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SINGULAR BGG SEQUENCES FOR THE EVEN ORTHOGONAL CASE

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Abstract. Locally exact complexes of invariant differential operators are constructed on the homogeneous model for a parabolic geometry for the even orthogonal group. The tool used for the construction is the Penrose transform developed by R. Baston and M. Eastwood. Complexes constructed here belong to the singular infinitesimal character.

1. Introduction

The purpose of the paper is to construct complexes of differential operators on a homogeneous space G/P, where $G = Spin(2, 2n; \mathbb{R})$ and P is a suitable maximal parabolic subgroup. A construction of sequences of standard invariant differential operators on general manifolds with a given parabolic structure was described in [6] and simplified and extended in [3]. Invariant differential operators can act, in general, only among sections of bundles induced by representations on the same orbit of the affine action of the corresponding Weyl group. The construction mentioned above covers the case of regular orbits. There is no such general construction available for orbits, which are singular.

A motivation to study the case of singular orbits for this particular homogenous space is coming from another field of mathematics. The Dirac operator acting on spinor fields is a natural generalization of the Cauchy-Riemann operator in the complex plane. A study of solutions of the Dirac equation in several variables is a higher-dimensional analogue of the theory of several complex variables. The question is whether there is an analogue of the Dolbeault complex for the case of the Dirac equation in several variables.

In higher dimensions, a key question is to understand well the symmetry group of the studied system of equation. For one Dirac equation, it is the conformal group. Conformal geometry belongs to parabolic geometries studied very intensively in recent decades (see e.g. ([12, 5, 13, 4]). An appropriate group for Dirac equations in two variables is again a parabolic geometry, its homogeneous model is

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G/P with $G = Spin(2, 2n; \mathbb{R})$ and P is a maximal parabolic subgroup, corresponding to the cross over the second node of the Dynkin diagram. The homogeneous model can be realized as the isotropic Grassmaniann, i.e. as the space of all two dimensional isotropic subspaces in $\mathbb{R}^{2,2n}$.

It can be shown that there is an invariant differential operator in this parabolic geometry with the property that the operator coincides, after the restriction to a suitable flat subspace, with the Dirac operator D in several variables. It is hence reasonable to look for a resolution starting with the operator D and composed entirely from invariant differential operators. A short computation shows that the spinor space belongs to a singular case.

Resolutions starting with D were already constructed in some cases by various methods. One is an algebraic approach summarized in [7], which covers quite a few cases. Another recent approach is based on a study of homomorphisms of (generalized) Verma modules, see [8]. Here we are starting an approach based on the Penrose transform on homogeneous spaces. The method of construction of the complexes based on the Penrose transform was developed by R. Baston and M. Eastwood ([2]). Inspiration for us came from the paper by R. Baston ([1]) who used it in the case of quaternionic manifolds. In the paper, we transfer his method to another parabolic geometry, appropriate for the problem formulated above.

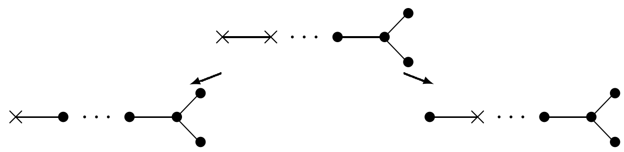
Geometry of the Penrose transform on homogeneous spaces is based on a double fibration of suitable homogenous spaces and the transform translates objects on the left bottom space (usually called the twistor space) to the right bottom space. The transform is formulated for complex Lie groups, i.e. spaces in the double fibration are given by a quotient of a complex simple Lie group G^c and its parabolic subgroup. We want to construct resolutions on the corresponding real spaces, where homogeneous spaces are quotients of a real form G of GA^c and its corresponding parabolic subgroup. Hence we shall formulate our problem in the real situation and for proofs, we shall use the machinery of the Penrose transform in the complexified situation.

In our case, the group G is $Spin(2n + 2, \mathbb{C})$ and its real form is $Spin(2, 2n; \mathbb{R})$. The Penrose transform makes it possible to interpret chosen cohomology groups with values in sections of suitable bundles on (a suitable part of) the Minkowski space using invariant differential operators on the complex isotropic Grassmaniann. We shall consider here first cohomology groups with values in certain line bundles induced by weights from singular orbits. This is just a first step in a more systematic study of invariant operators on singular orbits. We shall find that, contrary to the quaternionic case studied by Baston, constructed complexes do not cover individual singular orbits completely. In all cases, the resulting complexes cover only a part of the corresponding singular orbit.

A geometric foundation of the Penrose transform on homogeneous spaces, as developed in [2], is a double fibration of homogeneous spaces

$$\begin{array}{ccc} & G/R & \\ \swarrow & & \searrow \\ G/Q & & G/P \end{array}$$

In the case we are going to study, particular homogeneous spaces are described by their Dynkin diagrams.



The transform is then restricted to the complexification of a chosen (topologically trivial) subspace of $U \subset G/P$, its preimage in $U'' \subset G/R$ and the image U' of U'' in G/Q. We shall denote the projection to the left factor by η and the projection to the right factor by τ.

All necessary information concerning the Penrose transform can be found in the book [2]. If E is a homogeneous bundle over the twistor space $(U')^c$, then the Penrose transform offers a possibility to describe, or to interpret, a cohomology group on $(U')^c$ with values in appropriate sheaves as the limit of certain hypercohomology spectral sequences on U^c. In our cases, the spectral sequence converges in at most two steps and we get resolutions of the cohomological groups by sequences of invariant differential operators on U.

The Penrose transform is usually split into two parts. Firstly, a relative BGG sequence is written down on the space $(U'')^c$, it is a resolution of the pull-back sheaf $\eta^{-1}(\mathcal{O}(E)$ on $(U'')^c$. Then we have to compute the direct images by τ and to write down the corresponding hypercohomology spectral sequence.

After fixing notation, we describe the corresponding resolutions on the top space in the twistor double fibration (the relative BGG sequences). Then we compute direct images of the corresponding sheaves. The main theorem, which describes the cohomology groups on U' by means of homology its resolution is formulated and proved then. At the end, we show how the obtained resolutions sit in the corresponding singular orbits. In the paper, we shall treat, as a representative example, the case of dimension $2n = 10$. A general case is an easy generalization of this example. The same methods can be used to construct sequences of resolutions depending on many parameters. A discussion of this more general case will be given elsewhere.

2. Notation

On each homogeneous space in the diagram, we shall consider representations of parabolic subgroups P, Q, R, resp. of their complexifications. Those which are irreducible are classified by irreducible representations of the corresponding Levi factor, which is a reductive Lie group. We can choose a Cartan subgroup in the complexification G^c in such a way that it is also a Cartan subgroup of all these three (complexified) Levi factors. The rank of (the complexification of) $G = Spin(2, 10)$ is 6. Hence irreducible representations of all these factors are characterized by their highest weights $\lambda = a_1e_1 + \cdots + a_6e_6 \simeq [a_1, \ldots, a_6]$, where $\{e_i\}$ is the standard Euclidean basis for weights. We shall use this notation instead of numbers over the Dynkin diagrams (used in [2]). An advantage of the chosen notation is that the action of reflections with respect to roots is realized by permutations and sign changes and is very easy to apply. Roots of G are $\pm\alpha_{ij} = \pm(e_i - e_j)$, $\pm\beta_{ij} = \pm(e_i + e_j)$, $i < j$.

Weights have the same number of components for all three parabolics. To indicate which parabolic we have in mind, we shall use the convention introduced by M. Eastwood; the position of crosses in the corresponding diagram will be indicated by vertical lines in appropriate positions of the weight. For the cross over the first simple root, the vertical line will be after the first component of the weight; and similarly for others. To compute easily the affine action of the Weyl group, we shall often add the canonical weight $\delta = [5, 4, 3, 2, 1, 0]$ to all weights.

3. Relative BGG sequences

According to the scheme of the Penrose transform, we shall compute first a resolution of the inverse image of a chosen sheaf over the twistor space by the corresponding relative BGG sequence. An algorithm for it can be found in [2]. The relative Hasse diagram (including roots over its arrows showing which reflections we have to apply) is the characteristic Hasse diagram of conformal geometry. Weights over individual arrows are in turn given by α_{23}, α_{24}, α_{25}, $(\alpha_{26}, \beta_{26})$, β_{25}, β_{24}, β_{23}. The reflection with respect to α_{ij} acts by transposition of the corresponding coordinates, while β_{ij} acts by the transposition composed with the sign change of both coordinates.

Hence the relative BGG for the case $\lambda_1 = [-1|0, 0, 0, 0, 0]$; $\lambda_1 + \delta = [4|4, 3, 2, 1, 0]$ has the form:

$$
\begin{array}{ccc}
 & [4|4|3,2,1,0] & \\
 & \downarrow & \\
 & [4|3|4,2,1,0] & \\
 & \downarrow & \\
 & [4|2|4,3,1,0] & \\
 & \downarrow & \\
 & [4|1|4,3,2,0] & \\
 \swarrow & & \searrow \\
 [4|0|4,3,2,1] & & [4|0|4,3,2,-1] \\
 \searrow & & \swarrow \\
 & [4|-1|4,3,2,0] & \\
 & \downarrow & \\
 & [4|-2|4,3,1,0] & \\
 & \downarrow & \\
 & [4|-3|4,2,1,0] & \\
 & \downarrow & \\
 & [4|-4|3,2,1,0] &
\end{array}
$$

It is easy to see that for $\lambda_k = [-k|0,0,0,0,0,0]$, $k = 2,\ldots,9$, the form of the relative BGG will be the same with $-k$ substituted by $5-k$.

4. Direct images

There is an algorithm for computing direct images (see again [2]). In our case, the fiber of τ is the projective sphere $P^1(\mathbb{C})$, hence only the first two images can be nontrivial. If the first two entries of the weight (shifted by δ) are equal, the both direct images vanishes. If the first entry is smaller than the second, then the first image is nontrivial and the corresponding weight is obtained by switching the first two entries. In the opposite case, the zeroth image is nontrivial and it corresponds to the same weight.

The case $k = 1$.

For $\lambda = [-1|0|0,0,0,0,0]$, we have $\lambda + \delta = [4|4|3,2,1,0]$ so that there is no nontrivial direct image. All remaining weights in the sequence are p-dominant, so $\tau_*^0([-1|-1|1,0,0,0,0]) = [-1,-1|1,0,0,0,0]$, etc.

The spectral sequence $E^1_{p,q}$ has the form

$$\begin{array}{|cccccc}
0 & 0 & 0 & 0 & 0 & \ldots \\
 & & & & 0 & \ldots \\
0 & [-1,-1|1,0,0,0] & [-1,-2|1,1,0,0] & [-1,-3|1,1,1,0] & [-1,-4|1,1,1,1] & \ldots \\
 & & & & [-1,-4|1,1,1,-1] & \\
\hline
\end{array}$$

where the first two rows contain the first images and the last two the zeroth images. A few other cases look as follows (the sequence converges in the second step).

The case $k = 2$.

$$\begin{array}{|cccccc}
[-1,-1|0,0,0,0] & 0 & 0 & 0 & 0 & \ldots \\
 & & & & 0 & \ldots \\
0 & 0 & [-2,-2|1,1,0,0] & [-2,-3|1,1,1,0] & [-2,-4|1,1,1,1] & \ldots \\
 & & & & [-2,-4|1,1,1,-1] & \\
\hline
\end{array}$$

The case $k = 3$.

$$\begin{array}{|cccccc}
[-1,-2|0,0,0,0] & [-2,-2|1,0,0,0] & 0 & 0 & 0 & \ldots \\
 & & & & 0 & \ldots \\
0 & 0 & 0 & [-3,-3|1,1,1,0] & [-3,-4|1,1,1,1] & \ldots \\
 & & & & [-3,-4|1,1,1,-1] & \\
\hline
\end{array}$$

The case $k = 4$.

$$\begin{array}{|cccccc}
[-1,-3|0,0,0,0] & [-2,-3|1,0,0,0] & [-3,-3|1,1,0,0] & 0 & 0 & \ldots \\
 & & & & 0 & \ldots \\
0 & 0 & 0 & 0 & [-4,-4|1,1,1,1] & \ldots \\
 & & & & [-4,-4|1,1,1,-1] & \\
\hline
\end{array}$$

The case $k = 5$.

$$\begin{array}{|cccccc}
[-1,-4|0,0,0,0] & [-2,-4|1,0,0,0] & [-3,-4|1,1,0,0] & [-4,-4|1,1,1,0] & 0 & \ldots \\
 & & & & 0 & \ldots \\
0 & 0 & 0 & 0 & 0 & \ldots \\
 & & & & 0 & \\
\hline
\end{array}$$

The other cases are similar (and in a sense dual to the previous ones), the scheme is clearly visible.

5. Resolutions

The machinery of the Penrose transform is producing a complex on U for each $k = 1, \dots, 9$. The homogeneous space G/P is a closure of a flat vector space V (we can take for V the big cell in G/P). Take now for U a ball in V and denote by U^c the corresponding complexification in V^c. Let us consider also U', U'' and their complexifications.

Let us denote by $\mathcal{O}(E_k)$ the sheaf of holomorphic sections of the homogeneous line bundle E_k induced by λ_k, restricted to $(U')^c$.

Then the cohomology groups $H^j\big((U')^c, \mathcal{O}(E_k)\big)$ can be computed using a suitable spectral sequence. In our case, there is exactly one nontrivial entry on each diagonal. Hence the cohomology groups $H^j\big((U')^c, \mathcal{O}(E_k)\big)$ are computed by the homology of the complex on U^c. The following important lemma has the same proof as in the quaternionic case ([1]).

Lemma 5.1. *The cohomology groups* $H^j\big((U')^c, \mathcal{O}(E_k)\big)$ *are vanishing for all* $j = 2, 3, \dots$

Hence we are getting the following theorem.

Theorem 5.2. *Fix a number k, $k = 1, \dots .9$. Let us denote by λ^k_j; $j = 1, \dots, 8$ the highest weight(s) for P^c, which were computed in the section on direct images (in some cases near the middle there are two irreducible pieces in one degree, then we take the sum of both) and by E^k_j the corresponding representations.*

Take now a ball $U \subset V \subset G/P$ in the flat dense subspace V. Let d^k_j be the differential in the spectral sequences above. Then the complex

$$\{\mathcal{C}^\infty(U, E^k_j), d^k_j\}\,, \qquad j = 1, \dots, 8\,,$$

is an exact sequence. At most one operator in the resolution is of the second order (if so, it is a nonstandard operator), all others are of the first order.

Proof. To use the machinery of the Penrose transform described in [2], we shall work in the complexification. Hence we shall consider the homogeneous spaces given by the complex versions of the groups G, P, R, Q. We shall work locally (i.e., we shall choose U^c to be a disk in the big cell of G^c/P^c). All sections will be holomorphic. It was pointed out by M. Eastwood that once the properties of the complex are established in the complex category, they hold in the smooth category by the work of Nacinovich ([11]).

Let us fix an integer $k = 1, \dots, 9$. The relative BGG sequence computed above gives a resolution of the sheaf $\eta^{-1}\mathcal{O}(E_k)$ by sections of bundles F^k_j. Orders of operators are visible from the inducing weights. The operators given by an arrow from the first direct images to the zeroth ones are of second order and nonstandard (they are defined by the second iteration $E^2_{p,q}$). The basic property of the

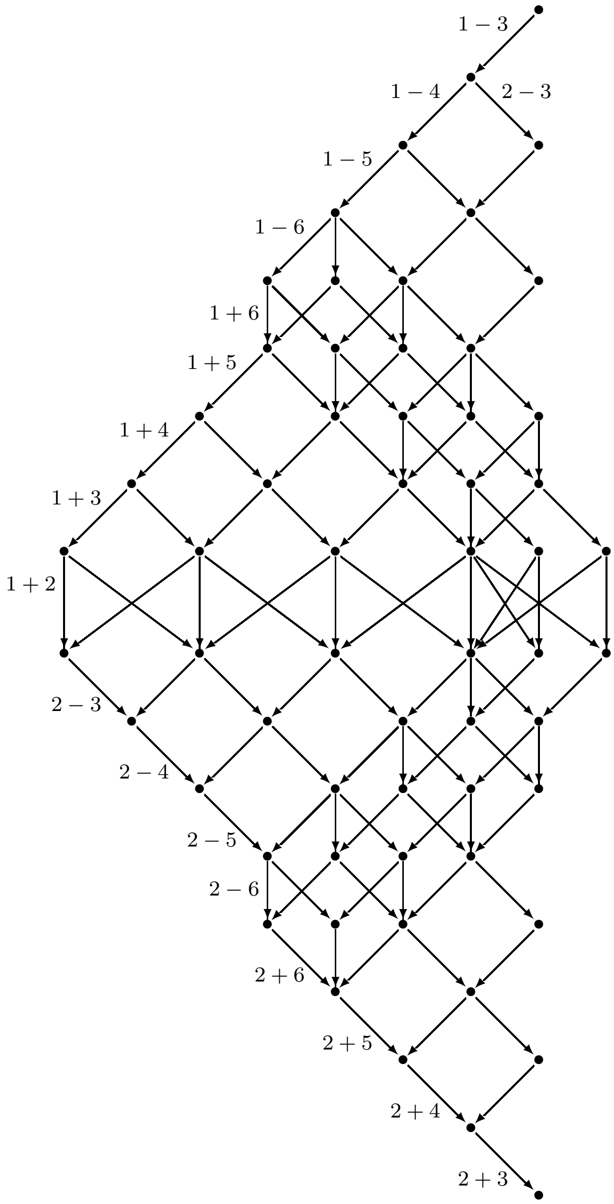
Penrose transform is that the spectral sequence $E^1_{p,q} = \Gamma\left(U^c, \tau^q_*(F^k_p)\right)$ converges to $H^{p+q}\big((U')^c, \mathcal{O}(E_k)\big)$. The vanishing lemma proves then the statement. □

6. Singular orbits

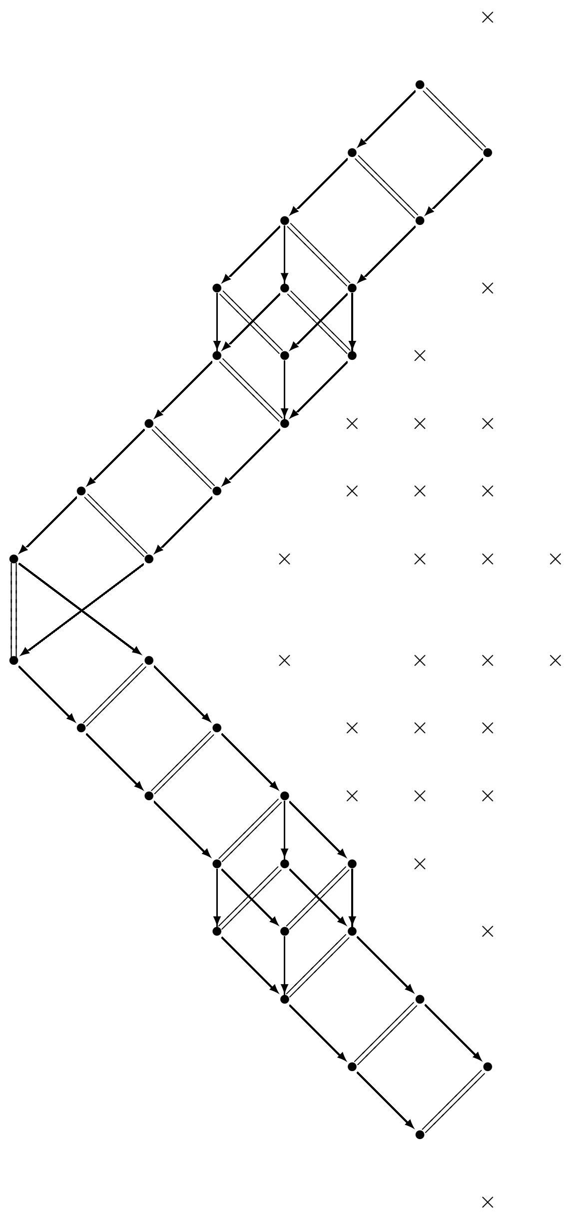
In the quaternionic case ([1]), all points of a singular orbit were contained in one singular BGG sequence. The situation is different in our case. All singular orbits can be obtained as degenerated cases of the regular orbit. Its form is given by the following Hasse diagram (for details see [14]).

In a few following diagrams, we illustrate the shape of regular and singular orbits. We first present a diagram for the Hasse diagram for a regular orbit.

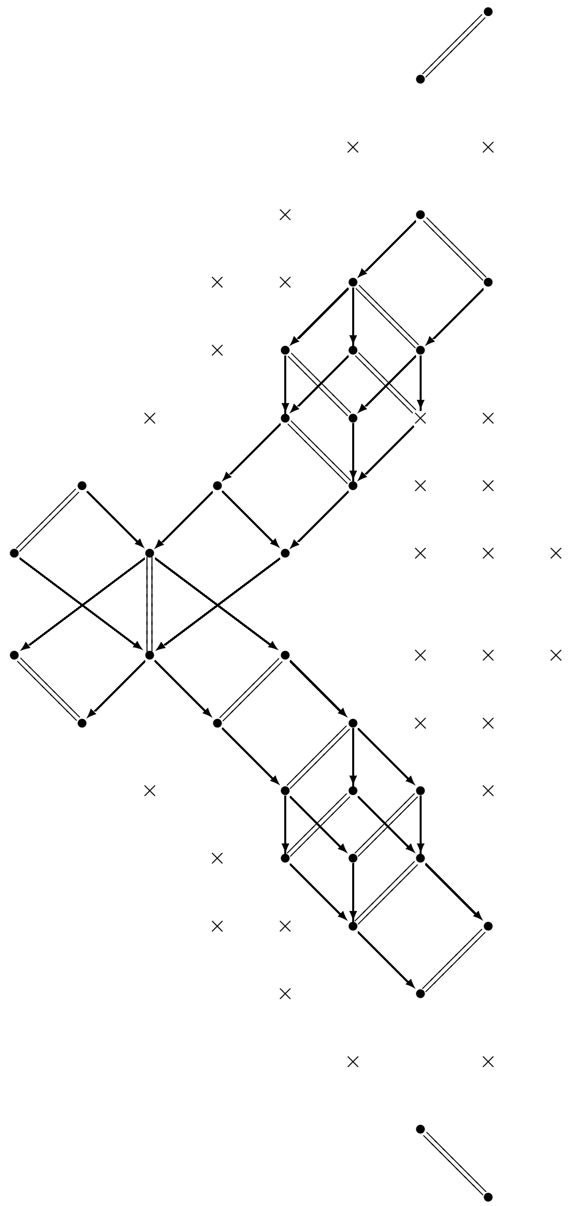
The Hasse diagram in regular character.



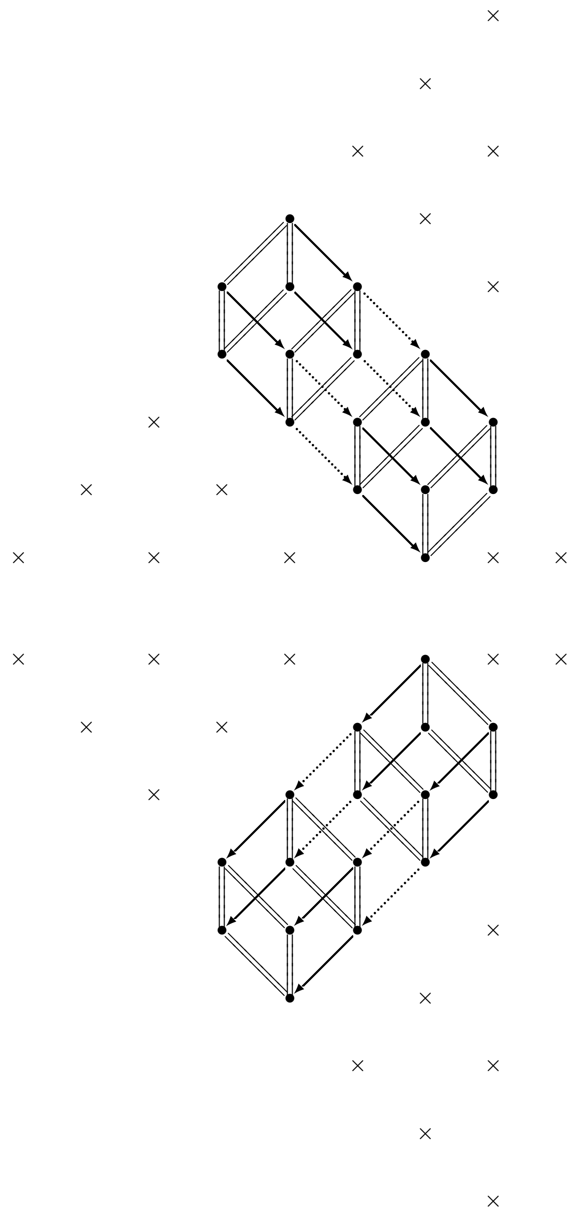
This is a diagram of the singular orbit containing the weights λ_1 and λ_9. The crosses indicate that the weights are not $\mathfrak{p}$-dominant, double lines mean that the modules coincide.



The diagram of the singular orbit containing the weight λ_2 and λ_8.



The diagram of the singular orbit containing the weight λ_4.



Note that the orbit of the weight λ_4 is doubly degenerate (the isotropic subgroup has four elements) and is covered by one resolution. Other weights are coming in pairs contained in the same orbit. The orbit is covered by two resolutions in these cases, they contain a common bundle. The resolutions always intersect each other in one point (one representation). This is a new feature, which was not present in the quaternionic case discussed in [1].

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