

**ON NATURAL METRICS ON TANGENT BUNDLES  
OF RIEMANNIAN MANIFOLDS**

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ABSTRACT. There is a class of metrics on the tangent bundle  $TM$  of a Riemannian manifold  $(M, g)$  (oriented, or non-oriented, respectively), which are 'naturally constructed' from the base metric  $g$  [15]. We call them " $g$ -natural metrics" on  $TM$ . To our knowledge, the geometric properties of these general metrics have not been studied yet. In this paper, generalizing a process of Musso-Tricerri (cf. [18]) of finding Riemannian metrics on  $TM$  from some quadratic forms on  $OM \times \mathbb{R}^m$  to find metrics (not necessary Riemannian) on  $TM$ , we prove that all  $g$ -natural metrics on  $TM$  can be obtained by Musso-Tricerri's generalized scheme. We calculate also the Levi-Civita connection of Riemannian  $g$ -natural metrics on  $TM$ . As application, we sort out all Riemannian  $g$ -natural metrics with the following properties, respectively: 1) The fibers of  $TM$  are totally geodesic. 2) The geodesic flow on  $TM$  is incompressible. We shall limit ourselves to the non-oriented situation.

## INTRODUCTION

Geometry of the tangent bundle  $TM$  of an  $m$ -dimensional Riemannian manifold  $(M, g)$  with Sasaki metric has been extensively studied since the 60's. Nevertheless, the rigidity of this metric (cf. [3], [18] and [22]) has incited some geometers to tackle the problem of the construction and the study of other metrics on  $TM$ . The Cheeger-Gromoll metric (cf. [8]) has appeared as a nicely fitted one to overcome this rigidity, and has been, thus, studied by many authors (see [2], [3] and [22]). Using the concept of naturality, O. Kowalski and M. Sekizawa [15] have given a full classification of metrics which are 'naturally constructed' from a metric  $g$  on the base  $M$ , supposing that  $M$  is oriented. Other presentations of the basic results from [15] (involving also the non-oriented case and something more) can be found in [13] or [17]. We call these metrics  *$g$ -natural metrics on  $TM$* . To our knowledge, the geometric properties of these general metrics on  $TM$  have not been studied yet (see also [16]).

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In this paper, we deal with  $g$ -natural metrics on  $TM$  in the case when the orientation of  $M$  is not taken into account. In fact, in the non-oriented case we only lose some special  $g$ -natural metrics over Riemannian manifolds of dimensions 2 and 3; in dimensions  $m > 3$ , the oriented case and the non-oriented case coincide. In § 2, we sort out from  $g$ -natural metrics on  $TM$  (which may even be degenerate) those which are regular and those which are Riemannian.

In § 3, we generalize, to the non Riemannian case, the process of construction of Riemannian metrics on  $TM$  from symmetric basic tensor fields of type  $(2, 0)$  on  $OM \times \mathbb{R}^m$ , presented in [18] by E. Musso and F. Tricerri, where  $OM$  is the bundle of orthonormal frames. We show then that all  $g$ -natural metrics can be obtained by the generalized Musso-Tricerri's process.

In § 4, we give explicit formulas of the Levi-Civita connection  $\bar{\nabla}$  of a  $g$ -natural metric on  $TM$  and we provide necessary and sufficient conditions on  $G$  to have the fibers of  $TM$  totally geodesic.

On the other hand, it is well known that with respect to Sasaki metric and Cheeger-Gromoll metric on  $TM$ , the geodesic flow of  $TM$  is incompressible (cf. [2] and [21]). In § 5, we give necessary and sufficient conditions on Riemannian  $g$ -natural metrics which let the geodesic flow of  $TM$  incompressible. As a consequence, it is particularly worth mentioning that some  $g$ -natural metrics present a kind of rigidity related to the geodesic flow. For instance, Let  $\mathcal{R}^3$  denote the vector space of all  $g$ -natural metrics of the form  $G = a \cdot g^s + b \cdot g^h + c \cdot g^v$  (i.e., linear combinations with constant coefficients of the three classical lifts  $g^s$ ,  $g^h$  and  $g^v$  of  $g$ ). Define  $\mathcal{C}$  as the 2-dimensional cone in  $\mathcal{R}^3$  characterized by the inequalities  $a > 0$ ,  $c > 0$  and  $b^2 - a(a + c) < 0$ . Then  $\mathcal{C}$  is just the subset of all Riemannian metrics in  $\mathcal{R}^3$ . Now, we can prove that, for every  $G$  from  $\mathcal{C}$  with  $b \neq 0$ , the geodesic flow on  $TM$  is incompressible, with respect to  $G$ , if and only if  $(M, g)$  is an Einstein space with vanishing scalar curvature.

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## 1. PRELIMINARIES

Let  $\nabla$  be the Levi-Civita connection of  $g$ . Then the tangent space of  $TM$  at any point  $(x, u) \in TM$  splits into the horizontal and vertical subspaces with respect to  $\nabla$ :

$$(TM)_{(x,u)} = H_{(x,u)} \oplus V_{(x,u)}.$$

If  $(x, u) \in TM$  is given then, for any vector  $X \in M_x$ , there exists a unique vector  $X^h \in H_{(x,u)}$  such that  $p_* X^h = X$ , where  $p : TM \rightarrow M$  is the natural projection. We call  $X^h$  the *horizontal lift* of  $X$  to the point  $(x, u) \in TM$ . The *vertical lift* of a vector  $X \in M_x$  to  $(x, u) \in TM$  is a vector  $X^v \in V_{(x,u)}$  such that  $X^v(df) = Xf$ , for all functions  $f$  on  $M$ . Here we consider 1-forms  $df$  on  $M$  as functions on  $TM$  (i.e.  $(df)(x, u) = uf$ ). Note that the map  $X \rightarrow X^h$  is an isomorphism between the vector spaces  $M_x$  and  $H_{(x,u)}$ . Similarly, the map  $X \rightarrow X^v$  is an isomorphism between the vector spaces  $M_x$  and  $V_{(x,u)}$ . Obviously,

each tangent vector  $\tilde{Z} \in (TM)_{(x,u)}$  can be written in the form  $\tilde{Z} = X^h + Y^v$ , where  $X, Y \in M_x$  are uniquely determined vectors.

If  $\varphi$  is a smooth function on  $M$ , then

$$(1.1) \quad X^h(\varphi \circ p) = (X\varphi) \circ p \quad \text{and} \quad X^v(\varphi \circ p) = 0$$

hold for every vector field  $X$  on  $M$ .

A system of local coordinates  $\{(U; x^i, i = 1, \dots, m)\}$  in  $M$  induces on  $TM$  a system of local coordinates  $\{(p^{-1}(U); x^i, u^i, i = 1, \dots, m)\}$ . Let  $X = \sum_i X^i \frac{\partial}{\partial x^i}$  be the local expression in  $U$  of a vector field  $X$  on  $M$ . Then, the horizontal lift  $X^h$  and the vertical lift  $X^v$  of  $X$  are given, with respect to the induced coordinates, by:

$$(1.2) \quad X^h = \sum X^i \frac{\partial}{\partial x^i} - \sum \Gamma_{jk}^i u^j X^k \frac{\partial}{\partial u^i},$$

and

$$(1.3) \quad X^v = \sum X^i \frac{\partial}{\partial u^i},$$

where  $(\Gamma_{jk}^i)$  denote the Christoffel's symbols of  $g$ .

Now, let  $r$  be the norm of a vector  $u$ . Then, for any function  $f$  of  $\mathbb{R}$  to  $\mathbb{R}$ , we get

$$(1.4) \quad X_{(x,u)}^h(f(r^2)) = 0,$$

$$(1.5) \quad X_{(x,u)}^v(f(r^2)) = 2f'(r^2)g_x(X_x, u),$$

and in particular, we have

$$(1.6) \quad X_{(x,u)}^h(r^2) = 0,$$

and

$$(1.7) \quad X_{(x,u)}^v(r^2) = 2g_x(X_x, u).$$

Let  $X, Y$  and  $Z$  be any vector fields on  $M$ . If  $F_Y$  is the function on  $TM$  defined by  $F_Y(x, u) = g_x(Y_x, u)$ , for all  $(x, u) \in TM$ , then we have

$$(1.8) \quad X_{(x,u)}^h(F_Y) = g_x((\nabla_X Y)_x, u) = F_{\nabla_X Y}(x, u),$$

$$(1.9) \quad X_{(x,u)}^v(F_Y) = g_x(X, Y),$$

$$(1.10) \quad X_{(x,u)}^h(g(Y, Z) \circ p) = X_x(g(Y, Z)),$$

$$(1.11) \quad X_{(x,u)}^v(g(Y, Z) \circ p) = 0.$$

The formulas (1.4)–(1.9) follow from (1.1) and

$$(1.12) \quad X^h u^i = - \sum X^\lambda u^\mu \Gamma_{\lambda\mu}^i \quad \text{and} \quad X^v u^i = X^i,$$

and the relations (1.10) and (1.11) follow easily from (1.1).

Next, we shall introduce some notations which will be used describing vectors getting from lifted vectors by basic operations on  $TM$ . Let  $T$  be a tensor field of type  $(1, s)$  on  $M$ . If  $X_1, X_2, \dots, X_{s-1} \in M_x$ , then  $h\{T(X_1, \dots, u, \dots, X_{s-1})\}$

(resp.  $v\{T(X_1, \dots, u, \dots, X_{s-1})\}$ ) is a horizontal (resp. vertical) vector at  $(x, u)$  which is introduced by the formula

$$h\{T(X_1, \dots, u, \dots, X_{s-1})\} = \sum u^\lambda (T(X_1, \dots, (\frac{\partial}{\partial x^\lambda})_{x, \dots, X_{s-1}}))^h$$

$$\left( \text{resp. } v\{T(X_1, \dots, u, \dots, X_{s-1})\} = \sum u^\lambda (T(X_1, \dots, (\frac{\partial}{\partial x^\lambda})_{x, \dots, X_{s-1}}))^v \right).$$

In particular, if  $T$  is the identity tensor of type  $(1, 1)$ , then we obtain the geodesic flow vector field at  $(x, u)$ ,  $\xi_{(x,u)} = \sum u^\lambda (\frac{\partial}{\partial x^\lambda})_{(x,u)}^h$ , and the canonical vertical vector at  $(x, u)$ ,  $U_{(x,u)} = \sum u^\lambda (\frac{\partial}{\partial x^\lambda})_{(x,u)}^v$ . Moreover  $h\{T(X_1, \dots, u, \dots, u, \dots, X_{s-1})\}$  and  $v\{T(X_1, \dots, u, \dots, u, \dots, X_{s-1})\}$  are introduced by similar way. Also we make the conventions  $h\{T(X_1, \dots, X_{s-1})\} = (T(X_1, \dots, X_{s-1}))^h$  and  $v\{T(X_1, \dots, X_{s-1})\} = (T(X_1, \dots, X_{s-1}))^v$ . Thus  $h\{X\} = X^h$  and  $v\{X\} = X^v$ , for each vector field  $X$  on  $M$ .

The bracket operation of vector fields on the tangent bundle is given by

$$(1.13) \quad [X^h, Y^h]_{(x,u)} = [X, Y]_{(x,u)}^h - v\{R(X_x, Y_x)u\},$$

$$(1.14) \quad [X^h, Y^v]_{(x,u)} = (\nabla_X Y)_{(x,u)}^v,$$

$$(1.15) \quad [X^v, Y^v]_{(x,u)} = 0,$$

for all vector fields  $X$  and  $Y$  on  $M$ , where  $R$  is the Riemannian curvature of  $g$  defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X, Y]}.$$

Finally, the following Koszul formula holds

$$(1.16) \quad \begin{aligned} g(\nabla_X Y, Z) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &+ g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X), \end{aligned}$$

for all vector fields  $X, Y$  and  $Z$  on  $M$ .

Now, if we write  $p_M : TM \rightarrow M$  for the natural projection and  $F$  for the natural bundle with  $FM = p_M^*(T^* \otimes T^*)M \rightarrow M$ ,  $Ff(X_x, g_x) = (Tf \cdot X_x, (T^* \otimes T^*)f \cdot g_x)$  for all manifolds  $M$ , local diffeomorphisms  $f$  of  $M$ ,  $X_x \in T_x M$  and  $g_x \in (T^* \otimes T^*)_x M$ . The sections of the canonical projection  $FM \rightarrow M$  are called *F-metrics* in literature. So, if we denote by  $\oplus$  the fibered product of fibered manifolds, then the *F-metrics* are mappings  $TM \oplus TM \oplus TM \rightarrow \mathbb{R}$  which are linear in the second and the third argument.

As generalization of the notion of *F-metrics*, we can define the notion of *F-tensor fields* of any type on a manifold. For  $(p, q) \in \mathbb{N}^2$ , we write  $p_M : TM \rightarrow M$  for the natural projection and  $F$  for the natural bundle with  $FM = p_M^*(\underbrace{T^* \otimes \dots \otimes T^*}_{p\text{-times}} \otimes \underbrace{T \otimes \dots \otimes T}_{q\text{-times}})M \rightarrow M$ ,  $Ff(X_x, S_x) = (Tf \cdot X_x, (T^* \otimes \dots \otimes T^* \otimes T \otimes \dots \otimes T)f \cdot S_x)$

for all manifolds  $M$ , local diffeomorphisms  $f$  of  $M$ ,  $X_x \in T_x M$  and  $S_x \in (T^* \otimes \dots \otimes T^* \otimes T \otimes \dots \otimes T)_x M$ . We call the sections of the canonical projection  $FM \rightarrow M$  *F-tensor fields of type  $(p, q)$* . So *F-tensor fields* are mappings  $A :$

$TM \oplus \underbrace{TM \oplus \cdots \oplus TM}_{q\text{-times}} \rightarrow \bigsqcup_{x \in M} \otimes^p M_x$  which are linear in the last  $q$  summands

such that  $\pi_2 \circ A = \pi_1$ , where  $\pi_1$  and  $\pi_2$  are the natural projections of the source and target fiber bundles of  $A$  respectively. For  $p = 0$  and  $q = 2$ , we obtain the classical notion of  $F$ -metrics.

If we fix an  $F$ -metric  $\delta$  on  $M$ , then there are three distinguished constructions of metrics on the tangent bundle  $TM$ , which are given as follows [15]:

(a) If we suppose that  $\delta$  is symmetric, then the *Sasaki lift*  $\delta^s$  of  $\delta$  is defined as follows:

$$\begin{cases} \delta_{(x,u)}^s(X^h, Y^h) = \delta(u; X, Y), & \delta_{(x,u)}^s(X^h, Y^v) = 0, \\ \delta_{(x,u)}^s(X^v, Y^h) = 0, & \delta_{(x,u)}^s(X^v, Y^v) = \delta(u; X, Y), \end{cases}$$

for all  $X, Y \in M_x$ . If  $\delta$  is non degenerate and positive definite, then the same holds for  $\delta^s$ .

(b) The *horizontal lift*  $\delta^h$  of  $\delta$  is a pseudo-Riemannian metric on  $TM$  which is given by:

$$\begin{cases} \delta_{(x,u)}^h(X^h, Y^h) = 0, & \delta_{(x,u)}^h(X^h, Y^v) = \delta(u; X, Y), \\ \delta_{(x,u)}^h(X^v, Y^h) = \delta(u; X, Y), & \delta_{(x,u)}^h(X^v, Y^v) = 0, \end{cases}$$

for all  $X, Y \in M_x$ . If  $\delta$  is positive definite, then  $\delta^s$  is of signature  $(m, m)$ .

(c) The *vertical lift*  $\delta^v$  of  $\delta$  is a degenerate metric on  $TM$  which is given by:

$$\begin{cases} \delta_{(x,u)}^v(X^h, Y^h) = \delta(u; X, Y), & \delta_{(x,u)}^v(X^h, Y^v) = 0, \\ \delta_{(x,u)}^v(X^v, Y^h) = 0, & \delta_{(x,u)}^v(X^v, Y^v) = 0, \end{cases}$$

for all  $X, Y \in M_x$ . The rank of  $\delta^v$  is exactly that of  $\delta$ . If  $\delta = g$  is a Riemannian metric on  $M$ , then the three lifts of  $\delta$  just constructed coincide with the three well-known classical lifts of the metric  $g$  to  $TM$ .

## 2. NATURAL METRICS ON TANGENT BUNDLES

Now, we shall describe all first order natural operators  $D : S_+^2 T^* \rightsquigarrow (S^2 T^*)T$  transforming Riemannian metrics on manifolds into metrics on their tangent bundles, where  $S_+^2 T^*$  and  $S^2 T^*$  denote the bundle functors of all Riemannian metrics and all symmetric two-forms over  $m$ -manifolds respectively. For the concept of naturality and related notions, see [13] for more details.

Let us call every section  $G : TM \rightarrow (S^2 T^*)TM$  a (possibly degenerate) *metric*. Then we can assert:

**Proposition 2.1** ([15]). *There is a bijective correspondence between the triples of natural  $F$ -metrics  $(\zeta_1, \zeta_2, \zeta_3)$ , where  $\zeta_1$  and  $\zeta_3$  are symmetric, and natural (possibly degenerate) metrics  $G$  on the tangent bundles given by*

$$G = \zeta_1^s + \zeta_2^h + \zeta_3^v.$$

Therefore, to find all first order natural operators  $S_+^2 T^* \rightsquigarrow (S^2 T^*)T$  transforming Riemannian metrics on manifolds into metrics on their tangent bundles, it



all  $t \in [0, \infty)$ . For  $m = 1$  the two conditions reduce to one, i.e.  $\alpha(t) \neq 0$ , for all  $t \in [0, \infty)$ .

**Proof.** Let  $x \in M$ ,  $u \in M_x \setminus \{0_x\}$  and  $X_1 = \frac{1}{r} \cdot u$ , where  $r = \|u\|$ .

For  $m = 1$ , the determinant of the matrix of  $G_{(x,u)}$ , with respect to the basis  $\{X_1^h, X_1^v\}$  of  $(TM)_{(x,u)}$ , is given by  $\alpha(t) \cdot \phi(t)$ .

For  $m > 1$ , choosing vectors  $X_2, \dots, X_m$  of  $M_x$  such that  $\{X_1, X_2, \dots, X_m\}$  is an orthonormal basis of  $(M_x, g_x)$ , then the matrix of  $G_{(x,u)}$ , with respect to the basis  $\{X_1^h, X_2^h, \dots, X_m^h, X_1^v, X_2^v, \dots, X_m^v\}$  of  $(TM)_{(x,u)}$ , is given by  $P_m(r^2)$ , where  $P_m$  is the  $(2m, 2m)$ -matrix-valued real function

$$(2.3) \quad P_m(t) = \begin{pmatrix} (\phi_1 + \phi_3)(t) & 0 \cdots 0 & \phi_2(t) & 0 \cdots 0 \\ 0 & & 0 & \\ \vdots & (\alpha_1 + \alpha_3)(t) \cdot I_{m-1} & \vdots & \alpha_2(t) \cdot I_{m-1} \\ 0 & & 0 & \\ \phi_2(t) & 0 \cdots 0 & \phi_1(t) & 0 \cdots 0 \\ 0 & & 0 & \\ \vdots & \alpha_2(t) \cdot I_{m-1} & \vdots & \alpha_1(t) \cdot I_{m-1} \\ 0 & & 0 & \end{pmatrix},$$

$I_{m-1}$  being the identity matrix of  $GL(m)$ , and we can prove by induction on  $m$  that the determinant of the last matrix is equal to  $\phi(t) \cdot \alpha^{m-1}(t)$ .

On the other hand, if  $u = 0$  then the determinant of the matrix of  $G_{(x,0)}$  with respect to any basis  $\{X_1^h, X_2^h, \dots, X_m^h, X_1^v, X_2^v, \dots, X_m^v\}$  of  $(TM)_{(x,u)}$ , where  $\{X_1, X_2, \dots, X_m\}$  is any orthonormal basis of  $(M_x, g_x)$ , is given by  $\alpha^m(0) = \phi(0) \cdot \alpha^{m-1}(0)$ . Now the result follows easily.  $\square$

Similarly, we can prove the following:

**Proposition 2.8** ([5]). *The necessary and sufficient conditions for a  $g$ -natural metric  $G$  on the tangent bundle of a Riemannian manifold  $(M, g)$  to be Riemannian are that the functions of Proposition 2.4, defining  $G$ , satisfy the inequalities*

$$(2.4) \quad \begin{cases} \alpha_1(t) > 0, & \phi_1(t) > 0, \\ \alpha(t) > 0, & \phi(t) > 0, \end{cases}$$

for all  $t \in [0, \infty)$ . For  $m = 1$  the system reduces to  $\alpha_1(t) > 0$  and  $\alpha(t) > 0$ , for all  $t \in [0, \infty)$ .

#### Important conventions:

1) In the sequel, when we consider an arbitrary Riemannian  $g$ -natural metric  $G$  on  $TM$ , we implicitly suppose that it is defined by the functions  $\alpha_i, \beta_i : [0, \infty) \rightarrow \mathbb{R}$ ,  $i = 1, 2, 3$ , given in Corollary 2.4 and satisfying (2.4).

2) Unless otherwise stated, all real functions  $\alpha_i, \beta_i, \phi_i, \alpha$  and  $\phi$  and their derivatives are evaluated at  $r^2 := g_x(u, u)$ .

3.  $g$ -NATURAL METRICS BY THE SCHEME OF MUSSO-TRICERRI

Considering  $TM$  as a vector bundle associated with the bundle of orthonormal frames  $OM$ , E. Musso and F. Tricerri have constructed an interesting class of *Riemannian* natural metrics on  $TM$  [18]. This construction is not a classification *per se*, but it is a construction process of *Riemannian* metrics on  $TM$  from symmetric, positive semi-definite tensor fields  $Q$  of type  $(2, 0)$  and rank  $2m$  on  $OM \times \mathbb{R}^m$ , which are basic for the natural submersion  $\Phi : OM \times \mathbb{R}^m \rightarrow TM$ ,  $\Phi(v, \varepsilon) = (x, \sum_i \varepsilon^i v_i)$ , for  $v = (x; v_1, \dots, v_m) \in OM$  and  $\varepsilon = (\varepsilon^1, \dots, \varepsilon^m) \in \mathbb{R}^m$ . Recall that  $Q$  is *basic* means that  $Q$  is  $O(m)$ -invariant and  $Q(X, Y) = 0$ , if  $X$  is tangent to a fiber of  $\Phi$ .

Given such a  $Q$ , there is a unique *Riemannian* metric  $G^Q$  on  $TM$  such that  $\Phi^*(G^Q) = Q$ . This metric is determined by the formula

$$(3.1) \quad G_{(x,u)}^Q(X, Y) = Q_{(v,\varepsilon)}(X', Y'),$$

where  $(v, \varepsilon)$  belongs to the fiber  $\Phi^{-1}(x, u)$ ,  $X, Y$  are elements of  $(TM)_{(x,u)}$ ,  $X', Y'$  are tangent vectors to  $OM \times \mathbb{R}^m$  at  $(v, \varepsilon)$  with  $d\Phi(X') = X$  and  $d\Phi(Y') = Y$ .

Now, we can check that this process can be generalized to construct also metrics on  $TM$  which are not necessarily Riemannian (even degenerate ones). Precisely, we have:

**Proposition 3.1.** *Let  $Q$  be a symmetric tensor field of type  $(2, 0)$  on  $OM \times \mathbb{R}^m$ , which is basic for the natural submersion  $\Phi : OM \times \mathbb{R}^m \rightarrow TM$ . Then there is a unique metric  $G^Q$  on  $TM$  such that  $\Phi^*(G^Q) = Q$ . It is given by (3.1).*

Furthermore, we have:

1. The rank of  $G^Q$  is equal to that of  $Q$ .
2.  $G^Q$  is Riemannian if and only if  $Q$  is positive semi-definite of rank  $2m$ .

Remark that the rank of  $Q$  is less than or equal to  $2m$ , since  $Q$  is basic, and that the second assertion of the previous proposition corresponds exactly to the original process of E. Musso and F. Tricerri given in [18]. Note also that such a generalization is possible due to the fact that the process of identification of  $TM$  as an associated bundle is natural.

Let  $(e_1, \dots, e_m)$  be an orthonormal frame field defined on an open set  $U \subset M$ , and let  $(x^1, \dots, x^m)$  be a local coordinate system on  $U$ . We define a local coordinate system  $(x^1, \dots, x^m, u^1, \dots, u^m)$  on  $p^{-1}(U)$  as follows:

$$x^i(x, u) = x^i(x), \quad u^i(x, u) = u^i, \quad (x, u) \in p^{-1}(U), \quad \text{where} \quad u = \sum_i u^i e_i(x).$$

We denote with  $\Gamma_j^i$  the local 1-forms defined by

$$\nabla_X e_i = \sum_j \Gamma_j^i(X) e_j.$$

Let  $e^i$  be the 1-forms on  $p^{-1}(U)$  defined by  $e^i(e_k) = \delta_k^i$ , and

$$Du^i = du^i + \sum_j u^j p^*(\Gamma_j^i).$$

Then  $(e_1^h, \dots, e_m^h, e_1^v, \dots, e_m^v)$  is a frame field on  $p^{-1}(U)$ , whose dual coframe is given by

$$(3.2) \quad p^* e^1, \dots, p^* e^m, Du^1, \dots, Du^m.$$

Let  $\theta = (\theta^1, \dots, \theta^m)$  denote the canonical 1-form on  $OM$ , and let  $\pi$  denote the natural projection  $OM \xrightarrow{\pi} M$ . Then

$$d\pi_v(X) = \sum_i \theta^i(X) v_i, \quad v = (x; v_1, \dots, v_m).$$

If we denote with  $\omega = (\omega_j^i)$  the  $so(m)$ -valued differential form defined by the Levi-Civita connection of  $g$ , then we find that

$$\theta^i, i = 1, \dots, m; \omega_j^i, 1 \leq i \leq j \leq m; d\varepsilon^i, i = 1, \dots, m,$$

is an absolute parallelism on  $OM \times \mathbb{R}^m$ . Note that we use here (and in the sequel) the abuse of notation  $\theta = \pi_1^* \theta$ , where  $\pi_1 : OM \times \mathbb{R}^m \rightarrow OM$  is the natural first projection. We put

$$D\varepsilon^i = d\varepsilon^i + \sum_j \varepsilon^j \omega_j^i.$$

On the other hand, we have [18]:

**Lemma 3.2.** *Any basic symmetric quadratic form  $Q$  on  $OM \times \mathbb{R}^m$  is a second order polynomial in  $\theta^i$  and  $D\varepsilon^i$  whose coefficients yield  $Q$  invariant under the  $O(m)$ -action.*

As an application of Proposition 3.1, we consider the two following symmetric quadratic forms on  $OM \times \mathbb{R}^m$ ,

$$Q^h = \sum_i \theta^i D\varepsilon^i \quad \text{and} \quad Q^v = \sum_i (\theta^i)^2,$$

which are basic by Lemma 3.2. They give rise, via the scheme of Proposition 3.1, to the classical lifts  $g^h$  and  $g^v$ , respectively. It is clear that  $Q^h$ , as  $g^h$ , is of signature  $(m, m)$ , and that  $Q^v$  is degenerate of rank  $m$  as the metric  $g^v$ .

Generally, we can assert the following:

**Proposition 3.3.** *Every  $g$ -natural metric on the tangent bundle  $TM$  of a Riemannian manifold  $(M, g)$  can be constructed by the Musso-Tricerri's generalized scheme, given by Proposition 3.1.*

**Proof.** Let  $G$  be a  $g$ -natural metric on the tangent bundle  $TM$  of a Riemannian manifold  $(M, g)$ . With respect to the coframe (3.2),  $G$  can be written as follows:

$$(3.3) \quad \begin{aligned} G = & (\alpha_1 + \alpha_3)(r^2) \sum_i (p^* e^i)^2 + (\beta_1 + \beta_3)(r^2) (\sum_i u^i (p^* e^i))^2 \\ & + \alpha_1(r^2) \sum_i (Du^i)^2 + \beta_1(r^2) (\sum_i u^i Du^i)^2 \\ & + \alpha_2(r^2) \sum_i (p^* e^i) Du^i + \beta_2(r^2) (\sum_i u^i (p^* e^i)) (\sum_i u^i Du^i), \end{aligned}$$

where  $r^2 = \sum_i (u^i)^2$  and  $\alpha_i, \beta_i, i = 1, 2, 3$ , are functions from  $[0, \infty)$  to  $\mathbb{R}$ .

Consider the symmetric tensor field  $Q$  of type  $(2, 0)$  on  $OM \times \mathbb{R}^m$

$$(3.4) \quad \begin{aligned} Q = & (\alpha_1 + \alpha_3)(r^2) \sum_i (\theta^i)^2 + (\beta_1 + \beta_3)(r^2) (\sum_i \varepsilon^i \theta^i)^2 \\ & + \alpha_1(r^2) \sum_i (D\varepsilon^i)^2 + \beta_1(r^2) (\sum_i \varepsilon^i D\varepsilon^i)^2 \\ & + \alpha_2(r^2) \sum_i \theta^i D\varepsilon^i + \beta_2(r^2) (\sum_i \varepsilon^i \theta^i) (\sum_i \varepsilon^i D\varepsilon^i), \end{aligned}$$

where  $r^2 = \sum_i (\varepsilon^i)^2$ . It is easy to see [18] that:

$$\begin{aligned} R_a^*(\theta^i) &= \sum_j (a^{-1})_j^i \theta^j, \\ R_a^*(\omega_j^i) &= \sum_{k,l} (a^{-1})_k^i \omega_l^k a_j^l, \\ R_a^*(D\varepsilon^i) &= \sum_j (a^{-1})_j^i D\varepsilon^j, \end{aligned}$$

for all  $a \in O(m)$ , where  $R_a$  is the natural translation by  $a$ . Then  $Q$  is  $O(m)$ -invariant.

On the other hand,  $Q$  is basic by Lemma 3.2. Therefore, by virtue of Proposition 3.1,  $Q$  induces a unique metric on  $TM$ . Furthermore, we claim that  $Q$  induces  $G$ . Indeed, if we denote by  $\Phi_U$  the  $O(m)$ -valued function on  $\pi^{-1}(U)$  given by

$$(\Phi_U)_j^i(v) = g(e_i(\pi(v)), v_j),$$

then the forms  $\omega_j^i$  are related to the local 1-forms  $\Gamma_j^i$  as follows

$$(3.5) \quad \omega_j^i = \sum_k (\Phi_U^{-1})_k^i d(\Phi_U)_j^k + \sum_{k,l} (\Phi_U^{-1})_k^i (p^* \Gamma_l^k) (\Phi_U)_j^l.$$

We can also check easily that

$$(3.6) \quad \Phi^*(u^i) = \sum_j (\Phi_U)_j^i \varepsilon^j.$$

Using formulas (3.5) and (3.6) and the following commutative diagram:

$$\begin{array}{ccc} OM \times \mathbb{R}^m & \xrightarrow{\Phi} & TM \\ \pi_1 \downarrow & & \downarrow p \\ OM & \xrightarrow{\pi} & M, \end{array}$$

we get

$$(3.7) \quad \Phi^*(p^* e^i) = \sum_j (\Phi_U)_j^i \theta^j,$$

and

$$(3.8) \quad \Phi^*(Dv^i) = \sum_j (\Phi_U)_j^i D\varepsilon^j.$$

Note that formula (4.10) in [18] should read our formula (3.7), i.e.  $\Phi^*(p^* e^i)$  instead of  $p^* e^i$ . Since  $\Phi_U$  is  $O(m)$ -valued, we have by virtue of formulas (3.3), (3.4), (3.6)–(3.8) and the  $O(m)$ -invariance of  $Q$ , the identity  $\Phi_U^*(G) = Q$ .  $\square$

4. THE LEVI-CIVITA CONNECTION OF  $(TM, G)$ 

In this section, we shall calculate the Levi-Civita connection of a Riemannian  $g$ -natural metric  $G$  on the tangent bundle of a Riemannian manifold  $(M, g)$ . We can assert the following:

**Proposition 4.1.** *Let  $(M, g)$  be a Riemannian manifold,  $\nabla$  its Levi-Civita connection and  $R$  its curvature tensor. Let  $G$  be a Riemannian  $g$ -natural metric on  $TM$ . Then the Levi-Civita connection  $\bar{\nabla}$  of  $(TM, G)$  is characterized by*

$$\begin{aligned}
 \text{(i)} \quad & (\bar{\nabla}_{X^h} Y^h)_{(x,u)} = (\nabla_X Y)_{(x,u)}^h + h\{A(u; X_x, Y_x)\} + v\{B(u; X_x, Y_x)\}, \\
 \text{(ii)} \quad & (\bar{\nabla}_{X^h} Y^v)_{(x,u)} = (\nabla_X Y)_{(x,u)}^v + h\{C(u; X_x, Y_x)\} + v\{D(u; X_x, Y_x)\}, \\
 \text{(iii)} \quad & (\bar{\nabla}_{X^v} Y^h)_{(x,u)} = h\{C(u; Y_x, X_x)\} + v\{D(u; Y_x, X_x)\}, \\
 \text{(iv)} \quad & (\bar{\nabla}_{X^v} Y^v)_{(x,u)} = h\{E(u; X_x, Y_x)\} + v\{F(u; X_x, Y_x)\},
 \end{aligned}$$

for all vector fields  $X, Y$  on  $M$  and  $(x, u) \in TM$ , where  $A, B, C, D, E$  and  $F$  are the  $F$ -tensor fields of type  $(2, 1)$  on  $M$  defined, for all  $u, X, Y \in M_x, x \in M$ , by:

$$\begin{aligned}
 A(u; X, Y) = & -\frac{\alpha_1 \alpha_2}{2\alpha} [R(X, u)Y + R(Y, u)X] \\
 & + \frac{\alpha_2(\beta_1 + \beta_3)}{2\alpha} [g_x(Y, u)X + g_x(X, u)Y] \\
 & + \frac{1}{\alpha\phi} \{ \alpha_2 [\alpha_1 (\phi_1(\beta_1 + \beta_3) - \phi_2\beta_2) + \alpha_2(\beta_1\alpha_2 \\
 & - \beta_2\alpha_1)] g_x(R(X, u)Y, u) + \phi_2\alpha(\alpha_1 + \alpha_3)' g_x(X, Y) \\
 & + [\alpha\phi_2(\beta_1 + \beta_3)' + (\beta_1 + \beta_3)[\alpha_2(\phi_2\beta_2 - \phi_1(\beta_1 + \beta_3)) \\
 & + (\alpha_1 + \alpha_3)(\alpha_1\beta_2 - \alpha_2\beta_1)] \} g_x(X, u)g_x(Y, u) \} u,
 \end{aligned}$$

$$\begin{aligned}
 B(u; X, Y) = & \frac{\alpha_2^2}{\alpha} R(X, u)Y - \frac{\alpha_1(\alpha_1 + \alpha_3)}{2\alpha} R(X, Y)u \\
 & - \frac{(\alpha_1 + \alpha_3)(\beta_1 + \beta_3)}{2\alpha} [g_x(Y, u)X + g_x(X, u)Y] \\
 & + \frac{1}{\alpha\phi} \{ \alpha_2 [\alpha_2(\phi_2\beta_2 - \phi_1(\beta_1 + \beta_3)) + (\alpha_1 + \alpha_3)(\beta_2\alpha_1 \\
 & - \beta_1\alpha_2)] g_x(R(X, u)Y, u) - \alpha(\phi_1 + \phi_3)(\alpha_1 + \alpha_3)' g_x(X, Y) \\
 & + [-\alpha(\phi_1 + \phi_3)(\beta_1 + \beta_3)' \\
 & + (\beta_1 + \beta_3)[(\alpha_1 + \alpha_3)[(\phi_1 + \phi_3)\beta_1 - \phi_2\beta_2] \\
 & + \alpha_2[\alpha_2(\beta_1 + \beta_3) - (\alpha_1 + \alpha_3)\beta_2] \} g_x(X, u)g_x(Y, u) \} u,
 \end{aligned}$$

$$\begin{aligned}
C(u; X, Y) = & -\frac{\alpha_1^2}{2\alpha}R(Y, u)X - \frac{\alpha_1(\beta_1+\beta_3)}{2\alpha}g_x(X, u)Y \\
& + \frac{1}{\alpha}[\alpha_1(\alpha_1 + \alpha_3)' - \alpha_2(\alpha_2' - \frac{\beta_2}{2})]g_x(Y, u)X \\
& + \frac{1}{\alpha\phi}\left\{\frac{\alpha_1}{2}[\alpha_2(\alpha_2\beta_1 - \alpha_1\beta_2) + \alpha_1(\phi_1(\beta_1 + \beta_3) \right. \\
& \left. - \phi_2\beta_2)]g_x(R(X, u)Y, u) + \alpha[\frac{\phi_1}{2}(\beta_1 + \beta_3) + \phi_2(\alpha_2' - \frac{\beta_2}{2})]g_x(X, Y) \right. \\
& + [\alpha\phi_1(\beta_1 + \beta_3)' + [\alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1) \\
& + \alpha_1(\phi_2\beta_2 - (\beta_1 + \beta_3)\phi_1)][(\alpha_1 + \alpha_3)' + \frac{\beta_1+\beta_3}{2}] \\
& + [\alpha_2(\beta_1(\phi_1 + \phi_3) - \beta_2\phi_2) + \alpha_1(\beta_2(\alpha_1 + \alpha_3) \\
& \left. - \alpha_2(\beta_1 + \beta_3))](\alpha_2' - \frac{\beta_2}{2})\right\}g_x(X, u)g_x(Y, u)u,
\end{aligned}$$

$$\begin{aligned}
D(u; X, Y) = & \frac{1}{\alpha}\left\{\frac{\alpha_1\alpha_2}{2}R(Y, u)X - \frac{\alpha_2(\beta_1+\beta_3)}{2}g_x(X, u)Y \right. \\
& + [-\alpha_2(\alpha_1 + \alpha_3)' + (\alpha_1 + \alpha_3)(\alpha_2' - \frac{\beta_2}{2})]g_x(Y, u)X \\
& + \frac{1}{\alpha\phi}\left\{\frac{\alpha_1}{2}[(\alpha_1 + \alpha_3)(\alpha_1\beta_2 - \alpha_2\beta_1) \right. \\
& + \alpha_2(\phi_2\beta_2 - \phi_1(\beta_1 + \beta_3))]g_x(R(X, u)Y, u) \\
& - \alpha[\frac{\phi_2}{2}(\beta_1 + \beta_3) + (\phi_1 + \phi_3)(\alpha_2' - \frac{\beta_2}{2})]g_x(X, Y) \\
& + [\alpha\phi_2(\beta_1 + \beta_3)' + [(\alpha_1 + \alpha_3)(\alpha_2\beta_1 - \alpha_1\beta_2) \\
& + \alpha_2(\phi_1(\beta_1 + \beta_3) - \phi_2\beta_2)][(\alpha_1 + \alpha_3)' + \frac{\beta_1+\beta_3}{2}] \\
& + [(\alpha_1 + \alpha_3)(\beta_2\phi_2 - \beta_1(\phi_1 + \phi_3)) + \alpha_2(\beta_2(\alpha_1 + \alpha_3) \\
& \left. - \alpha_2(\beta_1 + \beta_3))](\alpha_2' - \frac{\beta_2}{2})\right\}g_x(X, u)g_x(Y, u)u,
\end{aligned}$$

$$\begin{aligned}
E(u; X, Y) = & \frac{1}{\alpha}[\alpha_1(\alpha_2' + \frac{\beta_2}{2}) - \alpha_2\alpha_1']g_x(Y, u)X + g_x(X, u)Y \\
& + \frac{1}{\alpha\phi}\left\{\alpha[\phi_1\beta_2 - \phi_2(\beta_1 - \alpha_1')]g_x(X, Y) \right. \\
& + [\alpha(2\phi_1\beta_2' - \phi_2\beta_1') + 2\alpha_1'[\alpha_1(\alpha_2(\beta_1 + \beta_3) \\
& - \beta_2(\alpha_1 + \alpha_3)) + \alpha_2(\beta_1(\phi_1 + \phi_3) - \beta_2\phi_2)] \\
& + (2\alpha_2' + \beta_2)[\alpha_1(\phi_2\beta_2 - \phi_1(\beta_1 + \beta_3)) \\
& \left. + \alpha_2(\alpha_1\beta_2 - \alpha_2\beta_1)]\right\}g_x(X, u)g_x(Y, u)u,
\end{aligned}$$

$$\begin{aligned}
F(u; X, Y) = & \frac{1}{\alpha}[-\alpha_2(\alpha_2' + \frac{\beta_2}{2}) + (\alpha_1 + \alpha_3)\alpha_1']g_x(Y, u)X + g_x(X, u)Y \\
& + \frac{1}{\alpha\phi}\left\{\alpha[(\phi_1 + \phi_3)(\beta_1 - \alpha_1') - \phi_2\beta_2]g_x(X, Y) \right. \\
& + [\alpha((\phi_1 + \phi_3)\beta_1' - 2\phi_2\beta_2') + 2\alpha_1'[\alpha_2(\beta_2(\alpha_1 + \alpha_3) \\
& - \alpha_2(\beta_1 + \beta_3)) + (\alpha_1 + \alpha_3)(\beta_2\phi_2 - \beta_1(\phi_1 + \phi_3))] \\
& + (2\alpha_2' + \beta_2)[\alpha_2(\phi_1(\beta_1 + \beta_3) - \phi_2\beta_2) \\
& \left. + (\alpha_1 + \alpha_3)(\alpha_2\beta_1 - \alpha_1\beta_2)]\right\}g_x(X, u)g_x(Y, u)u.
\end{aligned}$$

For  $m = 1$  the same holds with  $\beta_i = 0$ ,  $i = 1, 2, 3$ .

**Proof.** At first, we can easily check the following formulas which relate the metric  $G$  to the base metric  $g$ . Let  $X$  and  $Y$  be vector fields on  $M$  and  $(x, u) \in TM$ , then, according to (2.2), we have

$$(4.1) \quad g_x(X, u) = \frac{1}{\phi_1 + \phi_3} G_{(x,u)}(X^h, h\{u\}),$$

$$(4.2) \quad g_x(X, Y) = \frac{1}{\alpha_1 + \alpha_3} \{G_{(x,u)}(X^h, Y^h) - (\beta_1 + \beta_3)g_x(X_x, u)g_x(Y_x, u)\},$$

and similarly, with respect to vertical lifts, we have

$$(4.3) \quad g_x(X, u) = \frac{1}{\phi_1} G_{(x,u)}(X^v, v\{u\}),$$

$$(4.4) \quad g_x(X, Y) = \frac{1}{\alpha_1} \{G_{(x,u)}(X^v, Y^v) - \beta_1 g_x(X_x, u)g_x(Y_x, u)\}.$$

Using Koszul formula (1.16), and then (1.4), (1.8), (1.10) and (1.13), we can write for each vector field  $Z$  on  $M$ ,

$$2G_{(x,u)}(\bar{\nabla}_{X^h} Y^h, Z^h) = 2G_{(x,u)}((\nabla_X Y)^h, Z^h) - 2\alpha_2 g_x(R(X_x, u)Y_x, Z_x),$$

and by virtue of (4.1) and (4.2), we have

$$2G_{(x,u)}(\bar{\nabla}_{X^h} Y^h, Z^h) = 2G_{(x,u)}(h\{T_{11}\}, Z^h(x, u)),$$

where  $T_{11}$  is given by

$$(4.5) \quad T_{11} = (\nabla_X Y)_x + \frac{\alpha_2}{\alpha_1 + \alpha_3} \{-R(X_x, u)Y_x + \frac{\beta_1 + \beta_3}{\phi_1 + \phi_3} g_x(R(X_x, u)Y_x, u)u\}.$$

By similar way, using Koszul formula (1.16), and then (1.4), (1.5), (1.8)-(1.10), (1.13), (1.14), (4.3) and (4.4), we can write for each vector field  $Z$  on  $M$ ,

$$2G_{(x,u)}(\bar{\nabla}_{X^h} Y^h, Z^v) = 2G_{(x,u)}(v\{T_{12}\}, Z^v(x, u)),$$

where  $T_{12}$  is given by

$$(4.6) \quad \begin{aligned} T_{12} = & \frac{1}{\alpha_1} \{ \alpha_2 (\nabla_X Y)_x - \frac{\beta_1 + \beta_3}{2} [g_x(Y_x, u)X_x + g_x(X_x, u)Y_x] \} \\ & - \frac{1}{2} R(X_x, Y_x)u + \frac{1}{\phi_1} \{ (\beta_2 - \frac{\alpha_2}{\alpha_1} \beta_1) g_x((\nabla_X Y)_x, u) \\ & - (\alpha_1 + \alpha_3)' g_x(X_x, Y_x) + [\frac{\beta_1(\beta_1 + \beta_3)}{\alpha_1} \\ & - (\beta_1 + \beta_3)'] g_x(X_x, u)g_x(Y_x, u) \} u. \end{aligned}$$

Similar formulas can be obtained using the same formulas and some formulas from § 1, i.e.

$$2G_{(x,u)}(\bar{\nabla}_{X^v} Y^h, Z^h) = 2G_{(x,u)}(h\{T_{21}\}, Z^h(x, u)),$$

$$2G_{(x,u)}(\bar{\nabla}_{X^v} Y^h, Z^v) = 2G_{(x,u)}(v\{T_{22}\}, Z^v(x, u)),$$

$$2G_{(x,u)}(\bar{\nabla}_{X^v} Y^v, Z^h) = 2G_{(x,u)}(h\{T_{31}\}, Z^h(x, u)),$$

$$2G_{(x,u)}(\bar{\nabla}_{X^v} Y^v, Z^v) = 2G_{(x,u)}(v\{T_{32}\}, Z^v(x, u)),$$

where  $T_{21}$ ,  $T_{22}$ ,  $T_{31}$  and  $T_{32}$  are given by

$$(4.7) \quad \begin{aligned} T_{21} = & \frac{1}{\alpha_1 + \alpha_3} \{ (\alpha_1 + \alpha_3)' g_x(X_x, u) Y_x + \frac{\beta_1 + \beta_3}{2} g_x(Y_x, u) X_x \\ & - \frac{\alpha_1}{2} R(X_x, u) Y_x + \frac{1}{2(\phi_1 + \phi_3)} \{ \alpha_1 (\beta_1 + \beta_3) g_x(R(X_x, u) Y_x, u) \\ & + (\alpha_1 + \alpha_3) (\beta_1 + \beta_3) g_x(X_x, Y_x) + [2(\alpha_1 + \alpha_3) (\beta_1 + \beta_3)' \\ & - (2(\alpha_1 + \alpha_3)' + (\beta_1 + \beta_3)) (\beta_1 + \beta_3) \} g_x(X_x, u) g_x(Y_x, u) \} u \}, \end{aligned}$$

$$(4.8) \quad T_{22} = \frac{1}{\alpha_1} (\alpha_2' - \frac{\beta_2}{2}) [g_x(X_x, u) Y_x - \frac{1}{\phi_1} G_{(x,u)}(X^v, Y^v) u],$$

$$(4.9) \quad \begin{aligned} T_{31} = & \frac{1}{\alpha_1 + \alpha_3} \{ (\alpha_2' + \frac{\beta_2}{2}) [g_x(Y_x, u) X_x + g_x(X_x, u) Y_x] \\ & + \frac{1}{\phi_1 + \phi_3} [\beta_2 (\alpha_1 + \alpha_3) g_x(X_x, Y_x) + (2(\alpha_1 + \alpha_3) \beta_2' \\ & - (\beta_1 + \beta_3) (\beta_2 + 2\alpha_2')] g_x(X_x, u) g_x(Y_x, u) \} u \}, \end{aligned}$$

$$(4.10) \quad \begin{aligned} T_{32} = & \frac{1}{\alpha_1} \{ \alpha_1' [g_x(Y_x, u) X_x + g_x(X_x, u) Y_x] + \frac{1}{\phi_1} [\alpha_1 (\beta_1 \\ & - \alpha_1') g_x(X_x, Y_x) + (\alpha_1 \beta_1' - 2\alpha_1' \beta_1) g_x(X_x, u) g_x(Y_x, u) \} u \}. \end{aligned}$$

If we put  $Q_1 = \bar{\nabla}_{X^h} Y^h$ ,  $Q_2 = \bar{\nabla}_{X^v} Y^h$  and  $Q_3 = \bar{\nabla}_{X^v} Y^v$ , then we can write

$$Q_i = h\{T_{i1}\} + v\{T_{i2}\} + h\{A_i\} + v\{B_i\}; \quad i = 1, 2, 3.$$

From the equalities

$$(4.11) \quad G_{(x,u)}(Q_i, Z^h(x, u)) = G_{(x,u)}(h\{T_{i1}\}, Z^h(x, u)),$$

$$(4.12) \quad G_{(x,u)}(Q_i, Z^v(x, u)) = G_{(x,u)}(v\{T_{i2}\}, Z^v(x, u)),$$

we obtain the following identities

$$(4.13) \quad (\alpha_1 + \alpha_3) A_i + \alpha_2 B_i = -\alpha_2 T_{i2} - [(\beta_1 + \beta_3) g_x(A_i, u) + \beta_2 (g_x(B_i, u) + g_x(T_{i2}, u))] u,$$

$$(4.14) \quad \alpha_2 A_i + \alpha_1 B_i = -\alpha_2 T_{i1} - [\beta_2 (g_x(A_i, u) + g_x(T_{i1}, u)) + \beta_1 g_x(B_i, u)] u.$$

Letting  $Z_x = u$  into the equations (4.11) and (4.12), we obtain

$$\begin{aligned} (\phi_1 + \phi_3) g_x(A_i, u) + \phi_2 g_x(B_i, u) &= -\phi_2 g_x(T_{i2}, u) \\ \phi_2 g_x(A_i, u) + \phi_1 g_x(B_i, u) &= -\phi_2 g_x(T_{i1}, u). \end{aligned}$$

Consequently, we can write

$$(4.15) \quad g_x(A_i, u) = \frac{\phi_2}{\phi} [\phi_2 g_x(T_{i1}, u) - \phi_1 g_x(T_{i2}, u)],$$

$$(4.16) \quad g_x(B_i, u) = \frac{\phi_2}{\phi} [-(\phi_1 + \phi_3) g_x(T_{i1}, u) + \phi_2 g_x(T_{i2}, u)].$$

Substituting from (4.15)–(4.16) into (4.11) and (4.12), we obtain

$$(4.17) \quad \begin{cases} (\alpha_1 + \alpha_3) A_i + \alpha_2 B_i = D_{i1}, \\ \alpha_2 A_i + \alpha_1 B_i = D_{i2}. \end{cases}$$

where  $D_{i1}$  and  $D_{i2}$  are given by

$$D_{i1} = -\alpha_2 T_{i2} - \frac{\alpha_2(\beta_1 + \beta_3) - \beta_2(\alpha_1 + \alpha_3)}{\phi} [\phi_2 g_x(T_{i1}, u) - \phi_1 g_x(T_{i2}, u)] u,$$

$$D_{i2} = -\alpha_2 T_{i1} - \frac{\alpha_1 \beta_2 - \beta_1 \alpha_2}{\phi} [(\phi_1 + \phi_3) g_x(T_{i1}, u) - \phi_2 g_x(T_{i2}, u)] u.$$

The resolution of the system (4.17) gives by routine calculations the result.  $\square$

**Remark 4.2.** Note that when we take into account the orientation of  $M$ , general formulas of  $g$ -natural metrics on  $TM$  become larger (precisely for the dimensions 2 and 3 of  $M$ ), as given explicitly in [15]. This yields very complicated formulas calculating the Levi-Civita connection of an arbitrary Riemannian  $g$ -natural metric. This question has been treated in detail in [4].

Now, among all Riemannian  $g$ -natural metrics on  $TM$ , we shall specify those with respect to which all the fibers of  $TM$  are totally geodesic.

**Theorem 4.3.** *Let  $(M, g)$  be a Riemannian manifold and  $G$  be a Riemannian  $g$ -natural metric on  $TM$ . The fibers of  $(TM, G)$  are totally geodesic if and only if there is a real constant  $c$  such that*

$$(4.18) \quad \begin{cases} \alpha_2(t) = \frac{c}{\sqrt{\phi_1(t)}} (t \cdot \alpha_1'(t) + \alpha_1(t)), \\ \beta_2(t) = \frac{c}{\sqrt{\phi_1(t)}} (\beta_1(t) - \alpha_1'(t)), \end{cases}$$

for all  $t \in \mathbb{R}^+$ .

Note that  $c = 0$ , in the system (4.18), corresponds to the case when horizontal and vertical distributions are orthogonal.

**Proof.** Remark first that the fibers of  $(TM, G)$  are totally geodesic if and only if  $\bar{\nabla}_{X^v} X^v$  is vertical, for all  $X \in \mathfrak{X}(M)$  (cf. [6], p.47). Hence, by virtue of Proposition 4.1, the fibers of  $(TM, G)$  are totally geodesic if and only if  $E(u; X, X) = 0$ , for all  $X \in \mathfrak{X}(M)$ . Since  $E$  is symmetric and linear in the second and third arguments, the last assertion is equivalent to  $E(u; u, u) = 0$  and  $E(u; X, X) = 0$ , for all  $u \in TM$  and  $X \perp u$ .

But, if  $X \perp u$  then we have by virtue of Proposition 4.1,

$$E(u; X, X) = \frac{1}{\phi} (\phi_1 \beta_2 - \phi_2 (\beta_1 - \alpha_1')) \cdot g_x(X, X) \cdot u.$$

Hence,  $E(u; X, X) = 0$ , for all  $u \in TM$  and  $X \perp u$ , is equivalent to

$$(4.19) \quad \phi_1 \beta_2 = \phi_2 (\beta_1 - \alpha_1'),$$

on  $\mathbb{R}^{+*}$ , and by continuity on  $\mathbb{R}^+$ .

On the other hand, we have for all  $u \in TM$ ,

$$\begin{aligned}
E(u; u, u) &= \frac{r^2}{\phi} \{ \phi_1 \beta_2 - \phi_2 (\beta_1 - \alpha'_1) + \frac{1}{\alpha} \{ 2\phi [\alpha_1 (\alpha'_2 + \frac{\beta_2}{2}) - \alpha_2 \alpha'_1] \\
&\quad + \alpha [2\phi_1 \beta'_2 - \phi_2 \beta'_1] \cdot r^2 + 2\alpha'_1 \cdot r^2 [\alpha_1 (\alpha_2 (\beta_1 + \beta_3) - \beta_2 (\alpha_1 + \alpha_3)) \\
&\quad + \alpha_2 (\beta_1 (\phi_1 + \phi_3) - \beta_2 \phi_2)] + (2\alpha'_2 + \beta_2) \cdot r^2 [\alpha_1 (\phi_2 \beta_2 - \phi_1 (\beta_1 + \beta_3)) \\
&\quad + \alpha_2 (\alpha_1 \beta_2 - \alpha_2 \beta_1)] \} \} u \\
&= \frac{r^2}{\phi} \{ \phi_1 \beta_2 - \phi_2 (\beta_1 - \alpha'_1) + \frac{1}{\alpha} \{ \alpha [2\phi_1 \beta'_2 - \phi_2 \beta'_1] \cdot r^2 \\
&\quad + 2\alpha'_1 [\alpha_2 (-\phi + \alpha_1 (\beta_1 + \beta_3)) \cdot r^2 + (\phi_1 + \phi_3) \beta_1 \cdot r^2 - \phi_2 \beta_2 \cdot r^2] \\
&\quad - \alpha_1 (\alpha_1 + \alpha_3) \beta_2 \cdot r^2 \} + (2\alpha'_2 + \beta_2) [\alpha_1 (\phi + (\phi_2 \beta_2 \\
&\quad - \phi_1 (\beta_1 + \beta_3)) \cdot r^2) + \alpha_2 (\alpha_1 \beta_2 - \alpha_2 \beta_1) \cdot r^2] \} \} u,
\end{aligned}$$

where  $r^2 = g(u, u)$ . But

$$\begin{aligned}
&\alpha_2 (-\phi + \alpha_1 (\beta_1 + \beta_3)) \cdot r^2 + (\phi_1 + \phi_3) \beta_1 \cdot r^2 - \phi_2 \beta_2 \cdot r^2 - \alpha_1 (\alpha_1 + \alpha_3) \beta_2 \cdot r^2 \\
&= \alpha_2 [(\phi_2^2 - \phi_2 \beta_2 \cdot r^2) + ((\phi_1 + \phi_3) \beta_1 \cdot r^2 - \phi_1 (\phi_1 + \phi_3)) + \alpha_1 (\beta_1 + \beta_3) \cdot r^2] \\
&\quad - \alpha_1 (\alpha_1 + \alpha_3) \beta_2 \cdot r^2 \\
&= \alpha_2 [\phi_2 \alpha_2 - \alpha_1 (\phi_1 + \phi_3) + \alpha_1 (\beta_1 + \beta_3) \cdot r^2] - \alpha_1 (\alpha_1 + \alpha_3) \beta_2 \cdot r^2 \\
&= \alpha_2 [\phi_2 \alpha_2 - \alpha_1 (\alpha_1 + \alpha_3)] - \alpha_1 (\alpha_1 + \alpha_3) \beta_2 \cdot r^2 \\
&= \alpha_2^2 \phi_2 - \alpha_1 (\alpha_1 + \alpha_3) (\alpha_2 + \beta_2 \cdot r^2) \\
&= -\alpha \cdot \phi_2.
\end{aligned}$$

By similar way, we find that

$$\alpha_1 (\phi + (\phi_2 \beta_2 - \phi_1 (\beta_1 + \beta_3)) \cdot r^2) + \alpha_2 (\alpha_1 \beta_2 - \alpha_2 \beta_1) \cdot r^2 = \alpha \cdot \phi_1,$$

so that, we obtain

$$\begin{aligned}
E(u; u, u) &= \frac{r^2}{\phi} \{ \phi_1 \beta_2 - \phi_2 (\beta_1 - \alpha'_1) + (2\phi_1 \beta'_2 - \phi_2 \beta'_1) \cdot r^2 \\
&\quad - 2\phi_2 \alpha'_1 + \phi_1 (2\alpha'_2 + \beta_2) \} u \\
&= \frac{r^2}{\phi} \{ 2\phi_1 (\beta_2 + \alpha'_2 + \beta'_2 \cdot r^2) - \phi_2 (\beta_1 + \alpha'_1 + \beta'_1 \cdot r^2) \} u \\
&= \frac{r^2}{\phi} \{ 2\phi_1 \phi'_2 - \phi_2 \phi'_1 \} u.
\end{aligned}$$

Hence,  $E(u; u, u) = 0$ , for all  $u \in TM$ , if and only if  $2\phi_1 \phi'_2 - \phi_2 \phi'_1 = 0$  on  $\mathbb{R}^{+*}$ , and by continuity on  $\mathbb{R}^+$ .

We deduce that the fibers of  $(TM, G)$  are totally geodesic if and only if

$$(4.20) \quad \begin{cases} \phi_1 \beta_2 = \phi_2 (\beta_1 - \alpha'_1), \\ 2\phi_1 \phi'_2 = \phi_2 \phi'_1, \end{cases}$$

on  $\mathbb{R}^+$ . Now, (4.20)<sub>2</sub> is equivalent to

$$(4.21) \quad \begin{cases} \alpha_2(t) = \beta_2(t) = 0 & \text{whenever } \phi_2(t) = 0, \\ \phi_2^2/\phi_1 & \text{is constant on each interval where } \phi_2(t) \neq 0 \text{ everywhere.} \end{cases}$$

Denote by  $J$  the complement of  $\phi_2^{-1}(0)$  in  $\mathbb{R}^+$ .  $J$  is an open subset of  $\mathbb{R}^+$ . We claim that, in the conditions of (4.21), either  $J = \emptyset$  or  $J = \mathbb{R}^+$ . If not, there is  $0 < a < b$  such that  $]a, b[ \subset J$  (since  $J$  is open) and  $a \notin J$ . Then there is a constant  $d > 0$  such that  $\phi_1 = d \cdot \phi_2^2$  on  $]a, b[$ . When  $t \rightarrow a$ , we have by continuity of  $\phi_1$  and  $\phi_2$ ,  $\phi_1(a) = d \cdot \phi_2^2(a)$ . Since  $a \notin J$ , then  $\phi_1(a) = 0$ , which contradicts the fact that  $G$  is Riemannian (Proposition 2.8). We deduce that either  $J = \emptyset$  or  $J = \mathbb{R}^+$ . Hence, (4.21) is equivalent to

$$\begin{cases} \text{either} & \alpha_2 = \beta_2 = 0 \quad \text{on} \quad \mathbb{R}^+, \\ \text{or} & \phi_2^2/\phi_1 \quad \text{is constant on} \quad \mathbb{R}^+, \end{cases}$$

or equivalently,

$$(4.22) \quad \phi_2^2/\phi_1 \quad \text{is a constant on} \quad \mathbb{R}^+.$$

Hence (4.20) holds if and only if

$$(4.23) \quad \begin{cases} \phi_1\beta_2 = \phi_2(\beta_1 - \alpha'_1), \\ \phi_2 = c \cdot \sqrt{\phi_1}, \end{cases}$$

on  $\mathbb{R}^+$ , where  $c$  is a constant, or equivalently,

$$(4.24) \quad \begin{cases} \beta_2 = \frac{c}{\sqrt{\phi_1}}(\beta_1 - \alpha'_1), \\ \alpha_2 = \phi_2 - t \cdot \beta_2 = \frac{c}{\sqrt{\phi_1}}(t \cdot \alpha'_1 + \alpha_1), \end{cases}$$

on  $\mathbb{R}^+$ . This completes the proof.  $\square$

**Remark 4.4.** As other applications of Proposition 4.1, we can check the following assertions:

1) Let  $(M, g)$  be a Riemannian manifold and  $G$  be a Riemannian  $g$ -natural metric on  $TM$ . Then  $M$ , considered as an embedded submanifold of  $TM$  by the zero section, is always totally geodesic.

2) Among all Riemannian  $g$ -natural metrics on the tangent bundle  $TM$  of a Riemannian manifold  $(M, g)$ , the only ones with respect to which the vertical lift preserves the parallelism of vector fields on  $M$  are those which belong to the 2-dimensional cone  $\mathcal{C} = \{a.g^s + b.g^h + c.g^v, c > 0, a > 0, b^2 - c(a + c) < 0\}$  in the 3-dimensional real vector space generated by the three classical lifts  $g^s, g^h$  and  $g^v$  of  $g$ .

Note that with respect to any element of  $\mathcal{C}$ , the horizontal lift also preserves the parallelism of vector fields on  $M$ .

## 5. THE GEODESIC FLOW IN $TM$ AND INCOMPRESSIBILITY

Let  $(M, g)$  be a Riemannian manifold, and  $G$  be a Riemannian  $g$ -natural metric on  $TM$ . In this section, we study the situations when the geodesic flow in  $TM$  is incompressible with respect to  $G$ .

Let  $\{(U; x^i, i = 1, \dots, m)\}$  be a local coordinate system in  $M$  and  $\{(p^{-1}(U); x^i, u^i, i = 1, \dots, m)\}$  the induced local coordinate system in  $TM$ . Let  $\{\Gamma_{ij}^k; i, j, k = 1, \dots, m\}$  and  $\{\bar{\Gamma}_{IJ}^K; I, J, K = 1, \dots, 2m\}$  be the Riemann-Christoffel symbols of  $(M, g)$  and  $(TM, G)$  respectively. If  $T$  is an  $F$ -tensor field on  $M$  of type  $(2, 1)$ ,

then we denote by  $T(u)_{ij}^k$  ( $1 \leq i, j \leq m$ ) the components of the  $(2, 1)$ -tensor on  $M_x$  determined by the bilinear mapping  $T(u; \cdot, \cdot) : M_x \times M_x \rightarrow M_x$ , i.e.  $T(u)_{ij}^k = dx^k[T(u; \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})]$ , ( $1 \leq i, j \leq m$ ). Now, the expressions of the identities of Proposition 4.1 in local coordinates yield the following:

**Lemma 5.1.** *The Riemann-Christoffel symbols of  $(TM, G)$  are given by*

$$\begin{aligned}\bar{\Gamma}_{m+im+j}^k(u) &= E(u)_{ij}^k, \\ \bar{\Gamma}_{m+im+j}^{m+k}(u) &= F(u)_{ij}^k - \Gamma_{\lambda\mu}^k(x)E(u)_{ij}^\lambda u^\mu, \\ \bar{\Gamma}_{im+j}^k(u) &= \Gamma_{i\mu}^\lambda(x)\bar{\Gamma}_{m+\lambda m+j}^k(u)u^\mu + C(u)_{ij}^k, \\ \bar{\Gamma}_{im+j}^{m+k}(u) &= \Gamma_{i\mu}^\lambda(x)\bar{\Gamma}_{m+\lambda m+j}^{m+k}(u)u^\mu + \Gamma_{ij}^k(x) - \Gamma_{l\mu}^k(x)C(u)_{ij}^l u^\mu, \\ \bar{\Gamma}_{ij}^k(u) &= \Gamma_{ij}^k(x) + A(u)_{ij}^k + \Gamma_{i\mu}^l(x)\bar{\Gamma}_{m+l j}^k(u)u^\mu \\ &\quad + \Gamma_{j\mu}^l(x)\bar{\Gamma}_{m+li}^k(u)u^\mu - \Gamma_{i\mu}^l(x)\Gamma_{j\lambda}^t(x)\bar{\Gamma}_{m+lm+t}^k(u)u^\lambda u^\mu, \\ \bar{\Gamma}_{ij}^{m+k}(u) &= -\Gamma_{\lambda\mu}^k(x)\Gamma_{ij}^\lambda(x)u^\mu - \Gamma_{i\mu}^l(x)\Gamma_{jl}^k(x)u^\mu - \Gamma_{\lambda\mu}^k(x)A(u)_{ij}^\lambda u^\mu \\ &\quad + B(u)_{ij}^k + \frac{\partial \Gamma_{j\mu}^k}{\partial x^i}(x)u^\mu + \Gamma_{i\mu}^l(x)\bar{\Gamma}_{m+l j}^{m+k}(u)u^\mu \\ &\quad + \Gamma_{j\mu}^l(x)\bar{\Gamma}_{m+li}^{m+k}(u)u^\mu - \Gamma_{i\mu}^l(x)\Gamma_{j\lambda}^t(x)\bar{\Gamma}_{m+lm+t}^{m+k}(u)u^\lambda u^\mu.\end{aligned}$$

for all  $x \in U$  and  $u \in p^{-1}(x)$ .

Note that we have been using, and we will be, the so-called Einstein's summation. Now, our main result in this section is:

**Theorem 5.2.** *Let  $(M, g)$  be a Riemannian manifold and  $G$  be a  $g$ -natural metric on the tangent bundle  $TM$ . Then the geodesic flow of  $(M, g)$  is incompressible with respect to  $G$  if and only if the following conditions are satisfied:*

- (i)  $\phi_1 + \phi_3$  is constant on each interval where  $\alpha_2 = 0$  and  $\beta_2 \neq 0$  everywhere;
- (ii)  $\text{Ric}_x(u, u) = \theta(r^2)g_x(u, u)$  whenever  $\alpha_2(r^2) \neq 0$ ;

where  $\theta$  is the function defined from  $\mathbb{R}^+ \setminus [(\alpha_2)^{-1}(0)]$  to  $\mathbb{R}$  by

$$(5.1) \quad \theta = \frac{1}{\alpha_1} \left\{ (m-1)(\beta_1 + \beta_3) + \frac{2\phi_2\alpha}{\alpha_2\phi} (\phi_1 + \phi_3)' \right\},$$

$r^2 = g_x(u, u)$  and  $\text{Ric}$  is the Ricci tensor field on  $(M, g)$ .

**Proof.** Let  $\xi$  be the geodesic flow vector of  $(M, g)$ . It is a vector field on  $TM$  which is locally expressed as  $\xi^i = u^i$ ,  $\xi^{m+i} = -\Gamma_{ij}^k u^j u^k$ . We shall compute the divergence of  $\xi$ ,  $\text{div}_G \xi$ , relatively to the metric  $G$  on  $TM$ . By the definition of the

divergence, we have

$$\begin{aligned}
 (\operatorname{div}_G \xi)_{(x,u)} &= \frac{\partial \xi^i}{\partial x^i} + \frac{\partial \xi^{m+i}}{\partial u^i} + \bar{\Gamma}^I_{IJ} \xi^J \\
 &= -2 \cdot \Gamma^i_{ij} + (\bar{\Gamma}^i_{ij} + \bar{\Gamma}^{m+i}_{m+ij}) u^j - (\bar{\Gamma}^i_{im+j} + \bar{\Gamma}^{m+i}_{m+im+j}) \Gamma^j_{kl} u^k u^l \\
 &= [\Gamma^i_{ij} + A(u)^i_{ji} + \Gamma^l_{i\mu} \Gamma^{\lambda}_{j\nu} \bar{\Gamma}^i_{m+\lambda m+l} u^\mu u^\nu + \Gamma^l_{i\mu} C(u)^i_{jl} u^\mu \\
 &\quad + \Gamma^l_{i\mu} \Gamma^{\lambda}_{j\nu} \bar{\Gamma}^i_{m+\lambda m+l} u^\mu u^\nu + \Gamma^l_{j\mu} C(u)^i_{il} u^\mu - \Gamma^l_{i\mu} \Gamma^{\lambda}_{j\nu} \bar{\Gamma}^i_{m+\lambda m+l} u^\mu u^\nu \\
 &\quad + \Gamma^l_{j\mu} \Gamma^{m+i}_{m+im+l} u^\mu + \Gamma^i_{ij} - \Gamma^i_{l\mu} C(u)^l_{ji} u^\mu] u^j \\
 &\quad - [\Gamma^l_{i\mu} \Gamma^j_{kl} \bar{\Gamma}^i_{m+l m+j} u^k u^l u^\mu + \Gamma^j_{kl} C(u)^i_{ij} u^k u^l + \Gamma^j_{kl} \Gamma^{m+i}_{m+im+j} u^k u^l] \\
 &= -2 \cdot \Gamma^i_{ij} u^j + [2 \cdot \Gamma^i_{ij} + A(u)^i_{ji}] \cdot u^j \\
 &= A(u)^i_{ji} \cdot u^j,
 \end{aligned}$$

so that  $(\operatorname{div}_G \xi)_{(x,u)}$  is the trace of the endomorphism of  $M_x$  given by  $X \rightarrow A(u; u, X)$ . But

$$\begin{aligned}
 A(u; u, X) &= \frac{1}{\alpha} \left\{ -\frac{\alpha_1 \alpha_2}{2} R(X, u) u + \frac{\alpha_2 (\beta_1 + \beta_3)}{2} g_x(u, u) X + \frac{1}{\phi} \left\{ \phi \frac{\alpha_2 (\beta_1 + \beta_3)}{2} \right. \right. \\
 &\quad \left. \left. + \phi_2 \alpha (\alpha_1 + \alpha_3)' + \alpha \phi_2 (\beta_1 + \beta_3)' \cdot r^2 + (\beta_1 + \beta_3) \cdot r^2 [\alpha_2 (\phi_2 \beta_2 \right. \right. \\
 &\quad \left. \left. - \phi_1 (\beta_1 + \beta_3)) + (\alpha_1 + \alpha_3) (\phi_1 \beta_2 - \phi_2 \beta_1)] \right\} g_x(X, u) u \right\},
 \end{aligned}$$

so that

$$\begin{aligned}
 A(u)^i_{ij} &= \frac{1}{\alpha} \left\{ -\frac{\alpha_1 \alpha_2}{2} R^i_{jik} u^j u^k + m \cdot \frac{\alpha_2 (\beta_1 + \beta_3)}{2} \cdot r^2 + \frac{1}{\phi} \left\{ \phi \frac{\alpha_2 (\beta_1 + \beta_3)}{2} \right. \right. \\
 &\quad \left. \left. + \phi_2 \alpha (\alpha_1 + \alpha_3)' + \alpha \phi_2 (\beta_1 + \beta_3)' \cdot r^2 + (\beta_1 + \beta_3) \cdot r^2 [\alpha_2 (\phi_2 \beta_2 \right. \right. \\
 &\quad \left. \left. - \phi_1 (\beta_1 + \beta_3)) + (\alpha_1 + \alpha_3) (\phi_1 \beta_2 - \phi_2 \beta_1)] \right\} r^2 \right\} \\
 &= \frac{1}{\alpha} \left\{ -\frac{\alpha_1 \alpha_2}{2} R_{jk} u^j u^k + (m-1) \cdot \frac{\alpha_2 (\beta_1 + \beta_3)}{2} \cdot r^2 + \frac{r^2}{\phi} \left\{ \phi_2 \alpha [(\alpha_1 + \alpha_3)' \right. \right. \\
 &\quad \left. \left. + (\beta_1 + \beta_3)' \cdot r^2] + \alpha_2 (\beta_1 + \beta_3) [\phi + \phi_2 \beta_2 \cdot r^2 - \phi_1 (\beta_1 + \beta_3) \cdot r^2 \right. \right. \\
 &\quad \left. \left. - (\alpha_1 + \alpha_3) \beta_1 \cdot r^2] + \alpha_1 (\alpha_1 + \alpha_3) \beta_2 (\beta_1 + \beta_3) \cdot r^2 \right\} \right\} \\
 &= \frac{1}{\alpha} \left\{ -\frac{\alpha_1 \alpha_2}{2} R_{jk} u^j u^k + (m-1) \cdot \frac{\alpha_2 (\beta_1 + \beta_3)}{2} \cdot r^2 + \frac{r^2}{\phi} \left\{ \phi_2 \alpha [(\alpha_1 + \alpha_3)' \right. \right. \\
 &\quad \left. \left. + (\beta_1 + \beta_3)' \cdot r^2] + \alpha_2 (\beta_1 + \beta_3) [\alpha_1 (\alpha_1 + \alpha_3) - \alpha_2 \phi_2] \right. \right. \\
 &\quad \left. \left. + \alpha_1 (\alpha_1 + \alpha_3) \beta_2 (\beta_1 + \beta_3) \cdot r^2 \right\} \right\} \\
 &= \frac{1}{\alpha} \left\{ -\frac{\alpha_1 \alpha_2}{2} R_{jk} u^j u^k + (m-1) \cdot \frac{\alpha_2 (\beta_1 + \beta_3)}{2} \cdot r^2 + \frac{r^2}{\phi} \left\{ \phi_2 \alpha [(\alpha_1 + \alpha_3)' \right. \right. \\
 &\quad \left. \left. + (\beta_1 + \beta_3)' \cdot r^2] + \alpha_1 (\alpha_1 + \alpha_3) (\beta_1 + \beta_3) \phi_2 - \alpha_2^2 \phi_2 (\beta_1 + \beta_3) \right\} \right\} \\
 &= -\frac{\alpha_1 \alpha_2}{2\alpha} R_{jk} u^j u^k + \left\{ (m-1) \cdot \frac{\alpha_2 (\beta_1 + \beta_3)}{2\alpha} + \frac{\phi_2 (\phi_1 + \phi_3)'}{\phi} \right\} \cdot r^2,
 \end{aligned}$$

where  $R_{jk}$  denote the components of the Ricci tensor field on  $M$ . We deduce then that

$$\begin{aligned}
 (5.2) \quad (\operatorname{div}_G \xi)_{(x,u)} &= -\frac{\alpha_1 \alpha_2}{2\alpha} \operatorname{Ric}_x(u, u) + \left\{ \frac{\phi_2 (\phi_1 + \phi_3)'}{\phi} + \frac{(m-1) \alpha_2 (\beta_1 + \beta_3)}{2\alpha} \right\} g_x(u, u).
 \end{aligned}$$

Now,  $(\operatorname{div}_G \xi) \equiv 0$  if and only if

$$\begin{cases} \frac{\beta_2 \cdot r^2}{\phi} \cdot (\phi_1 + \phi_3)'(r^2) = 0 & \text{if } \alpha_2(r^2) = 0, \\ \operatorname{Ric}_x(u, u) = \theta(r^2)g_x(u, u) & \text{if } \alpha_2(r^2) \neq 0, \end{cases}$$

or equivalently,

$$\begin{cases} (\phi_1 + \phi_3)'(r^2) = 0 & \text{if } \alpha_2(r^2) = 0, \quad \beta_2(r^2) \neq 0 \quad \text{and} \quad r \neq 0, \\ \operatorname{Ric}_x(u, u) = \theta(r^2)g_x(u, u) & \text{if } \alpha_2(r^2) \neq 0. \end{cases}$$

Using the continuity of  $(\phi_1 + \phi_3)'$  at 0, the last system is equivalent to

$$\begin{cases} (\phi_1 + \phi_3)' = 0 & \text{whenever } \alpha_2 = 0 \quad \text{and} \quad \beta_2 \neq 0, \\ \operatorname{Ric}_x(u, u) = \theta(r^2)g_x(u, u) & \text{if } \alpha_2(r^2) \neq 0. \end{cases}$$

□

**Corollary 5.3.** *Let  $(M, g)$  be a Riemannian manifold and  $G$  be a  $g$ -natural metric on the tangent bundle  $TM$ , with respect to which horizontal and vertical distributions are orthogonal. Then the geodesic flow of  $(M, g)$  is incompressible with respect to  $G$ .*

**Proof.** According to (2.2), the orthogonality of the horizontal and vertical distributions is equivalent to the vanishing of the functions  $\alpha_2$  and  $\beta_2$  identically. Since the conditions (i) and (ii) of Proposition 5.2 deal with values where  $\beta_2$  and  $\alpha_2$  don't vanish respectively, then they are automatically satisfied. This completes the proof. □

**Corollary 5.4.** *Let  $(M, g)$  be a Riemannian manifold and  $G$  be a  $g$ -natural metric on the tangent bundle  $TM$ , such that  $I_0 = \mathbb{R}^+ \setminus [(\alpha_2)^{-1}(0)]$  is dense in  $\mathbb{R}^+$ . Then the geodesic flow of  $(M, g)$  is incompressible with respect to  $G$  if and only if the function  $\theta$  of Theorem 5.2 is a constant  $\theta_0$  on  $I_0$  and  $(M, g)$  is an Einstein manifold with  $\operatorname{Ric} = \theta_0 g$ .*

**Proof.** Fix  $x \in M$ ,  $t_0 \in I_0 \setminus \{0\}$  and  $u_0 \in M_x$  such that  $g_x(u_0, u_0) = t_0$ . Suppose that  $\xi$  is incompressible with respect to  $G$ . Then from Theorem 5.2,

$$\operatorname{Ric}_x(\lambda u_0, \lambda u_0) = \theta(\lambda^2 t_0)g_x(\lambda u_0, \lambda u_0),$$

for all  $\lambda$  such that  $\lambda^2 t_0 \in I_0$ . By virtue of bilinearity of  $\operatorname{Ric}_x$  and  $g_x$ , we see that  $\theta(t) = \theta(t_0) =: \theta_0$ , for all  $t \in I_0$ . Furthermore,

$$(5.3) \quad \operatorname{Ric}_x(tu_0, tu_0) = \theta_0 g_x(tu_0, tu_0),$$

for all  $t \in I_0$ . Since  $I_0$  is dense in  $\mathbb{R}^+$ , then by continuity, (5.3) is valid for all  $t \in \mathbb{R}$ . Now,  $x$  and  $u_0$  being arbitrary and since  $\theta$  depends only on the norms of vectors under consideration,  $\operatorname{Ric}_x(u, u) = \theta_0 g_x(u, u)$ , for all  $(x, u) \in TM$ . Using once again the bilinearity and the symmetry of  $\operatorname{Ric}_x$  and  $g_x$ , we have  $\operatorname{Ric} = \theta_0 g$ .

Conversely, if  $\theta$  is a constant  $\theta_0$  on  $I_0$  and  $\operatorname{Ric} = \theta_0 g$ , then we have, by (5.2),  $(\operatorname{div}_G)_{(x, u)} = 0$ , for all  $(x, u)$  such that  $g_x(u, u) \in I_0$ . Fix  $t_0$  and  $u_0$  as before. Then

$(\operatorname{div}_G)_{(x,\lambda u_0)} = 0$ , for all  $\lambda$  such that  $\lambda^2 t_0 \in I_0$ . Since  $\{(x, \lambda u_0)/\lambda^2 t_0 \in I_0\}$  is dense in  $\{(x, \lambda u_0)/\lambda \in \mathbb{R}\}$ , the continuity of  $\operatorname{div}_G$  implies that  $(\operatorname{div}_G)_{(x,\lambda u_0)} = 0$ , for all  $\lambda \in \mathbb{R}$ .  $u_0$  and  $x$  being arbitrary, we have  $(\operatorname{div}_G)_{(x,u)} = 0$ , for all  $(x, u) \in TM$ .  $\square$

**Remark 5.5.** If  $I_0 = \mathbb{R}^+ \setminus [(\alpha_2)^{-1}(0)]$  is dense in  $\mathbb{R}^+$  and  $\theta$  is constant on  $I_0$ , then Corollary 5.4 shows that the metric  $G$  presents a certain kind of rigidity. For example, if we consider  $G$  Riemannian such that

- (a)  $\alpha_1, \alpha_3$  and  $\beta_1 + \beta_3$  are constant on  $\mathbb{R}^+$ ;
- (b)  $\alpha_2$  doesn't vanish on  $\mathbb{R}^+$ , except may be at isolated points of  $\mathbb{R}^+$ ,

then  $\theta = 0$  on  $I_0$ . It follows from Corollary 5.4 that the geodesic flow of  $(M, g)$  is incompressible with respect to  $G$  if and only if  $(M, g)$  is an Einstein manifold with vanishing scalar curvature.

As examples of such Riemannian  $g$ -natural metrics  $G$ , we mention:

- the elements  $G = a \cdot g^s + b \cdot g^h + c \cdot g^v$  of  $\mathcal{C}$ , such that  $b \neq 0$ ;
- $G = a \cdot g^s + \alpha_2 \cdot g^h + c \cdot g^v$ , where  $\alpha_2 = \sqrt{a(a+c)} \cdot \sin \circ \rho$  on  $\mathbb{R}^+$ ,  $\rho$  being any function on  $\mathbb{R}^+$  and  $a, a+c > 0$ .

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