is spanned. Then*

$$g_2(X,L) = -1 + h^1(\mathcal{O}_X) + \frac{1}{12}((K_X + (n-1)L)(K_X + (n-2)L) + c_2)L^{n-2} + \frac{n-3}{24}(2K_X + (n-2)L)L^{n-1}.$$

Proof. Here we use Notation 1.7. Then (X_{n-3}, L_{n-3}) is a quasi-polarized manifold with $\dim X_{n-3} = 3$ and $\mathrm{Bs}|L_{n-3}| = \emptyset$. Then we can prove that by the adjunction formula

$$(K_X + (n-1)L)(K_X + (n-2)L)L^{n-2} = (K_{X_{n-3}} + 2L_{n-3})(K_{X_{n-3}} + L_{n-3})L_{n-3}.$$

By the exact sequence

$$0 \to \mathcal{T}_{X_{i+1}} \to \rho^*(\mathcal{T}_{X_i}) \to \mathcal{O}(L_i)|_{X_{i+1}} \to 0,$$

we get that

$$c(\rho^*(\mathcal{T}_{X_i})) = c(\mathcal{T}_{X_{i+1}})c(\mathcal{O}(L_i)|_{X_{i+1}}),$$

where $\rho : X_{i+1} \to X_i$ is the embedding, $\mathcal{T}_{X_j}$ is the tangent bundle of X_j for $j = i, i+1$, and $c(\mathscr{E})$ denotes the total Chern class of a vector bundle $\mathscr{E}$. So we obtain that

$$\begin{aligned} c_2(X_i)|_{X_{i+1}} &= c_1(X_{i+1})\mathcal{O}(L_i)|_{X_{i+1}} + c_2(X_{i+1}) \\ &= -K_{X_{i+1}}L_{i+1} + c_2(X_{i+1}). \end{aligned}$$

Here we note that

$$\frac{n-3}{2}(2K_X + (n-2)L)L^{n-1} = (K_X + L)L^{n-1} + \cdots + (K_X + (n-3)L)L^{n-1}.$$

Therefore

$$\begin{aligned} & c_2(X)L^{n-2} + \frac{n-3}{2}(2K_X + (n-2)L)L^{n-1} \\ &= c_2(X)|_{X_1}L_1^{n-3} + K_{X_1}L_1^{n-2} + (K_{X_1} + L_1)L_1^{n-2} + \cdots + (K_{X_1} + (n-4)L_1)L_1^{n-2} \\ &= c_2(X_1)L_1^{n-3} + (K_{X_1} + L_1)L_1^{n-2} + \cdots + (K_{X_1} + (n-4)L_1)L_1^{n-2} \\ &= \cdots \\ &= c_2(X_{n-4})L_{n-4}^2 + (K_{X_{n-4}} + L_{n-4})L_{n-4}^3 \\ &= c_2(X_{n-4})|_{X_{n-3}}L_{n-3} + K_{X_{n-3}}L_{n-3}^2 \\ &= c_2(X_{n-3})L_{n-3}. \end{aligned}$$

We also note that $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_{X_{n-3}})$. By [6, Theorem 2.4] we get that $g_2(X_{n-3}, L_{n-3}) = g_2(X, L)$. Therefore

$$\begin{aligned}
&-1 + h^1(\mathcal{O}_X) + \frac{1}{12}((K_X + (n-1)L)(K_X + (n-2)L) + c_2)L^{n-2} \\
&\qquad + \frac{n-3}{24}(2K_X + (n-2)L)L^{n-1} \\
&\quad = -1 + h^1(\mathcal{O}_{X_{n-3}}) + \frac{1}{12}((K_{X_{n-3}} + 2L_{n-3})(K_{X_{n-3}} + L_{n-3}) + c_2(X_{n-3}))L_{n-3} \\
&\quad = g_2(X_{n-3}, L_{n-3}) \\
&\quad = g_2(X, L).
\end{aligned}$$

This completes the proof of Proposition 2.2. □

Corollary 2.3. *Let (X, L) be a quasi-polarized manifold of $n = \dim X \geqslant 3$. Then*

$$\begin{aligned}
g_2(X, L) = -1 + h^1(\mathcal{O}_X) &+ \frac{1}{12}((K_X + (n-1)L)(K_X + (n-2)L) + c_2)L^{n-2} \\
&+ \frac{n-3}{24}(2K_X + (n-2)L)L^{n-1}.
\end{aligned}$$

Proof. Let A be an ample line bundle on X. We put

$$\begin{aligned}
f(t) = -1 + h^1(\mathcal{O}_X) &+ \frac{1}{12}((K_X + (n-1)(L + tA))(K_X + (n-2)(L + tA)) + c_2) \\
&\times (L + tA)^{n-2} + \frac{n-3}{24}(2K_X + (n-2)(L + tA))(L + tA)^{n-1}.
\end{aligned}$$

Here we note that $g_2(X, L + tA)$ is a polynomial in one indeterminate t by Theorem 1.6 and Definition 1.9, and $f(t)$ is also a polynomial in one indeterminate t. If $\mathrm{Bs}|L + tA| = \emptyset$, then $g_2(X, L + tA) = f(t)$ by Proposition 2.2. But since there are infinitely many t with $\mathrm{Bs}|L + tA| = \emptyset$, we have $g_2(X, L + tA) = f(t)$ for any t. In particular $g_2(X, L) = f(0)$ and we get the assertion. □

3 Properties of the second sectional geometric genus of polarized manifolds

In this section, we assume that X is smooth and L is ample. We study the second sectional geometric genus of a polarized manifold (X, L). First we prove the following lemma.

Lemma 3.1. *Let X be a smooth projective variety of $\dim X = n \geqslant 3$, and let L, $H_1, \dots, H_{n-2}$ be ample Cartier divisors on X. We put $\mathcal{U} = (H_1, \dots, H_{n-2})$. Let $\mathcal{E}$ be a vector bundle on X such that $\mathcal{E}$ is generically $\mathcal{U}$-semipositive. Then $\mathcal{E} \otimes L$ is also generically $\mathcal{U}$-semipositive.*

Proof. Let D be a numerically effective $\mathbb{Q}$-divisor on X and let $\mathscr{W} = (H_1, \dots, H_{n-2}, D)$. Let

$$\Sigma_{\mathscr{W}} : 0 = (\mathscr{E}^*)_0 \subsetneq (\mathscr{E}^*)_1 \subsetneq \cdots \subsetneq (\mathscr{E}^*)_s = \mathscr{E}^*$$

be the Harder–Narashimhan filtration of $\mathscr{E}^*$ with respect to $\mathscr{W}$. Then by Theorem 1.2 for any integer i with $1 \leqslant i \leqslant s$ the following are satisfied:

(♠) $\mathrm{Gr}_i(\Sigma_{\mathscr{W}}) := (\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1}$ is a torsion free $\mathscr{W}$-semistable sheaf,

(♠♠) $\delta_{\mathscr{W}}(\mathrm{Gr}_i(\Sigma_{\mathscr{W}})) := \delta_{\mathscr{W}}((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1})$ is a strictly decreasing function on i.

Since $\mathscr{E}$ is generically $\mathscr{U}$-semipositive, we get that

$$\frac{c_1((\mathscr{E}^*)_1)H_1 \dots H_{n-2}D}{\mathrm{rank}(\mathscr{E}^*)_1} \leqslant 0.$$

Claim 3.1.1. *The Harder–Narashimhan filtration of* $(\mathscr{E} \otimes L)^*$ *with respect to* $\mathscr{W}$ *is the following*:

$$\begin{aligned}\Sigma_{\mathscr{W}} \otimes L^* : 0 = (\mathscr{E}^*)_0 \otimes L^* &\subsetneq (\mathscr{E}^*)_1 \otimes L^* \subsetneq \cdots \subsetneq (\mathscr{E}^*)_s \otimes L^* \\ &= \mathscr{E}^* \otimes L^* = (\mathscr{E} \otimes L)^*.\end{aligned}$$

Proof. (A) First we prove that $\mathrm{Gr}_i(\Sigma_{\mathscr{W}} \otimes L^*)$ is a torsion free $\mathscr{W}$-semistable sheaf. We find that $\mathrm{Gr}_i(\Sigma_{\mathscr{W}} \otimes L^*) = ((\mathscr{E}^*)_i \otimes L^*)/((\mathscr{E}^*)_{i-1} \otimes L^*) = ((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1}) \otimes L^*$ is torsion free. For any subsheaf $\mathscr{F}$ of $((\mathscr{E}^*)_i \otimes L^*)/((\mathscr{E}^*)_{i-1} \otimes L^*)$ we obtain that $\mathscr{F} \otimes L$ is a subsheaf of $(\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1}$ and by using (♠) we get that

$$\frac{c_1(\mathscr{F} \otimes L)H_1 \dots H_{n-2}D}{\mathrm{rank}(\mathscr{F} \otimes L)} \leqslant \frac{c_1((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1})H_1 \dots H_{n-2}D}{\mathrm{rank}((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1})}.$$

Since

$$\frac{c_1(\mathscr{F} \otimes L)H_1 \dots H_{n-2}D}{\mathrm{rank}(\mathscr{F} \otimes L)} = \frac{c_1(\mathscr{F})H_1 \dots H_{n-2}D}{\mathrm{rank}\,\mathscr{F}} + LH_1 \dots H_{n-2}D,$$

we get that

$$\begin{aligned}\frac{c_1(\mathscr{F})H_1 \dots H_{n-2}D}{\mathrm{rank}\,\mathscr{F}} &= \frac{c_1(\mathscr{F} \otimes L)H_1 \dots H_{n-2}D}{\mathrm{rank}(\mathscr{F} \otimes L)} - LH_1 \dots H_{n-2}D \\ &\leqslant \frac{c_1((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1})H_1 \dots H_{n-2}D}{\mathrm{rank}((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1})} - LH_1 \dots H_{n-2}D \\ &= \frac{c_1(((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1}) \otimes L^*)H_1 \dots H_{n-2}D}{\mathrm{rank}((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1})} \\ &= \frac{c_1(((\mathscr{E}^*)_i \otimes L^*)/((\mathscr{E}^*)_{i-1} \otimes L^*))H_1 \dots H_{n-2}D}{\mathrm{rank}(((\mathscr{E}^*)_i \otimes L^*)/((\mathscr{E}^*)_{i-1} \otimes L^*))}.\end{aligned}$$

Therefore $((\mathscr{E}^*)_i \otimes L^*)/((\mathscr{E}^*)_{i-1} \otimes L^*)$ is $\mathscr{W}$-semistable.

(B) Next we prove that $\delta_{\mathscr{W}}(\mathrm{Gr}_i(\Sigma_{\mathscr{W}} \otimes L^*))$ is a strictly decreasing function on i. By using (♠♠), we get that

$$\begin{aligned}
&\delta_{\mathscr{W}}(\mathrm{Gr}_i(\Sigma_{\mathscr{W}} \otimes L^*)) - \delta_{\mathscr{W}}(\mathrm{Gr}_{i+1}(\Sigma_{\mathscr{W}} \otimes L^*)) \\
&\quad = \frac{c_1(((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1}) \otimes L^*)H_1 \dots H_{n-2}D}{\mathrm{rank}(((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1}) \otimes L^*)} \\
&\qquad - \frac{c_1(((\mathscr{E}^*)_{i+1}/(\mathscr{E}^*)_i) \otimes L^*)H_1 \dots H_{n-2}D}{\mathrm{rank}(((\mathscr{E}^*)_{i+1}/(\mathscr{E}^*)_i) \otimes L^*)} \\
&\quad = \frac{c_1((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1})H_1 \dots H_{n-2}D}{\mathrm{rank}(((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1}) \otimes L^*)} - \frac{c_1((\mathscr{E}^*)_{i+1}/(\mathscr{E}^*)_i)H_1 \dots H_{n-2}D}{\mathrm{rank}(((\mathscr{E}^*)_{i+1}/(\mathscr{E}^*)_i) \otimes L^*)} \\
&\quad = \frac{c_1((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1})H_1 \dots H_{n-2}D}{\mathrm{rank}((\mathscr{E}^*)_i/(\mathscr{E}^*)_{i-1})} - \frac{c_1((\mathscr{E}^*)_{i+1}/(\mathscr{E}^*)_i)H_1 \dots H_{n-2}D}{\mathrm{rank}((\mathscr{E}^*)_{i+1}/(\mathscr{E}^*)_i)} > 0.
\end{aligned}$$

Therefore $\Sigma_{\mathscr{W}} \otimes L^*$ is the Harder–Narashimhan filtration of $\mathscr{E}^* \otimes L^*$ with respect to $\mathscr{W}$. This completes the proof of Claim 3.1.1. □

By Claim 3.1.1, the maximal $\mathscr{W}$-destabilizing subsheaf of $\mathscr{E}^* \otimes L^*$ is $(\mathscr{E}^*)_1 \otimes L^*$. Since $\mathscr{E}$ is generically $\mathscr{U}$-semipositive and L is ample, we have

$$\begin{aligned}
\delta_{(\mathscr{U},D)}((\mathscr{E}^* \otimes L^*)_1) &= \delta_{(H_1,\dots,H_{n-2},D)}((\mathscr{E}^*)_1 \otimes L^*) \\
&= \frac{c_1((\mathscr{E}^*)_1 \otimes L^*)H_1 \dots H_{n-2}D}{\mathrm{rank}((\mathscr{E}^*)_1 \otimes L^*)} \\
&= \frac{c_1((\mathscr{E}^*)_1)H_1 \dots H_{n-2}D}{\mathrm{rank}((\mathscr{E}^*)_1 \otimes L^*)} - LH_1 \dots H_{n-2}D \\
&= \frac{c_1((\mathscr{E}^*)_1)H_1 \dots H_{n-2}D}{\mathrm{rank}(\mathscr{E}^*)_1} - LH_1 \dots H_{n-2}D \leqslant 0.
\end{aligned}$$

Hence $\mathscr{E} \otimes L$ is generically $\mathscr{U}$-semipositive. □

By Theorem 1.5 and Lemma 3.1 we get the following.

Corollary 3.2. *Let X be a smooth projective variety of* $\dim X = n \geqslant 3$, *and let L, $H_1, \dots, H_{n-2}$ be ample Cartier divisors on X. Then $\Omega_X^1 \otimes L$ is generically $(H_1, \dots, H_{n-2})$-semipositive unless X is uniruled.*

Corollary 3.3. *Let X be a smooth projective variety of* $\dim X = n \geqslant 3$, *and let L be an ample divisor on X. If X is not uniruled, then $c_2(\Omega_X^1 \otimes L)L^{n-2} \geqslant 0$.*

Proof. By Corollary 3.2, we get that $\Omega_X^1 \otimes L$ is generically $(L, \dots, L)$-semipositive. On the other hand $c_1(\Omega_X^1 \otimes L) = K_X + nL$ is nef unless X is uniruled. (See Remark 1.13.1.) Hence by Theorem 1.4, we have $c_2(\Omega_X^1 \otimes L)L^{n-2} \geqslant 0$. □

Proposition 3.4. *Let X be a smooth projective variety of* $\dim X = n \geqslant 3$. *Let L be an ample Cartier divisor on X. If X is not uniruled, then*

$$c_2(X)L^{n-2} \geqslant -\binom{n}{2}L^n - (n-1)K_X L^{n-1}.$$

Proof. By [8, Example 3.2.2], we get that

$$\begin{aligned} c_2(\Omega_X^1 \otimes L) &= \sum_{i=0}^{2} \binom{n-i}{2-i} c_i(\Omega_X^1)L^{2-i} \\ &= \binom{n}{2}L^2 + \binom{n-1}{1} c_1(\Omega_X^1)L + c_2(\Omega_X^1) \\ &= \binom{n}{2}L^2 + \binom{n-1}{1} K_X L + c_2(\Omega_X^1). \end{aligned}$$

Therefore

$$c_2(\Omega_X^1 \otimes L)L^{n-2} = \binom{n}{2}L^n + \binom{n-1}{1}K_X L^{n-1} + c_2(\Omega_X^1)L^{n-2}.$$

Because $c_2(\Omega_X^1)L^{n-2} = c_2(X)L^{n-2}$, by Corollary 3.3 we have

$$0 \leqslant c_2(\Omega_X^1 \otimes L)L^{n-2} = \binom{n}{2}L^n + \binom{n-1}{1}K_X L^{n-1} + c_2(X)L^{n-2}.$$

Namely

$$c_2(X)L^{n-2} \geqslant -\binom{n}{2}L^n - (n-1)K_X L^{n-1}.$$

This completes the proof of Proposition 3.4. □

Theorem 3.5. *Let (X, L) be a polarized manifold of* $\dim X = n$. *Assume that $\kappa(X) \geqslant 0$. Let (M, A) be the first reduction of (X, L), and let γ be the number of points blown up under the first reduction map.*

(1) *If $n \geqslant 4$, then*

$$g_2(X, L) \geqslant -1 + h^1(\mathcal{O}_X) + \frac{L^n + \gamma}{12}(n^2 - 5n + 5).$$

(2) *If $n \geqslant 3$ and K_X is nef, then*

$$g_2(X, L) \geqslant -1 + h^1(\mathcal{O}_X) + \frac{1}{24}(3n^2 - 11n + 10)L^n.$$

Proof. (1) First we note that by Proposition 1.14, $g_2(X, L) = g_2(M, A)$. So we calculate $g_2(M, A)$. Here we note that M is not uniruled because $\kappa(M) \geqslant 0$. So by Proposition 3.4 we have

$$c_2(M)A^{n-2} \geqslant -\binom{n}{2}A^n - (n-1)K_M A^{n-1}.$$

On the other hand,

$$\begin{aligned}
&(K_M + (n-1)A)(K_M + (n-2)A)A^{n-2} + c_2(M)A^{n-2} \\
&\quad \geqslant (K_M + (n-1)A)(K_M + (n-2)A)A^{n-2} - \binom{n}{2}A^n - (n-1)K_M A^{n-1} \\
&\quad = (K_M + (n-1)A)(K_M + (n-2)A)A^{n-2} \\
&\qquad - (n-1)(K_M + (n-2)A)A^{n-1} + \left((n-1)(n-2) - (n-1)\frac{n}{2}\right)A^n \\
&\quad = K_M(K_M + (n-2)A)A^{n-2} + (n-1)\left(\frac{n}{2} - 2\right)A^n. \qquad (3.5.a)
\end{aligned}$$

Let $d_j(A)$ be the j-th pluridegree of (M, A), that is,

$$d_j(A) := (K_M + (n-2)A)^j A^{n-j}.$$

By (1) and (2) in Remark 1.16.1, we get that

$$\begin{aligned}
K_M(K_M + (n-2)A)A^{n-2} &+ (n-1)\left(\frac{n}{2} - 2\right)A^n \\
&= d_2(A) - (n-2)d_1(A) + \frac{(n-1)(n-4)}{2}d_0(A), \qquad (3.5.b)
\end{aligned}$$

$$(2K_M + (n-2)A)A^{n-1} = 2d_1(A) - (n-2)d_0(A). \qquad (3.5.c)$$

Therefore by Corollary 2.3 and by (3.5.a), (3.5.b), and (3.5.c), we obtain that

$$\begin{aligned}
g_2(M, A) &= -1 + h^1(\mathcal{O}_M) + \frac{1}{12}(K_M + (n-1)A)(K_M + (n-2)A)A^{n-2} \\
&\quad + \frac{1}{12}c_2(M)A^{n-2} + \frac{n-3}{24}(2K_M + (n-2)A)A^{n-1} \\
&\geqslant -1 + h^1(\mathcal{O}_M) + \frac{1}{12}K_M(K_M + (n-2)A)A^{n-2} \\
&\quad + \frac{1}{12}(n-1)\left(\frac{n}{2} - 2\right)A^n + \frac{n-3}{24}(2K_M + (n-2)A)A^{n-1}
\end{aligned}$$

$$= -1 + h^1(\mathcal{O}_M) + \frac{1}{12}(d_2(A) - (n-2)d_1(A)) + \frac{(n-1)(n-4)}{24}d_0(A) + \frac{n-3}{24}(2d_1(A) - (n-2)d_0(A))$$

$$= -1 + h^1(\mathcal{O}_M) + \frac{1}{12}(d_2(A) - d_1(A) - d_0(A)).$$

Since $\kappa(X) \geqslant 0$, by Lemma 1.17 we get that for $j = 1, \dots, n$

$$d_j(A) \geqslant (n-2)d_{j-1}(A).$$

Therefore

$$\begin{aligned} g_2(M, A) &\geqslant -1 + h^1(\mathcal{O}_M) + \frac{1}{12}(d_2(A) - d_1(A) - d_0(A)) \\ &\geqslant -1 + h^1(\mathcal{O}_M) + \frac{1}{12}((n-2)d_1(A) - d_1(A) - d_0(A)) \\ &\geqslant -1 + h^1(\mathcal{O}_M) + \frac{1}{12}((n-3)(n-2) - 1)d_0(A) \\ &= -1 + h^1(\mathcal{O}_M) + \frac{1}{12}(n^2 - 5n + 5)d_0(A). \end{aligned}$$

Since $d_0(A) = L^n + \gamma$ and $h^1(\mathcal{O}_M) = h^1(\mathcal{O}_X)$, we get the assertion (1).

(2) Assume that $n \geqslant 3$ and K_X is nef. In this case $(X, L) \cong (M, A)$ because K_X is nef. We also note that $c_2(X)L^{n-2} \geqslant 0$ by Miyaoka's theorem ([10, Theorem 6.6]).

Hence by Corollary 2.3

$$g_2(X, L) \geqslant -1 + h^1(\mathcal{O}_X) + \frac{1}{12}(K_X + (n-1)L)(K_X + (n-2)L)L^{n-2} + \frac{n-3}{24}(2K_X + (n-2)L)L^{n-1}.$$

By using (1) and (2) in Remark 1.16.1 we obtain that

$$(K_X + (n-1)L)(K_X + (n-2)L)L^{n-2} = d_2(L) + d_1(L),$$

and

$$(2K_X + (n-2)L)L^{n-1} = 2d_1(L) - (n-2)L^n.$$

Hence

$$\begin{aligned} g_2(X, L) &\geqslant -1 + h^1(\mathcal{O}_X) + \frac{d_2(L) + d_1(L)}{12} + \frac{n-3}{24}(2d_1(L) - (n-2)L^n) \\ &= -1 + h^1(\mathcal{O}_X) + \frac{d_2(L)}{12} + \frac{n-2}{12}d_1(L) - \frac{n^2 - 5n + 6}{24}L^n. \end{aligned}$$

By using Lemma 1.17, we obtain that

$$\begin{aligned} g_2(X,L) &\geqslant -1 + h^1(\mathcal{O}_X) + \left(\frac{(n-2)^2}{6} - \frac{n^2-5n+6}{24}\right)L^n \\ &= -1 + h^1(\mathcal{O}_X) + \frac{3n^2-11n+10}{24}L^n. \end{aligned}$$

We get the assertion (2). □

Remark 3.5.1. In both cases of Theorem 3.5, if $\kappa(X) \geqslant 1$, then the inequalities are strict by Lemma 1.17.

Corollary 3.5.2. Let (X,L) be a polarized manifold of $\dim X = n$.

(1) If $n \geqslant 4$ and $\kappa(X) \geqslant 0$, then $g_2(X,L) \geqslant h^1(\mathcal{O}_X)$.

(2) If $n = 3$, $\kappa(X) \geqslant 0$, and K_X is nef, then $g_2(X,L) \geqslant h^1(\mathcal{O}_X)$.

Proof. (1) By Theorem 3.5 (1), we obtain that

$$g_2(X,L) \geqslant -1 + h^1(\mathcal{O}_X) + \frac{L^n+\gamma}{12}(n^2-5n+5).$$

Since $n \geqslant 4$, $\gamma \geqslant 0$, and $L^n \geqslant 1$, we get that

$$\frac{L^n+\gamma}{12}(n^2-5n+5) > 0.$$

Hence $g_2(X,L) > h^1(\mathcal{O}_X) - 1$. Because $g_2(X,L)$ is an integer, we obtain the assertion (1).

(2) Assume that $n = 3$, $\kappa(X) \geqslant 0$, and K_X is nef. Then by Theorem 3.5 (2), we obtain that

$$g_2(X,L) \geqslant -1 + h^1(\mathcal{O}_X) + \frac{1}{6}L^3.$$

Since $L^3 \geqslant 1$, we get that $g_2(X,L) > h^1(\mathcal{O}_X) - 1$. Because $g_2(X,L)$ is an integer, we obtain the assertion (2). □

Remark 3.5.3. (1) Let (X,L) be a polarized manifold of $\dim X = n$ such that $\kappa(X) \geqslant 0$.

(1.1) If $n \geqslant 7$, then by Theorem 3.5 (1) we get that $g_2(X,L) \geqslant h^1(\mathcal{O}_X) + 1$.

(1.2) If K_X is nef and $n \geqslant 5$, then by Theorem 3.5 (2) we get that $g_2(X,L) \geqslant h^1(\mathcal{O}_X) + 1$.

(2) The inequality in Corollary 3.5.2 (2) is best possible. Namely, there exists an example of (X, L) such that $\dim X = 3$, $\kappa(X) \geqslant 0$, K_X is nef, and $g_2(X, L) = h^1(\mathcal{O}_X)$. Let $X = C^{(3)}$ be a symmetric product of a smooth projective curve C of genus three. Let $\pi : C \times C \times C \to C^{(3)}$ be the natural map and let $p : C \times C \times C \to C$ be the first projection. We put $L = \pi_*(p^*(x))$ for $x \in C$. Then $\kappa(X) \geqslant 0$ and K_X is nef. Furthermore $g_2(X, L) = 3 = h^1(\mathcal{O}_X)$.

By Theorem 3.5 (1), we can give numerical conditions for polarized manifolds (X, L) with $g_2(X, L) = 0$, $\kappa(X) \geqslant 0$, and $\dim X \geqslant 4$.

Corollary 3.5.4. *Let (X, L) be a polarized manifold of $\dim X = n \geqslant 4$. Assume that $\kappa(X) \geqslant 0$. Let γ be the number of points blown up under the first reduction map. If $g_2(X, L) = 0$, then $h^1(\mathcal{O}_X) = 0$ and*

(1) *$L^n + \gamma \leqslant 12$ for $n = 4$;*

(2) *$L^n + \gamma \leqslant 2$ for $n \geqslant 5$;*
 (2.a) *If $n \geqslant 5$ and $L^n + \gamma = 2$, then $n = 5$;*
 (2.b) *If $n \geqslant 5$ and $L^n + \gamma = 1$, then $n = 5, 6$.*

Proof. Assume that $g_2(X, L) = 0$. By Corollary 3.5.2 (1), we get that $h^1(\mathcal{O}_X) = 0$.

(1) If $n = 4$, then

$$\frac{L^n + \gamma}{12}(n^2 - 5n + 5) = \frac{L^4 + \gamma}{12}.$$

Because by Theorem 3.5 (1)

$$0 = g_2(X, L) \geqslant -1 + \frac{L^4 + \gamma}{12},$$

we obtain that $L^4 + \gamma \leqslant 12$ and we get the assertion (1).

(2) If $n \geqslant 5$, then

$$\frac{L^n + \gamma}{12}(n^2 - 5n + 5) \geqslant \frac{5}{12}(L^n + \gamma).$$

Hence by Theorem 3.5 (1)

$$0 = g_2(X, L) \geqslant -1 + \frac{5}{12}(L^n + \gamma),$$

and we obtain that $L^n + \gamma \leqslant 2$ because $L^n + \gamma$ is an integer.

(2.a) If $L^n + \gamma = 2$, then

$$0 = g_2(X, L) \geqslant -1 + \frac{1}{6}(n^2 - 5n + 5).$$

Namely $n^2 - 5n - 1 \leqslant 0$. Since $n \geqslant 5$, we get that $n = 5$.

(2.b) If $L^n + \gamma = 1$, then

$$0 = g_2(X, L) \geqslant -1 + \frac{1}{12}(n^2 - 5n + 5).$$

Namely $n^2 - 5n - 7 \leqslant 0$. Since $n \geqslant 5$, we get that $5 \leqslant n \leqslant 6$.
This completes the proof of Corollary 3.5.4. □

Here we consider the case where $\kappa(X) = -\infty$.

Proposition 3.6. *Let (X, L) be a polarized manifold of $\dim X = n$ such that $\kappa(X) = -\infty$ and X is not uniruled. Then*

(1) *If $K_X + ((n-2)/2)L$ is nef and $n \geqslant 6$, then $g_2(X, L) \geqslant h^1(\mathcal{O}_X)$.*

(2) *If $K_X + L$ is nef and $n = 5$, then $g_2(X, L) \geqslant h^1(\mathcal{O}_X)$.*

Proof. (1) By the same argument as in the proof of Theorem 3.5 (1) (see (3.5.a)), we get that

$$\begin{aligned}(K_X + (n-1)L)(K_X + (n-2)L)L^{n-2} + c_2(X)L^{n-2} \\ \geqslant K_X(K_X + (n-2)L)L^{n-2} + (n-1)\left(\frac{n}{2} - 2\right)L^n\end{aligned}$$

because X is not uniruled. Furthermore

$$\begin{aligned}&K_X(K_X + (n-2)L)L^{n-2} + (n-1)\left(\frac{n}{2} - 2\right)L^n \\ &= \left(K_X + \frac{n-2}{2}L\right)^2 L^{n-2} - \frac{(n-2)^2}{4}L^n + (n-1)\left(\frac{n}{2} - 2\right)L^n \\ &= \left(K_X + \frac{n-2}{2}L\right)^2 L^{n-2} + \frac{n^2 - 6n + 4}{4}L^n.\end{aligned}$$

Since $K_X + ((n-2)/2)L$ is nef and $n \geqslant 6$, we get that

$$\left(K_X + \frac{n-2}{2}L\right)^2 L^{n-2} + \frac{n^2 - 6n + 4}{4}L^n \geqslant 1$$

and

$$(2K_X + (n-2)L)L^{n-1} \geqslant 0.$$

Hence by Corollary 2.3 $g_2(X, L) \geqslant h^1(\mathcal{O}_X)$ because $g_2(X, L) \in \mathbb{Z}$.

(2) Assume that $n = 5$ and $K_X + L$ is nef. Then by the same argument as in the proof of Theorem 3.5 (1) (see (3.5.a)), we get that

$$(K_X + 4L)(K_X + 3L)L^3 + c_2(X)L^3 \geqslant K_X(K_X + 3L)L^3 + 2L^5.$$

Hence

$$\begin{aligned} g_2(X,L) + 1 - h^1(\mathcal{O}_X) &\geqslant \frac{1}{12}K_X(K_X + 3L)L^3 + \frac{1}{6}L^5 + \frac{1}{12}(2K_X + 3L)L^4 \\ &= \frac{1}{12}K_X(K_X + 3L)L^3 + \frac{1}{6}L^5 + \frac{1}{12}K_XL^4 + \frac{1}{12}(K_X + 3L)L^4 \\ &= \frac{1}{12}(K_X + L)(K_X + 3L)L^3 + \frac{1}{12}(K_X + 2L)L^4. \end{aligned}$$

Since $K_X + L$ is nef, we obtain that

$$(K_X + L)(K_X + 3L)L^3 \geqslant 0 \quad \text{and} \quad (K_X + 2L)L^4 > 0.$$

Therefore $g_2(X,L) > h^1(\mathcal{O}_X) - 1$. Because $g_2(X,L)$ is an integer, we get that $g_2(X,L) \geqslant h^1(\mathcal{O}_X)$. This completes the proof of Proposition 3.6. □

Next we consider a lower bound of $g_2(X,2L)$ for the case where $\dim X = 3$.

Theorem 3.7. *Let (X,L) be a polarized manifold of* $\dim X = 3$.

(1) *Assume that $\kappa(X) \geqslant 0$. Let (M,A) be the first reduction of (X,L), and let γ be the number of points blown up under the first reduction map. Then*

$$g_2(X,2L) \geqslant -1 + h^1(\mathcal{O}_X) + \frac{5}{6}(L^3 + \gamma).$$

(2) *Assume that $\kappa(X) = -\infty$. Then*

$$g_2(X,2L) \geqslant h^2(\mathcal{O}_X) \geqslant 0.$$

Proof. (I) The case where $\kappa(X) \geqslant 0$.

Let (M,A) be the first reduction of (X,L). By Theorem 1.10, we get that

$$g_2(X,2L) = h^0(K_X + 2L) - h^3(\mathcal{O}_X) + h^2(\mathcal{O}_X).$$

On the other hand, since $h^0(K_X + 2L) = h^0(K_M + 2A)$, $h^3(\mathcal{O}_X) = h^3(\mathcal{O}_M)$, and $h^2(\mathcal{O}_X) = h^2(\mathcal{O}_M)$, we obtain that $g_2(X,2L) = g_2(M,2A)$.

Next we calculate $g_2(M,2A)$. Since M is not uniruled, we have $c_2(M)A \geqslant -3A^3 - 2K_MA^2$ by Proposition 3.4. Hence

$$\begin{aligned} &((K_M + 4A)(K_M + 2A) + c_2(M))A \\ &\quad = (K_M + 4A)(K_M + 2A)A + c_2(M)A \\ &\quad \geqslant (K_M + 4A)(K_M + 2A)A - 3A^3 - 2K_MA^2. \end{aligned} \tag{3.7.a}$$

Since $K_M + A$ is nef and $\kappa(M) \geqslant 0$, we get that

$$\begin{aligned}(K_M+4A)(K_M+2A)A-3A^3-2K_MA^2 \\ &\geqslant 4(K_M+2A)A^2-2(K_M+A)A^2-A^3 \\ &= 4(K_M+A)A^2+4A^3-2(K_M+A)A^2-A^3 \\ &= 2(K_M+A)A^2+3A^3. \qquad (3.7.b)\end{aligned}$$

By Lemma 1.17 we get that $(K_M + A)A^2 \geqslant A^3$. Hence

$$(K_M+4A)(K_M+2A)A-3A^3-2K_MA^2 \geqslant 5A^3.$$

Therefore by Proposition 2.1

$$\begin{aligned}g_2(X,2L) &= g_2(M,2A) \\ &= -1+h^1(\mathcal{O}_M)+\frac{1}{12}((K_M+4A)(K_M+2A)+c_2(M))(2A) \\ &\geqslant -1+h^1(\mathcal{O}_M)+\frac{5}{6}A^3 = -1+h^1(\mathcal{O}_X)+\frac{5}{6}(L^3+\gamma).\end{aligned}$$

Hence we get the assertion (1).

(II) The case where $\kappa(X) = -\infty$.

By Theorem 1.10 and the Serre duality, we have

$$\begin{aligned}g_2(X,2L) &= h^0(K_X+2L)-k^0(K_X)+h^2(\mathcal{O}_X) \\ &= h^0(K_X+2L)+h^2(\mathcal{O}_X) \geqslant h^2(\mathcal{O}_X) \geqslant 0.\end{aligned}$$

This completes the proof of Theorem 3.7. □

Corollary 3.7.1. Let (X, L) be a polarized manifold of $\dim X = 3$. Assume that $\kappa(X) \geqslant 0$. Then $g_2(X, 2L) \geqslant h^1(\mathcal{O}_X) \geqslant 0$.

Proof. By Theorem 3.7 (1), we get that

$$g_2(X,2L) \geqslant -1+h^1(\mathcal{O}_X)+\frac{5}{6}(L^3+\gamma),$$

where γ is the number of points blown up under the first reduction map.

Since $L^3 + \gamma > 0$, we get that $g_2(X, 2L) > h^1(\mathcal{O}_X) - 1$. Hence we get the assertion because $g_2(X, 2L)$ is an integer. □

Here we note that if L is nef and big, $\dim X = 3$ and $h^0(L) \geqslant 1$, then we get the following:

Proposition 3.8. *Let* (X, L) *be a quasi-polarized manifold. If* $\dim X = 3$ *and* $h^0(L) \geqslant 1$, *then* $g_2(X, L) \geqslant h^2(\mathcal{O}_X) \geqslant 0$.

Proof. By Theorem 1.10 and the Serre duality we get that

$$g_2(X,L) = h^0(K_X + L) - h^0(K_X) + h^2(\mathcal{O}_X).$$

If $h^0(K_X) = 0$, then

$$g_2(X,L) = h^0(K_X + L) + h^2(\mathcal{O}_X) \geqslant h^2(\mathcal{O}_X).$$

If $h^0(K_X) \geqslant 1$, then $h^0(K_X + L) - h^0(K_X) \geqslant h^0(L) - 1 \geqslant 0$ and so we get that

$$g_2(X,L) = h^0(K_X + L) - h^0(K_X) + h^2(\mathcal{O}_X) \geqslant h^2(\mathcal{O}_X).$$

This completes the proof of Proposition 3.8. □

Remark 3.8.1. By the same method as in the proof of Proposition 3.8, we can prove that $g_{n-1}(X,L) \geqslant h^{n-1}(\mathcal{O}_X)$ if X is a smooth projective variety of $\dim X = n$, and L is a nef and big line bundle on X with $h^0(L) \geqslant 1$.

Here we assume that $\dim X = n \geqslant 3$, K_X is nef, and $\kappa(X) \geqslant 0$. In this case, by Theorem 3.5 (2), we get that

$$g_2(X,L) \geqslant -1 + h^1(\mathcal{O}_X) + \frac{1}{24}(3n^2 - 11n + 10)L^n.$$

By using this inequality, we study (X,L) with $g_2(X,L) = 0$.

Proposition 3.9. *Let* (X,L) *be a polarized manifold of* $\dim X = n \geqslant 3$. *Assume that* K_X *is nef and* $\kappa(X) \geqslant 0$. *If* $g_2(X,L) = 0$, *then* $n = 3$, $h^1(\mathcal{O}_X) = 0$, *and we obtain the following*:

L^3	K_XL^2	K_X^2L	$c_2(X)L$	$g(L)$
3	2	0	0	5
2	2	0	2	4
2	2	1	1	4
2	2	2	0	4
1	2	0	4	3
1	2	1	3	3
1	2	2	2	3
1	2	3	1	3
1	2	4	0	3
6	0	0	0	7
5	0	0	2	6
4	0	0	4	5
3	0	0	6	4
2	0	0	8	3
1	0	0	10	2

Proof. Here we note that (X, L) is the first reduction of itself. We also note that by Miyaoka's theorem ([10, Theorem 6.6])

$$c_2(X)L^{n-2} \geqslant 0. \tag{3.9.1}$$

Furthermore since K_X is nef we get that

$$K_X^2 L^{n-2} \geqslant 0 \quad \text{and} \quad K_X L^{n-1} \geqslant 0. \tag{3.9.2}$$

Assume that $g_2(X, L) = 0$. By Corollary 3.5.2, we get that $h^1(\mathcal{O}_X) = 0$. Hence by Theorem 3.5 (2)

$$g_2(X, L) \geqslant -1 + \frac{1}{24}(3n^2 - 11n + 10)L^n.$$

Claim. $n = 3$.

Proof. If $n \geqslant 5$, then

$$\frac{1}{24}(3n^2 - 11n + 10)L^n \geqslant \frac{5}{4}L^n$$

and $g_2(X, L) \geqslant 1$. Therefore this is impossible.

Assume that $n = 4$. Then

$$\frac{1}{24}(3n^2 - 11n + 10)L^n = \frac{7}{12}L^4.$$

Since

$$0 = g_2(X, L) \geqslant -1 + \frac{7}{12}L^4,$$

we obtain that $L^4 = 1$. In this case by Corollary 2.3 the second sectional geometric genus of (X, L) is the following:

$$g_2(X, L) = -1 + \frac{1}{12}K_X^2 L^2 + \frac{1}{2}K_X L^3 + \frac{7}{12} + \frac{1}{12}c_2(X)L^2.$$

By (3.9.1) and (3.9.2), we obtain that $K_X L^3 = 0$ because $g_2(X, L) = 0$. But then $(K_X + 3L)L^3 = 3$ and this is impossible because $(K_X + 3L)L^3$ is even. This completes the proof of this claim. □

Since $n = 3$ and $h^1(\mathcal{O}_X) = 0$, by Corollary 2.3 we get that

$$12 = ((K_X + 2L)(K_X + L) + c_2(X))L = K_X^2 L + 3K_X L^2 + 2L^3 + c_2(X)L. \tag{3.9.3}$$

By (3.9.1) and (3.9.2) we have $L^3 \leqslant 6$. Here we note that $K_X L^2 + 2L^3$ is even. Hence

$K_X L^2$ is even and we obtain that $K_X L^2 = 0$ or 2 by (3.9.3), and by (3.9.1), (3.9.2), and (3.9.3) we get the list in Proposition 3.9. (Here we note that by Theorem 1.11 $K_X^2 L = 0$ if $K_X L^2 = 0$ because K_X is nef.) □

Problem 3.9.1. Find an example of (X, L) such that $\dim X = 3$, K_X is nef, $\kappa(X) \geqslant 0$, and $g_2(X, L) = 0$.

Remark 3.9.2. There exists a Calabi–Yau 3-fold X such that there is an ample divisor L on X with $g_2(X, L) = 0$ and $L^3 = 1$ or 2. (See [6, Example 4.3.3].)

In Theorem 3.7 and Corollary 3.7.1, we proved that $g_2(X, 2L) \geqslant 0$ if (X, L) is a polarized 3-fold. Here we study a polarized 3-fold (X, L) with $g_2(X, 2L) = 0$.

Proposition 3.10. *Let* (X, L) *be a polarized manifold of* $\dim X = 3$. *If* $g_2(X, 2L) = 0$, *then* $\kappa(X) = -\infty$.

Proof. Assume that $\kappa(X) \geqslant 0$. Let (M, A) be the first reduction of (X, L), and let γ be the number of points blown up under the first reduction map. Assume that $g_2(X, 2L) = 0$. By Theorem 3.7 (1), we obtain that $L^3 = 1$, $\gamma = 0$, and $h^1(\mathcal{O}_X) = 0$. Hence $(X, L) \cong (M, A)$.

By (3.7.a) and (3.7.b) in the proof of Theorem 3.7, we get that

$$\begin{aligned} g_2(X, 2L) &= -1 + \frac{1}{6}(K_X + 4L)(K_X + 2L)L + \frac{1}{6}c_2(X)L \\ &\geqslant -1 + \frac{1}{3}(K_X + L)L^2 + \frac{1}{2}L^3 \\ &= -1 + \frac{1}{3}K_X L^2 + \frac{5}{6}L^3. \end{aligned}$$

Hence $K_X L^2 = 0$ because $g_2(X, 2L) = 0$.

In this case

$$g(L) = 1 + \frac{1}{2}(K_X + 2L)(L)^2 = 2.$$

By [2, Theorem (1.10) and Remark (2.2)], $\mathcal{O}(K_X) = \mathcal{O}_X$ and $\Delta(L) \leqslant 3$, where $\Delta(L) = 3 + L^3 - h^0(L)$. So we obtain that $h^0(L) \geqslant 1$. On the other hand by Theorem 1.10 and the Serre duality,

$$\begin{aligned} 0 &= g_2(X, 2L) \\ &= h^0(K_X + 2L) - h^0(K_X) + h^2(\mathcal{O}_X) \\ &\geqslant h^0(K_X + 2L) - h^0(K_X) \\ &= h^0(2L) - 1 \end{aligned}$$

since $\mathcal{O}(K_X) = \mathcal{O}_X$. Hence $h^0(2L) = 1$ since $h^0(2L) > 0$. On the other hand by the Riemann–Roch theorem and the Kodaira vanishing theorem, we get that

$$h^0(2L) = h^0(K_X + 2L) = L^3 + 2h^0(K_X + L) = L^3 + 2h^0(L) \geqslant 3.$$

So this is impossible. Therefore we get the assertion. □

Proposition 3.11. *Let* (X, L) *be a polarized manifold of* $\dim X = 3$. *Assume that* $g_2(X, 2L) = 0$ *and* $h^0(2L) \geqslant 2$. *Then* (X, L) *is one of the following type*:

(1) $(X, L) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$,

(2) $(X, L) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(1))$,

(3) (X, L) *is a scroll over a smooth curve*.

Proof. First we prove the following claim:

Claim 3.11.1. $h^0(K_X + 2L) = 0$.

Proof. By Theorem 1.10 and the Serre duality, we obtain that

$$\begin{aligned} g_2(X, 2L) &= h^0(K_X + 2L) - h^0(K_X) + h^2(\mathcal{O}_X) \\ &\geqslant h^0(K_X + 2L) - h^0(K_X). \end{aligned}$$

If $h^0(K_X) > 0$, then $0 = g_2(X, 2L) \geqslant h^0(K_X + 2L) - h^0(K_X) \geqslant h^0(2L) - 1 \geqslant 1$ and this is a contradiction. Hence $h^0(K_X) = 0$ and $0 = g_2(X, 2L) \geqslant h^0(K_X + 2L) \geqslant 0$. Therefore $h^0(K_X + 2L) = 0$. This completes the proof of Claim 3.11.1. □

Hence we obtain that $K_X + 2L$ is not nef by [5, Corollary 2.7]. So (X, L) is one of the above types by Theorem 1.13 and Remark 1.13.1. □

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