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NONLINEAR STABILITY OF A QUADRATIC FUNCTIONAL EQUATION WITH COMPLEX INVOLUTION

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Abstract. Let X, Y be complex vector spaces. Recently, Park and Th.M. Rassias showed that if a mapping $f : X \to Y$ satisfies

$$f(x+iy) + f(x-iy) = 2f(x) - 2f(y) \tag{1}$$

for all x, $y \in X$, then the mapping $f\colon X \to Y$ satisfies $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all x, $y \in X$. Furthermore, they proved the generalized Hyers-Ulam stability of the functional equation (1) in complex Banach spaces. In this paper, we will adopt the idea of Park and Th. M. Rassias to prove the stability of a quadratic functional equation with complex involution via fixed point method.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [19] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h\colon G_1 \to G_2$ satisfies the inequality $d(h(x*y), h(x) \diamond h(y)) < \delta$ for all x, $y \in G_1$, then there is a homomorphism $H\colon G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x*y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation? Hyers [7] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f\colon X \to Y$ satisfies $\|f(x+y) - f(x) - f(y)\| \le \varepsilon$ for all x, $y \in X$ and some $\varepsilon \ge 0$. Then there exists a unique additive mapping $T\colon X \to Y$ such that $\|f(x) - T(x)\| \le \varepsilon$ for all $x \in X$.

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A square norm on an inner product space satisfies the important parallelogram equality $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$. The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation.* In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping.* A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [18] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [3] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1]–[16] and [17]).

Let X be a set. A function $d\colon X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(3) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 1.1 ([4]). *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \qquad \forall n \ge n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) $d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ *for all $y \in Y$.*

In this paper, we solve the functional equation (1) and by using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation (1) in complex Banach spaces.

In 1996, G. Isac and Th. M. Rassias [9] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

2. Quadratic functional equations

Throughout this section, assume that X and Y are complex vector spaces. If an additive mapping $\varrho\colon X \to Y$ satisfies $\varrho(\varrho(x)) = -x$ for all $x \in X$, then ϱ is called complex involution on X. For example $\varrho(x) = ix$ is a complex involution.

Proposition 2.1. *If a mapping $f\colon X \to Y$ satisfies*

$$f(x + \varrho(y)) + f(x - \varrho(y)) = 2f(x) - 2f(y) \tag{2}$$

for all x, $y \in X$, then the mapping $f\colon X \to Y$ is quadratic, i.e.,

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

holds for all x, $y \in X$. If a mapping $f\colon X \to Y$ is quadratic and $f(\varrho(x)) = -f(x)$ holds for all $x \in X$, then the mapping $f\colon X \to Y$ satisfies (2)*.*

Proof. Assume that $f\colon X \to Y$ satisfies the functional equation (2).

Letting $x = y$ in (2), we get $f(x+\varrho(x)) + f(x-\varrho(x)) = 0$ for all $x \in X$. So $f(\varrho(x)) + f(x) = 0$ for all $x \in X$. Hence $f(\varrho(x)) = -f(x)$ for all $x \in X$. Thus

$$f\big(x+\varrho(y)\big)+f\big(x-\varrho(y)\big)=2f(x)-2f(y)=2f(x)+2f\big(\varrho(y)\big) \tag{3}$$

for all x, $y \in X$. Letting $z = \varrho(y)$ in (3), we get

$$f(x+z)+f(x-z)=2f(x)+2f(z)$$

for all x, $z \in X$.

Assume that a quadratic mapping $f\colon X \to Y$ satisfies $f(\varrho(x)) = -f(x)$ for all $x \in X$.

$$f\big(x+\varrho(y)\big)+f\big(x-\varrho(y)\big)=2f(x)+2f\big(\varrho(y)\big)=2f(x)-2f(y)$$

for all $x, y \in X$. So the mapping $f\colon X \to Y$ satisfies (2). □

3. Fixed points and generalized Hyers-Ulam stability of a quadratic functional equation

Throughout this section, assume that X is a normed vector space with norm $\|\cdot\|$ and that Y is a Banach space with norm $\|\cdot\|$.

For a given mapping $f\colon X \to Y$, we define

$$F(x,y) := f\big(x+\varrho(y)\big)+f\big(x-\varrho(y)\big)-2f(x)+2f(y)$$

for all x, $y \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the quadratic functional equation $F(x,y) = 0$.

Theorem 3.1. *Let $f\colon X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\Phi\colon X^2 \to [0,\infty)$ and an $0 < \alpha < 4$ such that*

$$\max\big\{\Phi(2x,2y), \Phi\big(2\varrho(x),2\varrho(y)\big)\big\} \le \alpha\Phi(x,y)\,, \tag{4}$$

$$\max\big\{\Phi\big(x,\varrho(x)\big), \Phi\big(\varrho(x),x\big)\big\} \le \Phi(x,x)\,, \tag{5}$$

$$\|F(x,y)\| \le \Phi(x,y) \tag{6}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q\colon X \to Y$ satisfying (2) *and*

$$\|f(x)-T(x)\| \le \frac{1}{4-\alpha}\Phi(x,x) \tag{7}$$

for all $x \in X$.

Proof. Since $f(\varrho(x)) = -f(x)$ for all $x \in X$, $f(0) = 0$. $f(-x) = f(\varrho(\varrho(x))) = -f(\varrho(x)) = f(x)$ for all $x \in X$.

Consider the set

$$S := \{g\colon X \to Y \; ; g(0) = 0\}$$

and introduce the *generalized metric* on S:

$$d(g,h) = \inf\{u \in \mathbb{R}^+ : \|g(x) - h(x)\| \le u\Phi(x,x), \quad \forall\, x \in X\}.$$

It is easy to show that (S,d) is complete.

Now we consider the mapping $J\colon S \to S$ such that

$$Jg(x) := \frac{1}{8}\big[g(2x) - g(2\varrho(x))\big] \tag{8}$$

for all $x \in X$.

First, we assert that J is strictly contractive on X. Given g, $h \in X$, let $u > 0$ be an arbitrary constant with $d(g,h) < u$, that is,

$$\|g(x) - h(x)\| \le u\Phi(x,x) \tag{9}$$

for all $x \in X$. If we replace y by $\varrho(x)$ in (6), then we obtain

$$\|f(2x) - 4f(x)\| \le \alpha\Phi\big(x, \varrho(x)\big)\,. \tag{10}$$

If we replace x by $\varrho(x)$ and y by x in (6), then we obtain

$$\|f(2\varrho(x)) + 4f(x)\| \le \alpha\Phi\big(\varrho(x), x\big)\,. \tag{11}$$

It follows from (4), (9) and (11) that

$$\begin{aligned}\|(Jg)(x) - (Jh)(x)\| &= \frac{1}{8}\big\|g(2x) - g(2\varrho(x)) - (h(2x) - h(2\varrho(x)))\,\big\| \\ &\le \frac{1}{8}\big\|g(2x) - h(2x)\| + \frac{1}{8}\|g(2\varrho(x)) - h(2\varrho(x))\big\| \\ &\le \frac{u}{8}\Phi(2x,2x) + \frac{u}{8}\Phi\big(2\varrho(x), 2\varrho(x)\big) \\ &\le \frac{\alpha}{4}u\Phi(x,x)\end{aligned} \tag{12}$$

for all $x \in X$, that is $d(Jg, Jh) \le \frac{\alpha}{4}$. We hence conclude that

$$d(Jg, Jh) \le \frac{\alpha}{4} d(g,h)$$

for all g, $h \in S$.

Next, we assert that $d(Jf, f) \leq \infty$. From (10), (11) and (8) we have

$$\begin{aligned}\|(Jf)(x) - f(x)\| &= \Big\|\frac{1}{8}[f(2x) - f(2\varrho(x)] - f(x)\Big\| \\ &= \frac{1}{8}\big\|f(2x) - f(2\varrho(x)) - 8f(x)\big\| \\ &\leq \frac{1}{8}\big\|f(2x) - 4f(x) - (f(2\varrho(x)) + 4f(x))\big\| \\ &\leq \frac{1}{8}\big\|f(2x) - 4f(x)\big\| + \frac{1}{8}\big\|f(2\varrho(x)) + 4f(x)\big\| \\ &\leq \frac{1}{8}\Phi(x, \varrho(x)) + \frac{1}{8}\Phi(\varrho(x), x) \\ &\leq \frac{1}{4}\Phi(x, x)\end{aligned}$$

for all $x \in X$, that is

$$d(Jf, f) \leq \frac{1}{4} < \infty \tag{13}$$

Now, it follows from Theorem 1.1 that there exists a mapping $T\colon X \to Y$ which is a fixed point of J, such that $d(J^n f, T) \to 0$ as $n \to \infty$.

By mathematical induction, we can easily show (and hence we can omit to show) that

$$(J^n f)(x) = \frac{1}{8^n}\Big[\sum_{i=0}^{n}(-1)^i\binom{n}{i}f\big(\varrho^i(2^n x)\big)\Big].$$

Since $d(J^n f, T) \to 0$ as $n \to \infty$, there exists a sequence $\{u_n\}$ such that $u_n \to 0$ as $n \to \infty$ and $d(J^n f, T) \leq u_n$ for every $n \in \mathbb{N}$. Hence, it follows from the definition of d that

$$\|(J^n f)(x) - T(x)\| \leq u_n \Phi(x, x)$$

for all $x \in X$. Thus, for each (fixed) $x \in X$, we have

$$\lim_{n\to\infty} \|(J^n f)(x) - T(x)\| = 0\,.$$

Therefore,

$$T(x) = \lim_{n\to\infty} \frac{1}{8^n}\Big[\sum_{i=0}^{n}(-1)^i\binom{n}{i}f\big(\varrho^i(2^n x)\big)\Big] \tag{14}$$

for all $x \in X$. It follows from (4), (5) and (14) that for every $n \in \mathbb{N}$,

$$\begin{aligned}
&\|T(x+\varrho(y)) + T(x-\varrho(y)) - 2T(x) + 2T(y)\| \\
&\quad = \lim_{n\to\infty} \frac{1}{8^n}\Big\| \sum_{i=0}^{n} (-1)^i \binom{n}{i} F\big(\varrho^i(2^n x), \varrho^i(2^n y)\big)\Big\| \\
&\quad \le \lim_{n\to\infty} \frac{1}{8^n}\Big| \sum_{i=0}^{n} (-1)^i \binom{n}{i}\Big| \alpha^n \Phi(x,y) \\
&\quad \le \lim_{n\to\infty} \frac{\alpha^n \Phi(x,y)}{8^n} \sum_{i=0}^{n} \binom{n}{i} \\
&\quad = \lim_{n\to\infty} \frac{2^n \alpha^n \Phi(x,y)}{8^n} = 0
\end{aligned}$$

for all $x, y \in X$, which implies that T is a solution of (2) and by Proposition 2.1 T is a quadratic mapping.

By Theorem 1.1 and (13), we obtain

$$d(f,T) \le \frac{1}{1-\frac{\alpha}{4}} d(Jf,f) \le \frac{1}{4-\alpha}\,,$$

and so

$$\|f(x) - T(x)\| \le \frac{1}{4-\alpha}\Phi(x,x) \tag{15}$$

for all $x \in X$. Assume that $T_1\colon X \to Y$ is another solution of (2) satisfying (7) (We know that T_1 is a fixed point of J). In view of (7) and the definition of d, we can conclude that (15) is true with T_1 in place of T. Due to Theorem 1.1, we get $T = T_1$. This proves the uniqueness of T. □

Theorem 3.2 (Compare with Theorem 3.1 of [14]). *Let $p < 2$ and θ be positive real numbers, and let $f\colon X \to Y$ be a mapping satisfying $f(ix) = -f(x)$ and*

$$\|F(x,y)\| \le \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $T\colon X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2\theta}{4-2^p}\|x\|^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 3.1 by taking $\Phi(x,y) = \theta(\|x\|^p + \|y\|^p)$, $\varrho(x) = ix$ and $\alpha = 2^p$ in which $p < 2$. Then all of the conditions of Theorem 3.1 hold and hence there exists a unique quadratic mapping $T\colon X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2\theta}{4-2^p}\|x\|^p$$

for all $x \in X$. □

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