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ESTIMATIONS OF NONCONTINUABLE SOLUTIONS OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH p-LAPLACIAN

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Abstract. We study asymptotic properties of solutions for a system of second differential equations with p-Laplacian. The main purpose is to investigate lower estimates of singular solutions of second order differential equations with p-Laplacian $(A(t)\Phi_p(y'))' + B(t)g(y') + R(t)f(y) = e(t)$. Furthermore, we obtain results for a scalar equation.

1. Introduction

Consider the differential equation

$$\big(A(t)\Phi_p(y')\big)' + B(t)g(y') + R(t)f(y) = e(t)\,, \tag{1}$$

where $p > 0$, $A(t)$, $B(t)$, $R(t)$ are continuous, matrix-valued function on $\mathbb{R}_+ := [0,\infty)$, $A(t)$ is regular for all $t \in \mathbb{R}_+$, $e\colon \mathbb{R}_+ \to \mathbb{R}^n$ and $f, g\colon \mathbb{R}^n \to \mathbb{R}^n$ are continuous mappings and $\Phi_p(u) = (|u_1|^{p-1}u_1, \dots, |u_n|^{p-1}u_n)$ for $u = (u_1, \dots, u_n) \in \mathbb{R}^n$. We shall use the norm $\|u\| = \max\limits_{1\le i\le n} |u_i|$ where $u = (u_1, \dots, u_n) \in \mathbb{R}^n$.

Definition 1. A solution y of (1) defined on $t \in [0,T)$ is called noncontinuable or nonextendable if $T < \infty$ and $\limsup\limits_{t\to T^-} \|y'(t)\| = \infty$. The solution y is called continuable if $T = \infty$.

Note, that noncontinuable solutions are also called singular of the second kind, see e.g. [3], [8], [13].

Definition 2. A noncontinuable solution $y\colon [0,T] \to \mathbb{R}^n$ is called oscillatory if there exists an increasing sequence $\{t_k\}_{k=1}^{\infty}$ of zeros of y such that $\lim\limits_{k\to\infty} t_k = T$; otherwise y is called nonoscillatory.

In the last two decades the existence and properties of noncontinuable solutions of special types of (1) are investigated. For the scalar case, see e.g. [3], [4], [5],

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[6], [9], [11], [12], [13], [15] and references therein. In particular, noncontinuable solutions do not exist if f and g satisfy the following conditions

$$\big|g(x)\big| \le |x|^p \quad \text{and} \quad \big|f(x)\big| \le |x|^p \quad \text{for } |x| \text{ large} \tag{2}$$

and R is positive. Hence, noncontinuable solutions may exist mainly in the case $|f(x)| \ge |x|^m$ with $m > p$.

As concern the system (1), see papers [7], [14], where sufficient conditions are given for (1) to have continuable solutions.

The scalar equation (1) can be applied in problems of radially symmetric solutions of the p-Laplace differential equation, see e.g. [14]; noncontinuable solutions appear e.g. in water flow problems (flood waves, a flow in sewerage systems), see e.g. [10].

The present paper deals with the estimations from bellow of norms of a noncontinuable solution of (1) and its derivative. Estimations of solutions are important e.g. in proofs of the existence of such solutions, see e.g. [4], [8] for

$$y^{(n)} = f\big(t, y, \dots, y^{(n-1)}\big) \tag{3}$$

with $n \ge 2$ and $f \in C^0(\mathbb{R}_+, \mathbb{R}^n)$. For generalized Emden-Fowler equation of the form (3), some estimation are proved in [1].

In the paper [14] the differential equation (1) is studied with the initial conditions

$$y(0) = y_0\,, \quad y'(0) = y_1 \tag{4}$$

where $y_0, y_1 \in \mathbb{R}^n$.

We will use results from [7, Theorem 1.2].

Theorem A. *Let $m > p$ and there exist positive constants K_1, K_2 such that*

$$\|g(u)\| \le K_1\|u\|^m\,, \quad \|f(v)\| \le K_2\|v\|^m\,, \quad u, v \in \mathbb{R}^n\,. \tag{5}$$

and $\int_0^\infty \|R(s)\| s^m \mathrm{d}s < \infty$. Denote

$$A_\infty := \sup_{0\le t<\infty} \|A(t)^{-1}\| < \infty\,, \quad E_\infty := \sup_{0\le t<\infty} \int_0^t \|e(s)\|\,\mathrm{d}s < \infty\,,$$

$$R_\infty := \int_0^\infty \|R(s)\|\,\mathrm{d}s\,, \quad B_\infty := \int_0^\infty \|B(t)\|\,\mathrm{d}t\,.$$

Let the following conditions be satisfied:

(i) *Let $m > 1$ and*

$$\frac{m-p}{p} A_\infty D_1^{\frac{m-p}{p}} \int_0^\infty \big(K_1\|B(s)\| + 2^{m-1}K_2 s^m \|R(s)\|\big)\,\mathrm{d}s < 1$$

for all $t \in \mathbb{R}_+$, where

$$D_1 = A_\infty \big\{\|A(0)\Phi_p(y_1)\| + 2^{m-1}K_2\|y_0\|^m R_\infty + E_\infty\big\}\,.$$

(ii) *Let $m \le 1$ and*

$$2^{m+1}\frac{m-p}{p} A_\infty D_2^{\frac{m-p}{p}} \int_0^\infty \big(K_1\|B(s)\| + K_2 s^m \|R(s)\|\big)\,\mathrm{d}s < 1$$

for all $t \in \mathbb{R}_+$, where

$$D_2 = A_\infty \big\{\|A(0)\Phi_p(y_1)\| + 2^m K_1\|y_1\|^m B_\infty + 2^{2m+1}K_2 R_\infty \|y_0\|^m + E_\infty\big\}\,.$$

Then any solution $y(t)$ of the initial value problem (1), (4) *is continuable.*

Proof. First let us prove the assertion (i). We will use [7, Theorem 1.2]. From (5) and its proof, it follows that equation (2.3) in [7] may have form

$$\big\|\Phi_p\big(u(t)\big)\big\| \le \|A(t)^{-1}\|\Big\{\|A(0)\Phi_p(y_1)\| + K_1\int_0^t \|B(s)\|\,\|u(s)\|^m\,\mathrm{d}s + K_2\int_0^t \|R(s)\|\,\|\,y_0 + \int_0^s u(\tau)\mathrm{d}\tau\,\|^m\,\mathrm{d}s\Big\} \tag{6}$$

where

$$c = A_\infty\big\{\|A(0)\Phi_p(y_1)\| + 2^{m-1}K_2\|y_0\|^m R_\infty\big\}$$

and

$$F(t) = 2^{m-1}K_2A_\infty\int_t^\infty \|R(s)\|s^{m-1}\mathrm{d}s + K_1A_\infty\|B(t)\|\,.$$

Now, the results follows from [7, Theorem 1.2].

The assertion (ii) follows from [7, Theorem 1.2]. □

2. Main results

In this chapter we will derive estimates for a noncontinuable solution y on the fixed definition interval $[T,\tau)\subset\mathbb{R}_+$, $\tau<\infty$.

Theorem 1. *Let y be a noncontinuable solution of the system* (1) *on the interval $[T,\tau)\subset\mathbb{R}_+$, $\tau - T\le 1$,*

$$A_0 := \max_{T\le t\le\tau}\|A(t)^{-1}\|\,,\quad B_0 := \max_{T\le t\le\tau}\|B(t)\|\,,\quad E_0 := \max_{T\le t\le\tau}\|e(t)\|\,,$$

$$R_0 := \max_{T\le t\le\tau}\|R(t)\|\,,\qquad \int_0^\infty \|R(s)\|s^m\,\mathrm{d}s<\infty$$

and let there exist positive constants K_1, K_2 and $m>p$ such that

$$\|g(u)\|\le K_1\|u\|^m\,,\quad \|f(v)\|\le K_2\|v\|^m\,,\quad u,v\in\mathbb{R}^n\,. \tag{7}$$

Then the following assertions hold:

(i) *If $p>1$ and $M=\frac{2^{2m+1}(2m+3)}{(m+1)(m+2)}$, then*

$$\|A(t)\Phi_p(y'(t))\| + 2^{m-1}K_2\|y(t)\|^mR_0 + 2E_0(\tau-t)\ge C_1(\tau-t)^{-\frac{p}{m-p}} \tag{8}$$

for $t\in[T,\tau)$, where

$$C_1 = A_0^{-\frac{m}{m-p}}\Big(\frac{m-p}{p}\Big)^{-\frac{p}{m-p}}\Big[\frac{3}{2}K_1B_0 + MK_2R_0\Big]^{-\frac{p}{m-p}}\,.$$

(ii) *If $p\le 1$, then*

$$\|A(t)\Phi_p(y'(t))\| + 2^mK_1B_0\|y'(t)\|^m + 2^{2m+1}K_2R_0\|y(t)\|^m + 2E_0(\tau-t)\ge C_2(\tau-t)^{-\frac{p}{p-m}} \tag{9}$$

for $t \in [T, \tau)$ *where*

$$C_2 = 2^{-\frac{p(m+1)}{m-p}} A_0^{-\frac{m}{m-p}} \Big(\frac{m-p}{p}\Big)^{-\frac{p}{m-p}} \Big[\frac{3}{2}K_1 B_0 + M K_2 R_0\Big]^{-\frac{p}{m-p}} .$$

Proof. First let us prove the assertion (i). Let y be a singular solution of system (1) on the interval $[T, \tau)$. We take t to be fixed in the interval $[T, \tau)$ and for the simplicity denote

$$D = A_0^{-\frac{p}{m-p}} \Big(\frac{m-p}{p}\Big)^{-\frac{p}{m-p}} . \tag{10}$$

Assume, by contradiction, that

$$\begin{aligned} &\|A(t)\Phi_p(y'(t))\| + 2^{m-1}K_2\|y(t)\|^m R_0 + 2E_0(\tau - t) \\ &< D\Big[\frac{3}{2}K_1 B_0 + M K_2 R_0\Big]^{-\frac{p}{m-p}} (\tau - t)^{-\frac{p}{m-p}} . \end{aligned} \tag{11}$$

Together with the Cauchy problem

$$\big(A(x)\Phi_p(y')\big)' + B(x)g(y') + R(x)f(y) = e(x) , \quad x \in [t, \tau) \tag{12}$$

and

$$y(t) = y_0 , \quad y'(t) = y_1 \tag{13}$$

we construct an auxiliary system

$$\big(\bar{A}(s)\Phi_p(z')\big)' + \bar{B}(s)g(z') + \bar{R}(s)f(z) = \bar{e}(s) , \tag{14}$$

$$z(0) = z_0 , \quad z'(0) = z_1 \tag{15}$$

where $s \in \mathbb{R}_+$, z_0, $z_1 \in \mathbb{R}^n$, $\bar{A}(s)$, $\bar{B}(s)$, $\bar{R}(s)$ are continuous, matrix-valued function on $\mathbb{R}_+$ given by

$$\bar{A}(s) = \begin{cases} A(s+t) & \text{if } \ 0 \le s < \tau - t , \\ A(\tau) & \text{if } \ \tau - t \le s < \infty , \end{cases} \tag{16}$$

$$\bar{B}(s) = \begin{cases} B(s+t) & \text{if } \ 0 \le s < \tau - t , \\ -\frac{B(\tau-t)}{\tau-t} s + 2B(\tau - t) & \text{if } \ \tau - t \le s < 2(\tau - t) , \\ 0 & \text{if } \ 2(\tau - t) \le s < \infty , \end{cases} \tag{17}$$

$$\bar{R}(s) = \begin{cases} R(s+t) & \text{if } \ 0 \le s < \tau - t , \\ -\frac{R(\tau-t)}{\tau-t} s + 2R(\tau - t) & \text{if } \ \tau - t \le s < 2(\tau - t) , \\ 0 & \text{if } \ 2(\tau - t) \le s < \infty , \end{cases} \tag{18}$$

$$\bar{e}(s) = \begin{cases} e(s) & \text{if } \ 0 \le s < \tau - t , \\ -\frac{e(\tau-t)}{\tau-t} s + 2e(\tau - t) & \text{if } \ \tau - t \le s < 2(\tau - t) , \\ 0 & \text{if } \ 2(\tau - t) \le s < \infty . \end{cases} \tag{19}$$

We can see that $\bar{A}(s)$ is regular for all $s \in \mathbb{R}_+$.

Hence, the systems (12) on $[t,\tau)$ and (14) on $[0,\tau-t)$ are equivalent with the change of independent variable $x-t\to s$. Let $z_0=y(t)$ and $z_1=y'(t)$. Then the definitions of the functions $\bar{A}$, $\bar{B}$, $\bar{R}$, $\bar{e}$ give that

$$z(s)=y(s+t)\,,\quad s\in[0,\tau-t)\quad \text{is a noncontinuable solution} \tag{20}$$

of the system (14), (15) on $[0,\tau-t)$. By the application of Theorem A (i) to the system (14), (15) we will see that every solution z of the system (14), (15) satisfying

$$\begin{aligned}\|\bar{A}(0)\Phi_p(z_1)\| + 2^{m-1}K_2\|z_0\|^m R_0 + \int_0^\infty \|\bar{e}(s)\|\,\mathrm{d}s\\ < D\Big[\int_0^\infty \big(K_1\|\bar{B}(w)\| + 2^{m-1}K_2\|\bar{R}(w)\|w^m\big)\,\mathrm{d}w\Big]^{-\frac{p}{m-p}}\end{aligned} \tag{21}$$

is continuable. Note, that according to (16)–(21) all assumptions of Theorem A are valid. Furthermore, we will show that (11) yields (21).

We estimate the right-hand side of inequality (21):

$$\begin{aligned}G &:= D\Big[\int_0^\infty \big(K_1\|\bar{B}(w)\| + 2^{m-1}K_2\|\bar{R}(w)\|w^m\big)\,\mathrm{d}w\Big]^{-\frac{p}{m-p}}\\
&\ge D\Big[\int_0^{2(\tau-t)} \big(K_1\|\bar{B}(w)\| + 2^{m-1}K_2\|\bar{R}(w)\|w^m\big)\,\mathrm{d}w\Big]^{-\frac{p}{m-p}}\\
&\ge D\Big[K_1\max_{0\le s\le\tau-t}\|B(s+t)\|(\tau-t)\\
&\quad + K_1\int_{\tau-t}^{2(\tau-t)}\Big\|-\frac{B(\tau-t)}{\tau-t}w + 2B(\tau-t)\Big\|\,\mathrm{d}w\\
&\quad + 2^{m-1}K_2\max_{0\le s\le(\tau-t)}\|R(s+t)\|\frac{(\tau-t)^{m+1}}{m+1}\,\mathrm{d}w\\
&\quad + 2^{m-1}K_2\int_{\tau-t}^{2(\tau-t)}\Big\|-\frac{R(\tau-t)}{\tau-t}w + 2R(\tau-t)\Big\|w^m\,\mathrm{d}w\Big]^{-\frac{p}{m-p}},\\
G &\ge D\Big[K_1\max_{T\le t\le\tau}\|B(t)\|(\tau-t) + \frac{1}{2}K_1\|B(\tau-t)\|(\tau-t)\\
&\quad + M_1K_2\max_{T\le t\le\tau}\|R(t)\|(\tau-t)^{m+1} + M_2K_2\|R(\tau-t)\|(\tau-t)^{m+1}\Big]^{-\frac{p}{m-p}},\end{aligned}$$

where

$$M_1=\frac{2^{m-1}}{m+1}\quad\text{and}\quad M_2=2^{m-1}\frac{2^{m+2}(2m+3)-3m-5}{(m+1)(m+2)}\,.$$

Hence,

$$G > D\Big[\frac{3}{2}K_1B_0(\tau-t) + MK_2R_0(\tau-t)^{m+1}\Big]^{-\frac{p}{m-p}} \tag{22}$$

as $M>M_1+M_2$.

As we assume that $\tau - t \le 1$, inequalities (11) and (22) imply

$$G > D\Big[\frac{3}{2}K_1B_0 + MK_2R_0\Big]^{-\frac{p}{m-p}}(\tau - t)^{-\frac{p}{m-p}} = C_1(\tau - t)^{-\frac{p}{m-p}}$$

$$\ge \|A(t)\Phi_p(y'(t))\| + 2^{m-1}K_2\|y(t)\|^m R_0 + 2E_0(\tau - t)$$

$$\ge \|\bar{A}(0)\Phi_p(z_1)\| + 2^{m-1}K_2\|z_0\|^m R_0 + \int_0^\infty \|\bar{e}(s)\|\,\mathrm{d}s\,, \tag{23}$$

where $C_1 = D\big[\frac{3}{2}K_1B_0 + MK_2R_0\big]^{-\frac{p}{m-p}}$. Hence (21) holds and the solution z of (14) satisfying the initial condition $z(0) = y_0$ and $z'(0) = y_1$ is continuable. This contradiction with (20) proves the statement.

Now we shall prove the assertion (ii). If $p \le 1$ then the proof is similar, we have to use only Theorem A (ii) instead of Theorem A (i). □

Now consider the following special case of equation (1):

$$\big(A(t)\Phi_p(y')\big)' + R(t)f(y) = 0 \tag{24}$$

for all $t \in \mathbb{R}_+$. In this case a better estimation than before can be proved.

Theorem 2. *Let $m > p$ and y be a noncontinuable solution of system* (24) *on interval $[T, \tau) \subset \mathbb{R}_+$. Let there exists a constant $K_2 > 0$ such that*

$$\|f(v)\| \le K_2\|v\|^m, \quad v \in \mathbb{R}^n\,. \tag{25}$$

Let R_0 and M to be given by Theorem 1. Then

$$\|A(t)\Phi_p(y'(t))\| + 2^{m+2}K_2\|y(t)\|^m R_0 \ge C_1(\tau - t)^{-\frac{p(m+1)}{m-p}} \tag{26}$$

where

$$C_1 = A_0^{-\frac{m}{m-p}}\Big(\frac{m-p}{p}\Big)^{-\frac{p}{m-p}}\big[MK_2R_0\big]^{-\frac{p}{m-p}} \quad \textit{in case} \quad p > 1$$

and

$$\|A(t)\Phi_p(y')\| + 2^{2m+1}K_2\|y(t)\|^m R_0 \ge C_2(\tau - t)^{-\frac{p(m+1)}{m-p}}$$

with

$$C_2 = 2^{-\frac{p(m+1)}{m-p}}A_0^{-\frac{m}{m-p}}\Big(\frac{m-p}{p}\Big)^{-\frac{p}{m-p}}\big[MK_2R_0\big]^{-\frac{p}{m-p}} \quad \textit{in case} \quad p \le 1\,.$$

Proof. Proof is similar the one of the Theorem 1 for $B(t) \equiv 0$ and $e(t) \equiv 0$. Let $p > 1$. We do not use assumption $\tau - t \le 1$ and we are able to improve an exponent of the estimation (8). The inequality (23) has changed to

$$G \ge C_1(\tau - t)^{-\frac{p(m+1)}{m-p}}$$

$$\ge \|A(t)\Phi_p(y'(t))\| + 2^{m-1}K_2\|y(t)\|^m R_0$$

$$\ge \|\bar{A}(0)\Phi_p(z'(0))\| + 2^{m-1}K_2\|z(0)\|^m R_0\,, \tag{27}$$

where $C_1 = D[MK_2R_0]^{-\frac{p}{(m-p)}}$. If $p \le 1$, the proof is similar. □

3. Applications

In this case we study the scalar differential equation

$$\big(a(t)\Phi_p(y')\big)' + r(t)f(y) = 0\,, \tag{28}$$

where $p > 0$, $a(t)$, $r(t)$ are continuous functions on $\mathbb{R}_+$, $a(t) > 0$ for $t \in \mathbb{R}_+$, $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous mapping and $\Phi_p(u) = |u|^{p-1}u$.

Corollary 3. *Let y be a noncontinuable oscillatory solution of equation* (28) *defined on $[T,\tau)$. Let there exist constants $K_2 > 0$ and $m > 0$ such that*

$$\big|f(v)\big| \le K_2|v|^m\,, \quad v \in \mathbb{R} \tag{29}$$

and let $\{t_k\}_1^\infty$ and $\{\tau_k\}_1^\infty$ be increasing sequences of all local extrema of the solution y and of $y^{[1]} = a(t)\Phi_p(y')$ on $[T,\tau)$, respectively. Then there exist constants C_1 and C_2 such that

$$\big|y(t_k)\big| \ge C_1(\tau - t_k)^{-\frac{p(m+1)}{m(m-p)}} \tag{30}$$

and, in the case $r \ne 0$ on $\mathbb{R}_+$,

$$\big|y^{[1]}(\tau_k)\big| \ge C_2(\tau - \tau_k)^{-\frac{p(m+1)}{m-p}} \tag{31}$$

for $k \ge 1, 2, \dots$.

Proof. Let $m > p$ and y be an oscillatory noncontinuable solution of equation (28) defined on $[T,\tau)$. An application of Theorem 2 to (28) gives

$$\big|y^{[1]}(t)\big| + 2^{2m+1}K_2\big|y(t)\big|^m r_0 \ge C(\tau - t)^{-\frac{p(m+1)}{m-p}}\,, \tag{32}$$

where C is a suitable constant and $r_0 = \max_{T\le t\le\tau}|r(t)|$. Note that according to (30), x ($x^{[1]}$) has a local extremum at $t_0 \in (T,\tau)$ if and only if $x^{[1]}(t_0) = 0$ ($x(t_0) = 0$). From this it follows that an accumulation point of zeros of x ($x^{[1]}$) does not exist in $[T,\tau)$. Otherwise, it holds $y(\tau) = 0$ and $y'(\tau) = 0$. That is in contradiction with (32). If $\{t_k\}_1^\infty$ is the sequence of all extrema of a solution y, then $y'(t_k) = 0$, i.e. $y^{[1]}(t_k) = 0$. We obtain the following estimate for $y(t_k)$ from (32)

$$\big|y(t_k)\big| \ge C_1(\tau - t_k)^{-\frac{p(m+1)}{m(m-p)}}\,, \tag{33}$$

where $C_1 = C^{\frac{1}{m}}(2^{2m+1}K_2r_0)^{-\frac{1}{m}}$ and (30) is valid. If $\{\tau_k\}_1^\infty$ is the sequence of all extrema of $y^{[1]}(\tau_k)$, then $y(\tau_k) = 0$. We obtain the following estimate for $y^{[1]}(\tau_k)$ from (32)

$$\big|y^{[1]}(\tau_k)\big| \ge C_2(\tau - \tau_k)^{-\frac{p(m+1)}{m-p}}\,, \tag{34}$$

where $C_2 = C$. □

Example 1. Consider (28) and (29) with $m = 2$, $p = 1$. Then from Corollary 3 we obtain the following estimates

$$\big|y(t_k)\big| \ge C_1(\tau - t_k)^{-\frac{3}{2}}\,, \quad \big|y^{[1]}(\tau_k)\big| \ge C_2(\tau - \tau_k)^{-3}\,,$$

where $M = \frac{56}{3}$, $C_1 = \frac{\sqrt{42}}{448K_2a_0r_0}$ and $C_2 = \frac{3}{448K_2a_0^2r_0}$.

Example 2. Consider (28) and (29) with $m = 3$, $p = 2$. Then from Corollary 3 we obtain the following estimates

$$\big|y(t_k)\big| \geq C_1(\tau - t_k)^{-\frac{8}{3}}\,, \quad \big|y^{[1]}(\tau_k)\big| \geq C_2(\tau - \tau_k)^{-8}\,,$$

where $M = \frac{288}{5}$, $C_1 = \frac{1}{32K_2 r_0}\big(\frac{10a_0}{9}\big)^{\frac{2}{3}}$ and $C_2 = \big(\frac{5a_0}{144K_2r_0}\big)^2$.

The following lemma is a special case of [13, Lemma 11.2].

Lemma 1. *Let $y \in C^2[a,b)$, $\delta \in (0, \frac{1}{2})$ and $y'(t)y(t) > 0$, $y''(t)y(t) \geq 0$ on $[a,b)$. Then*

$$(35) \qquad \big(y'(t)y(t)\big)^{-\frac{1}{1-2\delta}} \geq \omega \int_t^b |y''(s)|^\delta \big|y(s)\big|^{3\delta-2}\,\mathrm{d}s\,, \quad t \in [a,b)\,,$$

where $\omega = [(1-2\delta)\delta^\delta(1-\delta)^{1-\delta}]^{-1}$.

Now, let us turn our attention to nonoscillatory solutions of (28).

Theorem 4. *Let $m > p$ and $M \geq 0$ be such that*

$$(36) \qquad \big|f(x)\big| \leq |x|^m \quad \text{for} \quad |x| \geq M\,.$$

If y is a nonoscillatory noncontinuable solution of (28) defined on $[T,\tau)$, then constants C, C_0 and a left neighborhood J of τ exist such that

$$(37) \qquad \big|y'(t)\big| \geq C(\tau - t)^{-\frac{p(m+1)}{m(m-p)}}\,, \quad t \in J\,.$$

Let, moreover, $m < p + \sqrt{p^2+p}$. Then

$$(38) \qquad \big|y(t)\big| \geq C_0(\tau - t)^{m_1} \quad \text{with} \quad m_1 = \frac{m^2 - 2mp - p}{m(m-p)} < 0\,.$$

Proof. Let y be a nonoscillatory noncontinuable solutions of (28) defined on $[T,\tau)$. Then there exists $t_0 \in [T,\tau)$ such that $y(t)y^{[1]}(t) > 0$ for $t \in [t_0,\tau)$. Let

$$y(t) > 0 \quad \text{and} \quad y'(t) > 0 \quad \text{for} \quad t \in J := [t_0,\tau);$$

the opposite case $y(t) < 0$ and $y'(t) < 0$ can be studied similarly. As y is noncontinuable, $\lim_{t\to\tau^-} y'(t) = \infty$. Moreover, $\lim_{t\to\infty} y(t) = \infty$ as, otherwise, $y^{[1]}$ and y are bounded on the finite interval J. Hence, there exists $t_1 \in J$ such that $y'(t) \geq 1$ for $[t_1,\tau)$, $y(t) \geq M$ for $t \geq t_1$ and

$$(39) \qquad y(t) = y(t_0) + \int_{t_0}^t y'(s)\,\mathrm{d}s \leq y(t_0) + \tau y'(t) \leq 2\tau y'(t)\,, \quad t \in [t_1,\tau)\,.$$

Note, that due to $y \geq M$ it is sufficient to suppose (36) instead of (25) for an application of Theorem 2. Hence, Theorem 2 applied to (28), (39) and $y' \geq 1$ imply

$$\begin{aligned} C_1(\tau - t)^{-\frac{p(m+1)}{m-p}} &\leq a(t)(y'(t))^p + C_2 y^m(t) \\ &\leq a(t)(y'(t))^p + C_2(2\tau)^m(y'(t))^m \\ &\leq C_3(y'(t))^m \end{aligned}$$

or

$$y'(t) \geq C_4(\tau - t)^{-\frac{p(m+1)}{m(m-p)}} \quad \text{on} \quad [t_1, \tau)\,,$$

where C_1, C_2, C_3 and C_4 are positive constants which do not depend on y. Moreover, the integration of (37) yields

$$\begin{aligned} y(t) &= y(t_0) + \int_{t_0}^{t} y'(s)\mathrm{d}s \geq C \int_{t_0}^{t} (\tau - s)^{-\frac{p(m+1)}{m(m-p)}}\mathrm{d}s \\ &\geq \frac{C}{|m_1|}[(\tau - t)^{m_1} - (\tau - t_0)^{m_1}] \geq \frac{C}{2|m_1|}(\tau - t)^{m_1} \end{aligned}$$

for t lying in a left neighbourhood I_1 of τ. Hence, (37) and (38) are valid. □

Our last application is devoted to the equation

$$y'' = r(t)|y|^m \operatorname{sgn} y\,, \tag{40}$$

where $r \in C^0(\mathbb{R}_+)$, $m > 1$.

Theorem 5. *Let $\tau \in (0, \infty)$, $T \in [0, \tau)$ and $r(t) > 0$ on $[t, \tau]$.*

(i) *Then* (40) *has a nonoscillatory noncontinuable solution which is defined in a left neighbourhood of τ.*

(ii) *Let y be a nonoscillatory noncontinuable solution of* (40) *defined on $[T, \tau)$. Then constants C, C_1, C_2 and a left neighbourhood I of τ exist such that*

$$|y(t)| \leq C(\tau - t)^{-\frac{2(m+3)}{m-1}} \quad \text{and} \quad |y'(t)| \geq C_1(\tau - t)^{-\frac{m+1}{m(m-1)}}\,, \quad t \in I\,.$$

If, moreover, $m < 1 + \sqrt{2}$, then

$$|y(t)| \leq C_2(\tau - t)^{m_1} \quad \text{with} \quad m_1 = \frac{m^2 - 2m - 1}{m(m-1)} < 0\,.$$

Proof. The assertion (i) follows from [2, Theorem 2].
Let us prove the assertion (ii). Let y be a noncontinuable solution of (40) defined on $[T, \tau)$. According to Theorem 4 and its proof we have $\lim_{t \to \tau^-} |y(t)| = \infty$ and (37) holds. Hence, suppose that $t_0 \in [T, \tau)$ is such that

$$y(t) \geq 1 \quad \text{and} \quad y'(t) > 0 \quad \text{on} \quad [t_0, \tau)\,.$$

Furthermore, there exists $t_1 \in [t_0, \tau)$ such that

$$y(t) = y(t_0) + \int_{t_0}^{t} y'(s)\,\mathrm{d}s \leq y(t_0) + y'(t)(\tau - t_0) \leq C_3 y'(t) \tag{41}$$

for $t \in [t_1, \tau)$ with $C_3 = 2(\tau - t_0)$. Now, we estimate y from below. By applying Lemma 1 with $[a, b) = [t_1, \tau)$ and $\delta = \frac{2}{m+3} \in (0, \frac{1}{2})$. We have $\delta m + 3\delta - 2 = 0$ and

$$\begin{aligned} C_3^{\frac{m+3}{m-1}} y^{-\frac{2(m+3)}{m-1}}(t)m &\geq (y'(t)y(t))^{-\frac{1}{1-2\delta}} \geq \omega \int_t^\tau (y''(s))^\delta (y(s))^{3\delta-2}\mathrm{d}s \\ &\geq C_4 \int_t^\tau y^{\delta m + 3\delta - 2}(s)\mathrm{d}s = C_4(\tau - t) \quad \text{on} \quad [t_1, \tau)\,, \end{aligned} \tag{42}$$

where $C_4 = \omega \min_{t_0 \le \sigma \le \tau} |r(\sigma)|$. From this we have

$$y(t) \le C(\tau - t)^{-\frac{m-1}{2(m+3)}} \quad \text{on} \quad [t_1, \tau)$$

with a suitable positive C. The rest of the statement follows from Theorem 4. □

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