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THE NEUMANN PROBLEM FOR QUASILINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this note we prove the existence of extremal solutions of the quasilinear Neumann problem $-(|x'(t)|^{p-2}x'(t))' = f(t, x(t), x'(t))$, a.e. on T, $x'(0) = x'(b) = 0$, $2 \le p < \infty$ in the order interval $[\psi, \varphi]$, where ψ and φ are respectively a lower and an upper solution of the Neumann problem.

1. Introduction

Recently several authors studied quasilinear ordinary differential equations of second order. People mainly studied the Dirichlet and the periodic problem using a variety of methods. We refer to the works of Boccardo-Drábek-Giachetti-Kučera [1], Drábek [4], Del Pino-Elgueta-Manasevich [3], Guo [9], O'Regan [13] and Wang-Jiang [16]. Of all these works only Guo in 1993 considers a Neumann problem, but he assumes that his vector fields f is independent of the derivative. Moreover, he establishes only the existence of solutions for the Neumann boundary value problem and his approach is different from ours: it is based on degree theoretic techniques. In contrast here we allow f to depend on the derivative and we use the method of upper and lower solutions coupled with the theory of nonlinear operators of monotone type. Assuming the existence of an upper solution φ and a lower solution ψ such that $\psi \le \varphi$, we obtain the existence of a solution, belonging to $C^1(T)$, of the quasilinear Neumann problem in the order interval $[\psi, \varphi]$ as a consequence of a fixed point result called the nonlinear alternative of Leray-Schauder. Then we also show the existence of extremal solutions in $[\psi, \varphi]$, i.e. of solutions x_* and x^* in $[\psi, \varphi]$ such that any other solution x in $[\psi, \varphi]$ of the boundary value problem satisfies $x_* \le x \le x^*$. The upper and lower solutions method was also used by Wang-Jiang in [16] and, in the context of semilinear second order periodic boundary value problems, by Gao-Wang in [6] and, for multivalued semilinear Sturm-Liouville problems, by Halidias-Papageorgiou in [10]. The existence

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of extremal solutions in $[\psi, \varphi]$ was addressed only by Halidias-Papageorgiou, using different approach.

2. Preliminaries

If $T = [0, b]$, let $W^{1,p}(T) = \{x \in L^p(T) \mid x' \in L^p(T)\}$ be the function space equipped with the norm $\|x\| = (\|x\|_p^p + \|x'\|_p^p)^{\frac{1}{p}}$: the space $W^{1,p}(T)$ is a separable, reflexive Banach space for all $1 < p < \infty$. It is well known that $W^{1,p}(T)$ embeds continuously into $C(T)$, i.e. every element in $W^{1,p}(T)$ has a unique representative into $C(T)$. Moreover $W^{1,p}(T) \hookrightarrow L^p(T)$ compactly for all $1 \leq p < \infty$.

Let X a reflexive Banach space and X^* its topological dual. In what follows by $(.,.)$ we denote the duality brackets of the pair (X, X^*). A map $A : X \to X^*$ is said to be 'monotone', if for all $x_1, x_2 \in X$, we have $(A(x_2) - A(x_1), x_2 - x_1) \geq 0$. $A(\cdot)$ is said to be 'strictly monotone' if it is monotone and $(A(x_2) - A(x_1), x_2 - x_1) = 0$ implies $x_2 = x_1$. We say that $A(\cdot)$ is maximal monotone, if its graph is maximal monotone with respect to inclusion among the graphs of all monotone maps from X into X^*. It follows from this definition that $A(\cdot)$ is maximal monotone if and only if A is monotone and $(v^* - A(x), v - x) \geq 0$ for all $x \in X$, implies $v^* = A(v)$. An operator $A : X \to X^*$ is said to be 'demicontinuous' at $x \in X$, if for every $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to x$ in X, we have $A(x_n) \xrightarrow{w^*} A(x)$ in X^*. A monotone demicontinuous everywhere defined operator is maximal monotone (see Hu-Papageorgiou [11]). A map $A : X \to X^*$ is said to be 'pseudomonotone', if for all $x \in X$ and for every sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ such that $x_n \xrightarrow{w} x$ in X and $\limsup(A(x_n), x_n - x) \leq 0$, we have that $(A(x), x - y) \leq \liminf(A(x_n), x_n - y)$ for all $y \in X$. A map $A : X \to X^*$ is said to be 'coercive' if $\frac{(A(x),x)}{\|x\|} \to \infty$ as $\|x\| \to \infty$. A pseudomonotone map which is also coercive is surjective. A single valued operator $A : X \to X^*$ is said to be 'compact' if it is continuous and maps bounded sets into relatively compact sets.

3. Existence result

Let $T = [0, b] \subset \mathbb{R}$. The problem under consideration is the following:

$$(1) \qquad \left\{ \begin{array}{ll} -(|x'(t)|^{p-2}x'(t))' = f(t, x(t), x'(t)) & \text{a.e. on } T \\ x'(0) = x'(b) = 0 & 2 \leq p < \infty \end{array} \right\}$$

Let us start by introducing the hypotheses on the right hand side function $f(t, x, y)$.

H(f): $f : T \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $(x, y) \in \mathbb{R} \times \mathbb{R}$, $t \mapsto f(t, x, y)$ is measurable;
(ii) for almost all $t \in T$, $(x, y) \mapsto f(t, x, y)$ is continuous;
(iii) for almost all $t \in T$, all $x \in [\psi(t), \varphi(t)]$ and all $y \in \mathbb{R}$, we have

$$|f(t, x, y)| \leq a(t) + c|y|^{p-1}$$

with $a \in L^q(T)$, $c > 0$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 1. By a solution of (1) we mean a function $x \in C^1(T)$ such that $|x^{'}(\cdot)|^{p-2}x^{'}(\cdot) \in W^{1,q}(T)$ and it satisfies (1).

As we already mentioned we shall employ the method of upper and lower solutions. So let us introduce these two concepts:

Definition 2. By an 'upper solution' of (1) we mean a function $\varphi \in C^1(T)$ such that $|\varphi^{'}(\cdot)|^{p-2}\varphi^{'}(\cdot) \in W^{1,q}(T)$ and it satisfies

$$\left\{ \begin{array}{ll} -(|\varphi^{'}(t)|^{p-2}\varphi^{'}(t))^{'} \geq f(t,\varphi(t),\varphi^{'}(t)) & \text{a.e. on } T \\ \varphi^{'}(0) \leq 0 \leq \varphi^{'}(b) & \end{array} \right\} \tag{2}$$

Definition 3. By a 'lower solution' of (1) we mean a function $\psi \in C^1(T)$ such that $|\psi^{'}(\cdot)|^{p-2}\psi^{'}(\cdot) \in W^{1,q}(T)$ and it satisfies

$$\left\{ \begin{array}{ll} -(|\psi^{'}(t)|^{p-2}\psi^{'}(t))^{'} \leq f(t,\psi(t),\psi^{'}(t)) & \text{a.e. on } T \\ \psi^{'}(b) \leq 0 \leq \psi^{'}(0) & \end{array} \right\} \tag{3}$$

We will assume the existence of an upper solution φ and a lower solution ψ. More precisely we make the following hypothesis:

$\mathbf{H_0}$: There exist an upper solution φ and a lower solution ψ of problem (1) such that $\psi(t) \leq \varphi(t)$ for all $t \in T$.

We introduce the operator $A : W^{1,p}(T) \to W^{1,p}(T)^*$ defined by

$$\langle A(x), y \rangle = \int_0^b |x^{'}(t)|^{p-2}x^{'}(t)y^{'}(t)\,dt$$

for all $y \in W^{1,p}(T)$.

Proposition 1. $A : W^{1,p}(T) \to W^{1,p}(T)^*$ *is maximal monotone.*

Proof. It suffices to show that A is monotone, demicontinuous.
First we show the monotonicity of A. For all $x, y \in W^{1,p}(T)$ we have

$$\langle A(x) - A(y), x - y \rangle =$$

$$\textstyle\int_0^b (|x^{'}(t)|^{p-2}x^{'}(t) - |y^{'}(t)|^{p-2}y^{'}(t))(x^{'}(t) - y^{'}(t))\,dt \geq 0$$

from the well-known inequality (see Hu-Papageorgiou [11], p. 303).

Next we show the demicontinuity of A. To this end, let $x_n \to x$ in $W^{1,p}(T)$ as $n \to \infty$. By passing to a subsequence if necessary, we may assume that $x_n^{'}(t) \to x^{'}(t)$ a.e. on T and there exists $k \in L^p(T)$ such that $|x_n^{'}(t)| \leq k(t)$ a.e. on T for all $n \in \mathbb{N}$. Thus we deduce that $|x_n^{'}|^{p-2}x_n^{'} \to |x^{'}|^{p-2}x^{'}$ in $L^q(T)$ as $n \to \infty$. So for every $y \in W^{1,p}(T)$ we have

$$\langle A(x_n) - A(x), y \rangle = \int_0^b (|x_n^{'}(t)|^{p-2}x_n^{'}(t) - |x^{'}(t)|^{p-2}x^{'}(t))y^{'}(t)\,dt \to 0$$

as $n \to \infty$.
We conclude that $A(x_n) \xrightarrow{w} A(x)$ in $W^{1,p}(T)^*$, i.e. A is demicontinuous.
Because A is monotone and demicontinuous, A is maximal monotone. $\square$

Let $A_1 : D_1 \subseteq L^p(T) \to L^q(T)$ be defined by $A_1(x) = A(x)$ for all $x \in D_1 = \{y \in W^{1,p}(T) \mid A(y) \in L^q(T)\}$.

Proposition 2. *$A_1 : D_1 \subseteq L^p(T) \to L^q(T)$ is maximal monotone.*

Proof. Let $J : L^p(T) \to L^q(T)$ be defined by $J(x)(\cdot) = |x(\cdot)|^{p-2}x(\cdot)$. To establish the result of the proposition, it suffices to show that $R(A_1 + J) = L^q(T)$. Indeed, let this surjectivity condition be true and let $v \in D_1$, $v^* \in L^q(T)$ be such that

$$(A_1(x) - v^*, x - v)_{pq} \geq 0 \tag{4}$$

for all $x \in D_1$. Since we have assumed that $R(A_1 + J) = L^q(T)$, we can find $x_0 \in D_1$ such that

$$A_1(x_0) + J(x_0) = v^* + J(v)$$

Using this in (4) above, we obtain

$$(J(v) - J(x_0), v - x_0)_{pq} \geq 0\,.$$

But $J(\cdot)$ is strictly monotone. Hence it follows that $x_0 = v$ and so $A_1(x_0) = v^*$. This proves the maximality of $A_1(\cdot)$. To show the previous range condition, we proceed as follows. Let $\hat{J} = i \circ J_{|W^{1,p}(T)}$ where i is the embedding operator and $J_{|W^{1,p}(T)}$ is the restriction of J on $W^{1,p}(T)$. Then $\hat{J} : W^{1,p}(T) \to W^{1,p}(T)^*$ is continuous, monotone, $D(\hat{J}) = W^{1,p}(T)$, hence maximal monotone (see Hu-Papageorgiou [11], Corollary III.1.35, p. 309). So from this fact together with Proposition 1 and Theorem III.3.3, p. 334 of Hu-Papageorgiou [11], we have that $A + \hat{J} : W^{1,p}(T) \to W^{1,p}(T)^*$ is maximal monotone. Moreover, we have

$$\langle A(x) + \hat{J}(x), x\rangle = \langle A(x), x\rangle + (\hat{J}(x), x)_{pq} = \|x'\|_p^p + \|x\|_p^p = \|x\|_{1,p}^p.$$

Thus $A + \hat{J}(\cdot)$ is coercive. But a maximal monotone coercive operator is surjective (see, for example, Corollary III.2.19, p. 332 of Hu-Papageorgiou [11]). Hence $R(A + \hat{J}) = W^{1,p}(T)^*$. Therefore given $g \in L^q(T)$ we can find $x \in W^{1,p}(T)$ such that $A(x) + \hat{J}(x) = g$. So $A(x) = g - \hat{J}(x) \in L^q(T)$ and then $A(x) = A_1(x)$. Hence $A_1(x) + J(x) = g$. Because $g \in L^q(T)$ was arbitrary, we conclude that $R(A_1 + J) = L^q(T)$. $\square$

In the next proposition we describe the range of A_1 by means of a boundary value problem.

Proposition 3. *If $g \in R(A_1)$ then, for every $x \in D_1$ such that $A_1(x) = g$, we have*

$$\left\{\begin{array}{ll} -(|x'(t)|^{p-2}x'(t))' = g(t) & \text{a.e. on } T \\ x'(0) = x'(b) = 0 & \end{array}\right\} \tag{5}$$

and $x \in C^1(T)$ with $|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,q}(T)$.

Proof. For every $\varphi \in C_0^\infty((0,b))$ we have

$$(A_1(x), \varphi)_{pq} = (g, \varphi)_{pq}$$

that is

$$\int_0^b |x'(t)|^{p-2}x'(t)\varphi'(t)\,dt = \int_0^b g(t)\varphi(t)\,dt \tag{6}$$

So from the definition of the distributional derivative we infer that

$$-(|x'|^{p-2}x')' = g \text{ and } |x'|^{p-2}x' \in W^{1,q}(T)\,.$$

Recall that $W^{1,q}(T)$ is embedded continuously in $C(T)$. So $|x'(\cdot)|^{p-2}x'(\cdot) \in C(T)$. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(r) = |r|^{p-2}r$. This is continuous, strictly monotone and so $h^{-1} : \mathbb{R} \to \mathbb{R}$ is well-defined and it is easily seen to be continuous. Hence $h^{-1}(|x'(\cdot)|^{p-2}x'(\cdot)) = x'(\cdot) \in C(T)$. Using Green's formula (integration by parts), for every $y \in W^{1,p}(T)$, we have

$$\begin{aligned}\int_0^b g(t)y(t)\,dt &= -\int_0^b (|x'(t)|^{p-2}x'(t))'y(t)\,dt \\ &= \int_0^b |x'(t)|^{p-2}x'(t)y'(t)\,dt \;-|x'(b)|^{p-2}x'(b)y(b) + |x'(0)|^{p-2}x'(0)y(0)\,.\end{aligned} \tag{7}$$

So, by equality (6), we have

$$|x'(b)|^{p-2}x'(b)y(b) = |x'(0)|^{p-2}x'(0)y(0)\,. \tag{8}$$

Let $y \in W^{1,p}(T)$ be such that $y(0) = y(b) = 1$ (for example take $y(t) = 1$ for all $t \in T$). We have

$$|x'(b)|^{p-2}x'(b) = |x'(0)|^{p-2}x'(0)\,.$$

Hence

$$h^{-1}(|x'(b)|^{p-2}x'(b)) = h^{-1}(|x'(0)|^{p-2}x'(0))$$

and so $x'(b) = x'(0)$. Using this in (8) and because $y \in W^{1,p}(T)$ is arbitrary, we conclude that $x'(b) = x'(0) = 0$. $\square$

Our application of the upper and lower solutions method will proceed via truncation and penalization techniques. So we introduce the truncation map $\tau :\ W^{1,p}(T) \to W^{1,p}(T)$ be defined by

$$\tau(x)(t) = \begin{cases} \varphi(t) & \text{if} \quad \varphi(t) \le x(t) \\ x(t) & \text{if} \quad \psi(t) \le x(t) \le \varphi(t) \\ \psi(t) & \text{if} \quad x(t) \le \psi(t)\,. \end{cases}$$

It is clear that $\tau(\cdot)$ is continuous.
We observe that $\tau(x)' \in L^p(T)$ being

$$\tau(x)'(t) = \begin{cases} \varphi'(t) & \text{if} \quad \varphi(t) \le x(t) \\ x'(t) & \text{if} \quad \psi(t) \le x(t) \le \varphi(t) \\ \psi'(t) & \text{if} \quad x(t) \le \psi(t)\,. \end{cases}$$

The penalty function $u : T \times \mathbb{R} \to \mathbb{R}$ is defined by

$$u(t,x) = \begin{cases} (x-\varphi(t))^{p-1} & \text{if} \quad \varphi(t) \le x \\ 0 & \text{if} \quad \psi(t) \le x \le \varphi(t) \\ -(\psi(t)-x)^{p-1} & \text{if} \quad x \le \psi(t)\,. \end{cases}$$

It is clear that $u(\cdot,\cdot)$ is a Carathéodory function such that

$$|u(t,x)| \le a_1(t) + c_1|x|^{p-1} \quad \text{a.e. on} \quad T$$

and

$$\int_0^b u(t,x(t))x(t)\,dt \ge \|x\|_p^p - c_2\|x\|_p^{p-1} \quad \text{for all} \quad x \in L^p(T)$$

with $a_1 \in L^q(T)$ and $c_1, c_2 > 0$. Using $\tau(\cdot)$ and $u(\cdot,\cdot)$, we introduce the following auxiliary problem:

$$(9) \quad \left\{ \begin{array}{ll} -(|x'(t)|^{p-2}x'(t))' = f(t,\tau(x)(t),\tau(x)'(t)) - \alpha\; u(t,x(t)) & \text{a.e. on } T \\ x'(0) = x'(b) = 0 & 2 \le p < \infty \end{array} \right\}$$

where $\alpha > 0$. In what follows let $K = [\psi, \varphi] = \{x \in C^1(T) \mid \psi(t) \le x(t) \le \varphi(t) \text{ for all } t \in T\}$.

Proposition 4. *If hypotheses H_0 and $H(f)$ hold, then problem* (1) *has a solution x in $K = [\psi, \varphi]$.*

Proof. First we establish the existence of solutions for problem (9) and then we show that every such solution belongs in K. Hence from the definition of the truncation map $\tau(\cdot)$ and the penalty function $u(\cdot,\cdot)$, we can conclude that these are also solutions of problem (1). From the proof of Proposition 2 we know that $G = A_1 + J : D_1 \subseteq L^p(T) \to L^q(T)$ is surjective and strictly monotone. So $G^{-1} : L^q(T) \to D_1$ is well defined.

Claim 1: G^{-1} is strongly continuous from $L^q(T)$ to $W^{1,p}(T)$ (see Zeidler [17], p.597; remark that strong continuity implies compactness and strong continuity is also referred to as complete continuity).

We need to show that if $y_n \xrightarrow{w} y$ in $L^q(T)$, then $G^{-1}(y_n) \to G^{-1}(y)$ in $W^{1,p}(T)$. Let $x_n = G^{-1}(y_n)$, $n \in \mathbb{N}$ and $x = G^{-1}(y)$. We have

$$A_1(x_n) + J(x_n) = y_n\,.$$

and so

$$(A_1(x_n), x_n)_{pq} + (J(x_n), x_n)_{pq} = (y_n, x_n)_{pq}\,.$$

By definition of A_1 and J, we obtain

$$\|x_n^{'}\|_p^p + \|x_n\|_p^p \ \le M_1 \|x_n\|_p$$

where $\|y_n\|_q \le M_1$ for all $n \in \mathbb{N}$. Then we conclude that

$$\|x_n\|_{1,p} \le M_2$$

for some $M_2 > 0$ and all $n \in \mathbb{N}$. Thus, by passing to a subsequence if necessary, we may assume $x_n \xrightarrow{w} \hat{x}$ in $W^{1,p}(T)$ and $x_n \to \hat{x}$ in $L^p(T)$ (since $W^{1,p}(T)$ is embedded compactly in $L^p(T)$). So we have

$$(A_1(x_n) + J(x_n), x_n - \hat{x})_{pq} = (y_n, x_n - \hat{x})_{pq} \to 0$$

as $n \to \infty$. Since $J(x_n) \to J(\hat{x})$ in $L^q(T)$, being $J : L^p(T) \to L^q(T)$ continuous, we have

$$\limsup (A_1(x_n), x_n - \hat{x})_{pq} = \limsup \langle A(x_n), x_n - \hat{x} \rangle = 0\,.$$

But from Proposition 1 we know that $A(\cdot)$ is maximal monotone and $D(A) = W^{1,p}(T)$, hence $A(\cdot)$ is pseudomonotone. So we have

$$\langle A(x_n), x_n \rangle \to \langle A(\hat{x}), \hat{x} \rangle$$

and so

$$\|x_n^{'}\|_p \to \|\hat{x}^{'}\|_p\,.$$

Since we also have that $x_n^{'} \xrightarrow{w} \hat{x}^{'}$ in $L^p(T)$ and the space $L^p(T)$ has the Kadec-Klee property (being uniformly convex), we have that $x_n^{'} \to \hat{x}^{'}$ in $L^p(T)$ and then $x_n \to \hat{x}$ in $W^{1,p}(T)$. So we have

$$A(x_n) + J(x_n) \xrightarrow{w} A(\hat{x}) + J(\hat{x}) \text{ in } W^{1,p}(T)^*$$

as $n \to \infty$. Hence $A(\hat{x})+J(\hat{x}) = y$ and then $A_1(\hat{x})+J(\hat{x}) = y$. Because $(A_1+J)(\cdot)$ is strictly monotone, we infer that $\hat{x} = x$. Therefore we conclude that $x_n \to x$ in $W^{1,p}(T)$, which proves the claim.

Next let $H_\alpha : W^{1,p}(T) \to L^q(T)$ be defined by

$$H_\alpha(x)(\cdot) = f(\cdot, \tau(x)(\cdot), \tau(x)^{'}(\cdot)) - \alpha \ u(\cdot, x(\cdot)) + J(x)(\cdot)\,.$$

By virtue of hypotheses $H(f)$, the properties of $\tau(\cdot)$ and $u(\cdot,\cdot)$, we infer that H_α is bounded and continuous.

Now using Proposition 3, we see that the solvability of problem (9) is equivalent to solving the fixed point problem

$$x = G^{-1} H_\alpha(x)\,. \tag{10}$$

Since $G^{-1}H_\alpha$ is compact, to solve problem (10), we shall use the Leray-Schauder principle. This require to show that the set

$$\hat{S}_\alpha = \{x \in W^{1,p}(T) \mid x = \lambda G^{-1} H_\alpha(x) \text{ for some } 0 < \lambda < 1\}$$

is bounded.

<u>Claim 2:</u> $\hat{S}_\alpha \subseteq W^{1,p}(T)$ is bounded for some $\alpha > 0$.

If $x \in \hat{S}_\alpha$, we have

$$G(\frac{x}{\lambda}) = H_\alpha(x)$$

for some $\lambda \in]0,1[$ and so

$$(G(\frac{x}{\lambda}), x)_{pq} = (H_\alpha(x), x)_{pq}\,.$$

By definition of G and H_α, we obtain

$$\begin{aligned}\int_0^b \Big|\frac{x^{'}(t)}{\lambda}\Big|^{p-2}\frac{x^{'}(t)}{\lambda}\; x^{'}(t)\;\; dt \quad &+ \int_0^b \Big|\frac{x(t)}{\lambda}\Big|^{p-2}\frac{x(t)}{\lambda}\; x(t)\;\; dt \\ &= \int_0^b f(t,\tau(x)(t),\tau(x)^{'}(t))x(t)\,dt \\ &\quad - \alpha\int_0^b u(t,x(t))x(t)\,dt + \int_0^b |x(t)|^{p-2}x(t)x(t)\,dt.\end{aligned}$$

Using hypotheses $H(f)$(iii), the properties of $\tau(\cdot)$ and $u(\cdot,\cdot)$ and Young's inequality, we have

$$\begin{aligned}&\frac{1}{\lambda^{p-1}}(\|x^{'}\|_p^p + \|x\|_p^p) \\ &\qquad \le\; \|a\|_q\|x\|_p + \tilde{c}\|x\|_p + \bar{c}\|x^{'}\|_p^{p-1}\|x\|_p -\; \alpha\|x\|_p^p + \alpha\; c_2\|x\|_p^{p-1} + \|x\|_p^p \\ &\qquad \le (\|a\|_q + \tilde{c})\|x\|_p +\; \frac{\bar{c}}{q\epsilon^q}\|x^{'}\|_p^p \;+\; \Big(\frac{\bar{c}\epsilon^p}{p} - \alpha + 1\Big)\|x\|_p^p +\; \alpha\; c_2\|x\|_p^{p-1} \\ &\qquad \le (\|a\|_q +\; \tilde{c})\|x\|_w +\; \frac{\bar{c}}{q\epsilon^q}\|x\|_w^p \;+\; \Big(\frac{\bar{c}\epsilon^p}{p} - \alpha + 1\Big)\|x\|_p^p +\; \alpha\; c_2\|x\|_w^{p-1}\end{aligned}$$

where $\tilde{c},\ \bar{c},\ c_2,\ \epsilon > 0$. So we obtain

$$(11)\quad \Big(\frac{1}{\lambda^{p-1}} - \frac{\bar{c}}{q\epsilon^q}\Big)\|x\|_w^p \le (\|a\|_q + \tilde{c})\|x\|_w + \Big(\frac{\bar{c}\epsilon^p}{p} - \alpha + 1\Big)\|x\|_p^p + \alpha\; c_2\|x\|_w^{p-1}$$

Now, choosing $\epsilon > 0$ such that

$$\frac{1}{\lambda^{p-1}} - \frac{\bar{c}}{q\epsilon^q} > 0$$

and $\alpha > 0$ such that

$$\alpha > \frac{\bar{c}\epsilon^p}{p} + 1\,,$$

we can conclude, by using the inequality (11), that $\hat{S}_\alpha$ is bounded in $W^{1,p}(T)$ and this proves Claim 2.

Claims 1 and 2 allow us to apply the Leray-Schauder principle and obtain $x \in D_1$ such that $x = G^{-1}H_\alpha(x)$. Then $G(x) = H_\alpha(x)$ and so $A_1(x) =\ \hat{H}_\alpha(x) \in L^q(T)$, where $\hat{H}_\alpha(x)(\cdot) = f(\cdot,\tau(x)(\cdot),\tau(x)^{'}(\cdot)) - \alpha\; u(\cdot,x(\cdot))$. So by virtue of Proposition 3, we have that x is a solution of the auxiliary problem (9), with $\alpha > \frac{\bar{c}\epsilon^p}{p} + 1$, where $\bar{c} > 0$ and $\epsilon > 0$ are such that $\frac{1}{\lambda^{p-1}} - \frac{\bar{c}}{q\epsilon^q} > 0$.

Next we show that $x \in K = [\psi, \varphi]$. Fixed α as above, we have

$$A_1(x) = \hat{H}_\alpha(x)\,.$$

and so

$$(A_1(x), (\psi - x)^+)_{pq} = (\hat{H}_\alpha(x), (\psi - x)^+)_{pq}\,.$$

By definition of A_1 and $\hat{H}_\alpha$, we obtain

$$\begin{aligned}(12)\qquad &\int_0^b |x^{'}(t)|^{p-2}x^{'}(t)((\psi - x)^+)^{'}(t)\,dt \\ &= \int_0^b f(t, \tau(x)(t), \tau(x)^{'}(t))(\psi - x)^+(t)\,dt - \alpha\int_0^b u(t, x(t))(\psi - x)^+(t)\,dt\,.\end{aligned}$$

Also since $\psi(\cdot)$ is a lower solution, by definition we have

$$(13)\qquad \int_0^b |\psi^{'}(t)|^{p-2}\psi^{'}(t)((\psi - x)^+)^{'}(t)\,dt \le \int_0^b f(t, \psi(t), \psi^{'}(t))(\psi - x)^+(t)\,dt\,.$$

From (12) and (13), we obtain

$$\begin{aligned}&\int_{\{\psi \ge x\}} (|x^{'}(t)|^{p-2}x^{'}(t) - |\psi^{'}(t)|^{p-2}\psi^{'}(t))(\psi - x)^{'}(t)dt \\ (14)\qquad &\ge \int_0^b (f(t, \tau(x)(t), \tau(x)^{'}(t)) - f(t, \psi(t), \psi^{'}(t)))(\psi - x)^+(t)\,dt \\ &\quad - \alpha\int_0^b u(t, x(t))(\psi - x)^+(t)\,dt\,.\end{aligned}$$

Note that

$$(15)\qquad \int_{\{\psi \ge x\}} (|x^{'}(t)|^{p-2}x^{'}(t) - |\psi^{'}(t)|^{p-2}\psi^{'}(t))(\psi - x)^{'}(t)dt \le 0$$

and

$$\begin{aligned}&\int_0^b (f(t, \tau(x)(t), \tau(x)^{'}(t)) - f(t, \psi(t), \psi^{'}(t)))(\psi - x)^+(t)\,dt \\ (16)\qquad &= \int_{\{\psi \ge x\}} (f(t, \psi(t), \psi^{'}(t)) - f(t, \psi(t), \psi^{'}(t)))(\psi - x)(t)dt = 0\,.\end{aligned}$$

Using (15) and (16) in (14) we obtain

$$-\alpha\int_0^b u(t, x(t))(\psi - x)^+(t)\,dt \le 0\,.$$

By definition of the penalty function, we have

$$\int_0^b (\psi - x)^{p-1}(t)(\psi - x)^+(t)\,dt \le 0$$

and so

$$\|(\psi - x)^+\|_p^p \le 0\,.$$

Then since $\psi - x \in C(T)$, we have $\psi(t) \le x(t)$ for all $t \in T$.

In a similar fashion we show that $x \leq \varphi$. So $x \in K = [\psi, \varphi]$. Therefore x is a solution of problem (1) in K. $\square$

Next we establish the existence of extremal solutions in the order interval $K = [\psi, \varphi]$

Theorem 1. *If hypotheses H_0 and $H(f)$ hold then problem* (1) *has extremal solutions in K.*

Proof. Let S be the set of solutions of problem (1) in the order interval $K = [\psi, \varphi]$. We show that $(S, \leq)$ is directed, i.e. for all $x_1, x_2 \in S$ there exists $x_3 \in S$ such that $x_1 \leq x_3$ and $x_2 \leq x_3$. For this purpose, fixed $x_1, x_2 \in S$, we consider the function $y = x_1 \vee x_2$.

For each $\epsilon > 0$, let $\xi_\epsilon \in Lip(\mathbb{R})$ be defined by

$$\xi_\epsilon(t) = \begin{cases} 0 & \text{if} \quad t \leq 0 \\ \frac{t}{\epsilon} & \text{if} \quad 0 < t < \epsilon \\ 1 & \text{if} \quad \epsilon \leq t\,. \end{cases}$$

Note that $\xi_\epsilon \to \chi_{\{t>0\}}$ as $\epsilon \downarrow 0$. Let $\theta \in C_0^\infty((0,b)),\ \theta \geq 0$ and set

$$\theta_1^\epsilon = (1 - \xi_\epsilon(x_2 - x_1))\theta \quad \text{and} \quad \theta_2^\epsilon = \xi_\epsilon(x_2 - x_1)\theta\,.$$

Evidently $\theta_1^\epsilon,\ \theta_2^\epsilon \geq 0$ and, from the chain rule for Sobolev functions, we have

$$(\theta_1^\epsilon)^{'} = \theta^{'} - \xi_\epsilon^{'}(x_2 - x_1)(x_2 - x_1)^{'}\theta - \xi_\epsilon(x_2 - x_1)\theta^{'}$$

and

$$(\theta_2^\epsilon)^{'} = \xi_\epsilon^{'}(x_2 - x_1)(x_2 - x_1)^{'}\theta + \xi_\epsilon(x_2 - x_1)\theta^{'}\,.$$

Since by hypothesis $x_1, x_2 \in S$ we have

$$(A_1(x_1), \theta_1^\epsilon)_{pq} = (\hat{f}(x_1), \theta_1^\epsilon)_{pq}$$

and

$$(A_1(x_2), \theta_2^\epsilon)_{pq} = (\hat{f}(x_2), \theta_2^\epsilon)_{pq}$$

with $\hat{f}(y(\cdot)) = f(\cdot, y(\cdot), y^{'}(\cdot))$ for all $y \in W^{1,p}(T)$. We have

$$\begin{aligned} &(A_1(x_1), \theta_1^\epsilon)_{pq} + (A_1(x_2), \theta_2^\epsilon)_{pq} \\ &\quad = \int_0^b (|x_2^{'}(t)|^{p-2}x_2^{'}(t) - |x_1^{'}(t)|^{p-2}x_1^{'}(t))\xi_\epsilon^{'}(x_2 - x_1)(t)(x_2 - x_1)^{'}(t)\theta(t)\,dt \\ &\qquad + \int_0^b (|x_2^{'}(t)|^{p-2}x_2^{'}(t) - |x_1^{'}(t)|^{p-2}x_1^{'}(t))\xi_\epsilon(x_2 - x_1)(t)\theta^{'}(t)\,dt \\ &\qquad + \int_0^b (|x_1^{'}(t)|^{p-2}x_1^{'}(t)\theta^{'}(t)\,dt\,. \end{aligned}$$

Note that $\xi_\epsilon^{'}(x_2 - x_1)\theta \geq 0$ and

$$(|x_2^{'}(t)|^{p-2}x_2^{'}(t) - |x_1^{'}(t)|^{p-2}x_1^{'}(t))(x_2 - x_1)^{'}(t) \geq 0\,.$$

Hence

$$\begin{aligned}
&(A_1(x_1),\theta_1^\epsilon)_{pq}+(A_1(x_2),\theta_2^\epsilon)_{pq}\\
&\quad\geq \int_0^b (|x_2^{'}(t)|^{p-2}x_2^{'}(t)-|x_1^{'}(t)|^{p-2}x_1^{'}(t))\xi_\epsilon(x_2-x_1)(t)\theta^{'}(t)\,dt\\
&\qquad+\int_0^b (|x_1^{'}(t)|^{p-2}x_1^{'}(t)\theta^{'}(t)\,dt\,.
\end{aligned}$$

Passing to the limit as $\epsilon\downarrow 0$, we obtain

$$\begin{aligned}
&\liminf[(A_1(x_1),\theta_1^\epsilon)_{pq}+(A_1(x_2),\theta_2^\epsilon)_{pq}]\\
&\quad\geq \int_0^b |x_2^{'}(t)|^{p-2}x_2^{'}(t)\chi_{\{x_2\geq x_1\}}(t)\theta^{'}(t)\,dt\\
&\qquad+\int_0^b |x_1^{'}(t)|^{p-2}x_1^{'}(t)\chi_{\{x_2\leq x_1\}}(t)\theta^{'}(t)\,dt\\
&\quad=\int_0^b |y^{'}(t)|^{p-2}y^{'}(t)\theta^{'}(t)\,dt=(A_1(y),\theta)_{pq}\,.
\end{aligned}$$

Also we have

$$(\hat f(x_1),\theta_1^\epsilon)_{pq}+(\hat f(x_2),\theta_2^\epsilon)_{pq}\to(\hat f(y),\theta)_{pq}$$

as $\epsilon\downarrow 0$. Thus finally in the limit as $\epsilon\downarrow 0$, we obtain

$$(A_1(y),\theta)_{pq}\leq(\hat f(y),\theta)_{pq}\,,$$

for all $\theta\in C_0^\infty((0,b)),\theta\geq 0$. Then

$$\int_0^b |y^{'}(t)|^{p-2}y^{'}(t)\theta^{'}(t)\,dt\leq\int_0^b f(t,y(t),y^{'}(t))\theta(t)\,dt$$

and so, by Green's identity, since $y^{'}(0)=y^{'}(b)=0$, we have

$$\int_0^b -(|y^{'}(t)|^{p-2}y^{'}(t))^{'}\theta(t)\,dt\leq\int_0^b f(t,y(t),y^{'}(t))\theta(t)\,dt\,.$$

Then, since $\theta\in C_0^\infty((0,b)),\theta\geq 0$ is arbitrary, we infer that the function $y=x_1\vee x_2$ satisfies the following properties

$$\left\{\begin{array}{ll}-(|y^{'}(t)|^{p-2}y^{'}(t))^{'}\leq f(t,y(t),y^{'}(t)) & \text{a.e. on } T\\ y^{'}(b)=0=y^{'}(0)\end{array}\right\}\tag{17}$$

and y is a continuous function such that $|y^{'}(\cdot)|^{p-2}y^{'}(\cdot)\in W^{1,q}(T)$. Therefore (we observe that in order to prove the existence of a function x_3 is not necessary $y\in C^1(T)$), as in the proof of Proposition 4, we can find $x_3\in S$ such that $y\leq x_3\leq\varphi$. Hence $(S,\leq)$ is directed.

Next let C be a chain of S. Let $x=\sup C$. Using Corollary 7, p. 336 of Dunford-Schwartz [5], we can find $\{x_n\}_{n\in\mathbb{N}}\subseteq C$ non decreasing such that $x_n\to x$ in $L^p(T)$. We have

$$A(x_n)=\hat f(x_n)$$

and so

$$(A(x_n), x_n)_{pq} = (\hat{f}(x_n), x_n)_{pq}\,.$$

Using hypothesis $H(f)$(iii), we obtain

$$\|x_n^{'}\|_p^p \ \le M(\|a\|_q + \ c\|x_n^{'}\|_p^{p-1})$$

where $M > 0$. Then $\{x_n\}_{n\in\mathbb{N}} \subseteq W^{1,p}(T)$ is bounded. So we may assume that $x_n \xrightarrow{w} x$ in $W^{1,p}(T)$. Also we have

$$\lim (A(x_n), x_n)_{pq} = \lim (\hat{f}(x_n), x_n)_{pq} = 0$$

and so, being A pseudomonotone, we can say

$$\|x_n^{'}\|_p \to \|x^{'}\|_p\,.$$

Hence, by the Kadec-Klee property, $x_n \to x$ in $W^{1,p}(T)$. So in the limit as $n \to \infty$, we have $A(x) = \hat{f}(x)$ a.e. on T and so, by Proposition 3,

$$\left\{ \begin{array}{ll} -(|x^{'}(t)|^{p-2}x^{'}(t))^{'} = f(t, x(t), x^{'}(t)) & \text{a.e. on } T \\ x^{'}(0) = x^{'}(b) = 0 & \end{array} \right\} \tag{18}$$

i.e. $x \in S$.

Apply Zorn's lemma to obtain a maximal element x^* of $(S, \le)$. Since $(S, \le)$ is directed, x^* is unique and it is the maximal solution of problem (1) in K. On the other hand, S is directed with respect to $\ge$, i.e. for all $x_1, x_2 \in S$ there exists $x_3 \in S$ such that $x_1 \ge x_3$ and $x_2 \ge x_3$ (see Peressini [15], p.3). Similarly we prove the existence of a maximal element x_* of $(S, \ge)$. Therefore x_* is the unique minimal solution of problem (1) in K. □

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