ARCHIVUM MATHEMATICUM (BRNO)
Tomus 47 (2011), 181–199

ON THE OSCILLATION OF THIRD-ORDER QUASI-LINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

E. Thandapani and Tongxing Li

Abstract. The aim of this paper is to study asymptotic properties of the third-order quasi-linear neutral functional differential equation

$$\text{(E)} \qquad \Big[a(t)\Big([x(t)+p(t)x(\delta(t))]''\Big)^{\alpha}\Big]' + q(t)x^{\alpha}(\tau(t)) = 0\,,$$

where $\alpha > 0$, $0 \le p(t) \le p_0 < \infty$ and $\delta(t) \le t$. By using Riccati transformation, we establish some sufficient conditions which ensure that every solution of (E) is either oscillatory or converges to zero. These results improve some known results in the literature. Two examples are given to illustrate the main results.

1. Introduction

We are concerned with the oscillation and asymptotic behavior of the third-order neutral differential equation

$$\text{(E)} \qquad \big[a(t)\left([x(t)+p(t)x(\delta(t))]''\right)^{\alpha}\big]' + q(t)x^{\alpha}(\tau(t)) = 0\,,$$

where $\alpha > 0$ is the quotient of odd positive integers, $a(t)$, $p(t)$, $q(t)$, $\tau(t)$, $\delta(t) \in C([t_0,\infty))$ and

(H) $a(t) > 0$, $\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)}\mathrm{d}t = \infty$, $0 \le p(t) \le p_0 < \infty$, $q(t) \ge 0$, $q(t)$ is not identically zero on any ray of the form $[t_*,\infty)$ for any $t_* \ge t_0$, $\delta(t) \le t$, $\delta'(t) \ge \delta_0 > 0$, $\tau \circ \delta = \delta \circ \tau$ and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \delta(t) = \infty$.

We set $z(t) = x(t) + p(t)x(\delta(t))$. By a solution of Eq. (E) we mean a function $x(t) \in C([T_x,\infty))$, $T_x \ge t_0$, which has the properties $z(t) \in C^2([T_x,\infty))$, $a(t)(z''(t))^{\alpha} \in C^1([T_x,\infty))$ and satisfies (E) on $[T_x,\infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$ and otherwise, it is said to be nonoscillatory. Equation (E) itself is said to be almost oscillatory if all its solutions are oscillatory or convergent to zero asymptotically.

In recent years, great attention has been devoted to the oscillation of differential equations; see for example [1 - 29], and the references cited therein. Especially, differential equations of the form (E) and its special cases have been the subject

---

2010 *Mathematics Subject Classification*: primary 34K11; secondary 34C10.

Key words and phrases: third-order, neutral functional differential equations, oscillation and asymptotic behavior.

Received December 15, 2010, revised March 2011. Editor O. Došlý.

of intensive research. Hartman and Wintner [9], Hanan [8] and Erbe [5] studied a particular case of (E), namely, the third-order differential equation

$$x'''(t) + q(t)x(t) = 0\,.$$

Baculíková et al. [4] considered the oscillation of third-order differential equation

$$\left[b(t)\left(\,[a(t)x'(t)]'\,\right)^{\alpha}\right]' + q(t)x^{\alpha}(t) = 0\,.$$

Baculíková and Džurina [3, 1], Grace et al. [6] and Saker and Džurina [12] investigated the nonlinear differential equation

$$\left[a(t)\left(x''(t)\right)^{\alpha}\right]' + q(t)x^{\alpha}\left(\tau(t)\right) = 0\,.$$

Regarding the oscillation of third-order neutral differential equations, Zhong et al. [13] used method given in [6] and extended some of their results to neutral differential equation (E) for the case when $0 \le p(t) < 1$. Baculíková and Džurina [2] examined the oscillation behavior of (E) under the case when $a'(t) \ge 0$ and $-1 < -p_1 \le p(t) \le p_2 < 1$. Han et al. [7] considered the oscillation nature of (E) for the case when $a(t) = 1$, $\alpha = 1$ and $-1 < -p_1 \le p(t) \le 0$. Karpuz et al. [10] studied the odd-order neutral delay differential equation

$$[x(t) + p(t)x(\delta(t))]^{(n)} + q(t)x(\tau(t)) = 0$$

under the condition when $-1 < p(t) < 1$.

It is interesting to study (E) under the condition $0 \le p(t) \le p_0 < \infty$. To the best of our knowledge, there are no results regarding the oscillation of (E) under the assumption $p(t) \ge 1$. So the purpose of this paper is to present some new oscillatory and asymptotic criteria for (E). We derive criteria for (E) to be oscillatory or for all its nonoscillatory solutions tend to zero as $t \to \infty$.

In order to prove our results, we give the following definition and remarks.

Definition 1 ([11])**.** Consider the sets $\mathbb{D}_0 = \{(t,s) : t > s \ge t_0\}$ and $\mathbb{D} = \{(t,s) : t \ge s \ge t_0\}$. Assume that $H \in C(\mathbb{D},\mathbb{R})$ satisfies the following assumptions:

- (A_1) $H(t,t) = 0$, $t \ge t_0$; $H(t,s) > 0$, $(t,s) \in \mathbb{D}_0$;
- (A_2) H has a non-positive continuous partial derivative with respect to the second variable in $\mathbb{D}_0$.

Then the function H has the property P.

Remark 1. All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

Remark 2. Without loss of generality we can deal only with the positive solutions of (E) since the proof for the opposite case is similar.

2. Main results

In this section, we obtain some new oscillatory criteria for (E). We begin with some useful lemmas, which will be used later.

Lemma 1. *Assume that* $\alpha \ge 1$, $x_1, x_2 \in [0,\infty)$. *Then*

$$x_1{}^{\alpha} + x_2{}^{\alpha} \ge \frac{1}{2^{\alpha-1}}(x_1+x_2)^{\alpha}\,. \tag{2.1}$$

Proof. (i) Suppose that $x_1 = 0$ or $x_2 = 0$. Then we have (2.1).
(ii) Suppose that $x_1 > 0,\ x_2 > 0$. Define the function f by $f(x) = x^{\alpha}$, $x \in (0,\infty)$. Then $f''(x) = \alpha(\alpha-1)x^{\alpha-2} \ge 0$ for $x > 0$. Thus, f is a convex function. By the definition of convex function, we have

$$f\Big(\frac{x_1+x_2}{2}\Big) \le \frac{f(x_1)+f(x_2)}{2}\,.$$

That is,

$$x_1{}^{\alpha} + x_2{}^{\alpha} \ge \frac{1}{2^{\alpha-1}}(x_1+x_2)^{\alpha}\,.$$

This completes the proof. □

Lemma 2. *Assume that* $0 < \alpha \le 1$, $x_1, x_2 \in [0,\infty)$. *Then*

$$x_1{}^{\alpha} + x_2{}^{\alpha} \ge (x_1+x_2)^{\alpha}\ . \tag{2.2}$$

Proof. Assume that $x_1 = 0$ or $x_2 = 0$. Then we have (2.2). Assume that $x_1 > 0$ and $x_2 > 0$. Define $f(x_1,x_2) := x_1{}^{\alpha} + x_2{}^{\alpha} - (x_1+x_2)^{\alpha}$, x_1, $x_2 \in (0,\infty)$. Fix x_1. Then

$$\begin{aligned}\frac{\mathrm{d}f(x_1,x_2)}{\mathrm{d}x_2} &= \alpha x_2{}^{\alpha-1} - \alpha(x_1+x_2)^{\alpha-1}\\ &= \alpha\left[x_2{}^{\alpha-1} - (x_1+x_2)^{\alpha-1}\right] \ge 0\,,\quad \text{since}\quad 0<\alpha\le 1\,.\end{aligned}$$

Thus, f is nondecreasing with respect to x_2, which yields $f(x_1,x_2) \ge 0$.
The proof of the lemma is complete. □

Lemma 3 ([10, Lemma 3])**.** *Let* f *and* $g \in C([t_0,\infty),\mathbb{R})$ *and* $\alpha \in C([t_0,\infty),\mathbb{R})$ *satisfies* $\lim_{t\to\infty}\alpha(t) = \infty$ *and* $\alpha(t) \le t$ *for all* $t \in [t_0,\infty)$*; further suppose that there exists* $h \in C([t_{-1},\infty),\mathbb{R}^+)$*, where* $t_{-1} := \min_{t\in[t_0,\infty)}\{\alpha(t)\}$*, such that* $f(t) = h(t) + g(t)h(\alpha(t))$ *holds for all* $t \in [t_0,\infty)$. *Suppose that* $\lim_{t\to\infty} f(t)$ *exists and* $\liminf_{t\to\infty} g(t) > -1$. *Then* $\limsup_{t\to\infty} h(t) > 0$ *implies* $\lim_{t\to\infty} f(t) > 0$.

Lemma 4. *Assume that* x *is a positive solution of* (E) *and* $\lim_{t\to\infty} x(t) \ne 0$. *If*

$$\int_{t_0}^{\infty}\int_v^{\infty}\Big(\frac{1}{a(\delta(u))}\int_u^{\infty}Q(s)\mathrm{d}s\Big)^{1/\alpha}\mathrm{d}u\mathrm{d}v = \infty\,, \tag{2.3}$$

where

$$Q(t) = \min\{q(t), q(\delta(t))\}\,, \tag{2.4}$$

then

$$z(t) > 0\,,\quad z'(t) > 0\,,\quad z''(t) > 0\,,\quad \left[a(t)\,(z''(t))^{\alpha}\right]' \le 0\,, \tag{2.5}$$

for $t \ge t_1$*, where* t_1 *is sufficiently large.*

Proof. Assume that x is a positive solution of (E). We may only prove the case when $\alpha \geq 1$, since the case when $0 < \alpha \leq 1$ is similar. From (E), we see that $z(t) \geq x(t) > 0$ and

$$\big[a(t)\,(z''(t))^{\alpha}\big]' = -q(t)x^{\alpha}\big(\tau(t)\big) \leq 0\,. \tag{2.6}$$

Thus, $a(t)\,(z''(t))^{\alpha}$ is nonincreasing and of one sign. Therefore, $z''(t)$ is also of one sign and so we have two possibilities: $z''(t) > 0$ or $z''(t) < 0$ for $t \geq t_1$. We claim that $z''(t) > 0$. If not, then there exists a constant $M > 0$ such that

$$a(t)\,(z''(t))^{\alpha} \leq -M < 0\,.$$

Integrating the above inequality from t_1 to t, we get

$$z'(t) \leq z'(t_1) - M^{1/\alpha} \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)}\,\mathrm{d}s\,.$$

Therefore, $\lim_{t\to\infty} z'(t) = -\infty$. Then, from $z''(t) < 0$ and $z'(t) < 0$, we obtain $\lim_{t\to\infty} z(t) = -\infty$. This contradiction proves that $z''(t) > 0$.

Next, we prove that $z'(t) > 0$. Otherwise, we assume that $z'(t) < 0$. From (E), we obtain

$$\begin{aligned}\big[a(t)\big(z''(t)\big)^{\alpha}\big]' &+ ({p_0}^{\alpha})\frac{\big[a(\delta(t))\,(z''(\delta(t)))^{\alpha}\big]'}{\delta_0}\\ &+ q(t)x^{\alpha}\big(\tau(t)\big) + ({p_0}^{\alpha})q\big(\delta(t)\big)x^{\alpha}\big(\tau(\delta(t))\big) \leq 0\,,\end{aligned}$$

which follows from (2.1), (2.4) and $\tau \circ \delta = \delta \circ \tau$ that

$$\big[a(t)\,(z''(t))^{\alpha}\big]' + ({p_0}^{\alpha})\frac{\big[a(\delta(t))\,(z''(\delta(t)))^{\alpha}\big]'}{\delta_0} + \frac{Q(t)}{2^{\alpha-1}}z^{\alpha}\big(\tau(t)\big) \leq 0\,. \tag{2.7}$$

Integrating the last inequality from t to ∞, we obtain

$$a(t)\big(z''(t)\big)^{\alpha} + ({p_0}^{\alpha})\frac{a(\delta(t))\,(z''(\delta(t)))^{\alpha}}{\delta_0} \geq \frac{1}{2^{\alpha-1}}\int_t^{\infty} Q(s)z^{\alpha}\big(\tau(s)\big)\mathrm{d}s\,.$$

In view of (2.6) and $\delta(t) \leq t$, we see that

$$a(t)\big(z''(t)\big)^{\alpha} \leq a\big(\delta(t)\big)\big(z''(\delta(t))\big)^{\alpha}\,.$$

Thus

$$a\big(\delta(t)\big)\big(z''(\delta(t))\big)^{\alpha} \geq \frac{1}{2^{\alpha-1}(1+\frac{{p_0}^{\alpha}}{\delta_0})}\int_t^{\infty} Q(s)z^{\alpha}\big(\tau(s)\big)\,\mathrm{d}s\,.$$

Since $\lim_{t\to\infty} x(t) \neq 0$, from Lemma 3, $\lim_{t\to\infty} z(t) = L > 0$ and $z^{\alpha}(\tau(t)) \geq L^{\alpha}$. Then, we obtain

$$z''\big(\delta(t)\big) \geq L\Big(\frac{1}{2^{\alpha-1}(1+\frac{{p_0}^{\alpha}}{\delta_0})}\Big)^{1/\alpha}\Big(\frac{1}{a(\delta(t))}\int_t^{\infty} Q(s)\,\mathrm{d}s\Big)^{1/\alpha}.$$

Integrating again from t to ∞, we get

$$-\frac{1}{\delta_0}z'(\delta(t)) \geq L\Big(\frac{1}{2^{\alpha-1}(1+\frac{{p_0}^{\alpha}}{\delta_0})}\Big)^{1/\alpha}\int_t^{\infty}\Big(\frac{1}{a(\delta(u))}\int_u^{\infty} Q(s)\,\mathrm{d}s\Big)^{1/\alpha}\mathrm{d}u\,.$$

Integrating the last inequality from t_1 to ∞, we have

$$\frac{1}{(\delta_0)^2} z(\delta(t_1)) \geq L\Big(\frac{1}{2^{\alpha-1}(1+\frac{p_0{}^\alpha}{\delta_0})}\Big)^{1/\alpha} \int_{t_1}^{\infty} \int_{v}^{\infty} \Big(\frac{1}{a(\delta(u))} \int_u^\infty Q(s)\mathrm{d}s\Big)^{1/\alpha} \mathrm{d}u\mathrm{d}v\,,$$

which contradicts (2.3). Thus $z'(t) > 0$. This completes the proof. $\square$

Lemma 5. *Assume that z satisfies (2.5) for $t \geq t_1$. Then*

$$z'(t) \geq \big(a^{1/\alpha}(t) z''(t)\big)\beta_1(t,t_1)\,, \tag{2.8}$$

and

$$z(t) \geq \big(a^{1/\alpha}(t) z''(t)\big)\beta_2(t,t_1)\,, \tag{2.9}$$

where

$$\beta_1(t,t_1) := \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)}\,\mathrm{d}s\,, \qquad \beta_2(t,t_1) := \int_{t_1}^{t}\int_{t_1}^{s} \frac{1}{a^{1/\alpha}(u)}\,\mathrm{d}u\mathrm{d}s\,.$$

Proof. Since $[a(t)(z''(t))^\alpha]' \leq 0$, $a(t)(z''(t))^\alpha$ is nondecreasing. Then we get

$$\begin{aligned} z'(t) \geq z'(t) - z'(t_1) &= \int_{t_1}^{t} \frac{\big[a(s)\,(z''(s))^\alpha\big]^{1/\alpha}}{a^{1/\alpha}(s)}\,\mathrm{d}s \\ &\geq \big(a^{1/\alpha}(t) z''(t)\big) \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)}\,\mathrm{d}s\,. \end{aligned}$$

Similarly, we have

$$z(t) \geq \big(a^{1/\alpha}(t) z''(t)\big) \int_{t_1}^{t}\int_{t_1}^{s} \frac{1}{a^{1/\alpha}(u)}\,\mathrm{d}u\mathrm{d}s\,.$$

$\square$

Next, we state and prove the main theorems.

Theorem 1. *Let $\alpha \geq 1$, $\tau(t) \in C^1([t_0,\infty))$ and $\tau'(t) > 0$. Assume that (2.3) holds and $\tau(t) \leq \delta(t)$. Moreover, assume that there exists a function $\rho \in C^1([t_0,\infty),(0,\infty))$, for all sufficiently large $t_1 \geq t_0$, there is a $t_2 > t_1$ such that*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \Big[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{(1+\frac{p_0{}^\alpha}{\delta_0})}{(\alpha+1)^{\alpha+1}} \frac{((\rho'(s))_+)^{\alpha+1}}{(\rho(s)\beta_1(\tau(s),t_1)\tau'(s))^\alpha}\Big]\,\mathrm{d}s = \infty\,, \tag{2.10}$$

where $(\rho'(t))_+ := \max\{0,\rho'(t)\}$. Then (E) is almost oscillatory.

Proof. Assume that x is a positive solution of (E), which does not tend to zero asymptotically. From the proof of Lemma 4, we obtain (2.5) and (2.7). Then, from Lemma 5, we have (2.8).

Define the function ω by

$$\omega(t) = \rho(t)\frac{a(t)(z''(t))^\alpha}{z^\alpha(\tau(t))}\,. \tag{2.11}$$

Then $\omega(t) > 0$ due to Lemma 4, and

$$\omega'(t) = \rho'(t)\frac{a(t)(z''(t))^\alpha}{z^\alpha(\tau(t))} + \rho(t)\Big(\frac{a(t)(z''(t))^\alpha}{z^\alpha(\tau(t))}\Big)'$$

$$= \rho'(t)\frac{a(t)(z''(t))^\alpha}{z^\alpha(\tau(t))} + \frac{\rho(t)\,[a(t)(z''(t))^\alpha]'}{z^\alpha(\tau(t))}$$

$$- \frac{\alpha\rho(t)a(t)(z''(t))^\alpha z^{\alpha-1}(\tau(t))z'(\tau(t))\tau'(t)}{z^{2\alpha}(\tau(t))}. \tag{2.12}$$

From (2.5), (2.8) and $\tau(t) \le t$, we have

$$z'\big(\tau(t)\big) \ge \big(a^{1/\alpha}(\tau(t))z''(\tau(t))\big)\beta_1\big(\tau(t),t_1\big) \ge \big(a^{1/\alpha}(t)z''(t)\big)\beta_1\big(\tau(t),t_1\big).$$

It follows from (2.11) and (2.12) that

$$\omega'(t) \le \frac{\rho(t)[a(t)(z''(t))^\alpha]'}{z^\alpha(\tau(t))} + \frac{\rho'(t)}{\rho(t)}\omega(t)$$

$$- \frac{\alpha\beta_1(\tau(t),t_1)\tau'(t)}{\rho^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t). \tag{2.13}$$

Similarly, define another function ν by

$$\nu(t) = \rho(t)\frac{a(\delta(t))(z''(\delta(t)))^\alpha}{z^\alpha(\tau(t))}. \tag{2.14}$$

Then $\nu(t) > 0$ due to Lemma 4, and

$$\nu'(t) = \rho'(t)\frac{a(\delta(t))(z''(\delta(t)))^\alpha}{z^\alpha(\tau(t))} + \rho(t)\Big(\frac{a(\delta(t))(z''(\delta(t)))^\alpha}{z^\alpha(\tau(t))}\Big)'$$

$$= \rho'(t)\frac{a(\delta(t))(z''(\delta(t)))^\alpha}{z^\alpha(\tau(t))} + \frac{\rho(t)\,[a(\delta(t))(z''(\delta(t)))^\alpha]'}{z^\alpha(\tau(t))}$$

$$- \frac{\alpha\rho(t)a(\delta(t))(z''(\delta(t)))^\alpha z^{\alpha-1}(\tau(t))z'(\tau(t))\tau'(t)}{z^{2\alpha}(\tau(t))}. \tag{2.15}$$

From (2.5), (2.8) and $\tau(t) \le \delta(t)$, we have

$$z'\big(\tau(t)\big) \ge \big(a^{1/\alpha}(\tau(t))z''(\tau(t))\big)\beta_1(\tau(t),t_1) \ge \big(a^{1/\alpha}(\delta(t))z''(\delta(t))\big)\beta_1(\tau(t),t_1),$$

which follows from (2.14) and (2.15) that

$$\nu'(t) \le \frac{\rho(t)[a(\delta(t))(z''(\delta(t)))^\alpha]'}{z^\alpha(\tau(t))} + \frac{\rho'(t)}{\rho(t)}\nu(t)$$

$$- \frac{\alpha\beta_1(\tau(t),t_1)\tau'(t)}{\rho^{1/\alpha}(t)}\nu^{(\alpha+1)/\alpha}(t). \tag{2.16}$$

Using (2.13) and (2.16), we get

$$\omega'(t)+\frac{{p_0}^{\alpha}}{\delta_0}\nu'(t)\le\rho(t)\frac{[a(t)(z''(t))^{\alpha}]'+\frac{{p_0}^{\alpha}}{\delta_0}[a(\delta(t))(z''(\delta(t)))^{\alpha}]'}{z^{\alpha}(\tau(t))}$$
$$+\frac{(\rho'(t))_+}{\rho(t)}\omega(t)-\frac{\alpha\beta_1(\tau(t),t_1)\tau'(t)}{\rho^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t)$$
$$+\frac{{p_0}^{\alpha}}{\delta_0}\Big[\frac{(\rho'(t))_+}{\rho(t)}\nu(t)-\frac{\alpha\beta_1(\tau(t),t_1)\tau'(t)}{\rho^{1/\alpha}(t)}\nu^{(\alpha+1)/\alpha}(t)\Big]. \tag{2.17}$$

By (2.7) and (2.17), we obtain

$$\omega'(t)+\frac{{p_0}^{\alpha}}{\delta_0}\nu'(t)\le-\rho(t)\frac{Q(t)}{2^{\alpha-1}}+\frac{(\rho'(t))_+}{\rho(t)}\omega(t)-\frac{\alpha\beta_1(\tau(t),t_1)\tau'(t)}{\rho^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t)$$
$$+\frac{{p_0}^{\alpha}}{\delta_0}\Big[\frac{(\rho'(t))_+}{\rho(t)}\nu(t)-\frac{\alpha\beta_1(\tau(t),t_1)\tau'(t)}{\rho^{1/\alpha}(t)}\nu^{(\alpha+1)/\alpha}(t)\Big]. \tag{2.18}$$

Using (2.18) and the inequality

$$Bu-Au^{(\alpha+1)/\alpha}\le\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}},\qquad A>0\,, \tag{2.19}$$

we have

$$\omega'(t)+\frac{{p_0}^{\alpha}}{\delta_0}\nu'(t)\le-\rho(t)\frac{Q(t)}{2^{\alpha-1}}+\frac{1}{(\alpha+1)^{\alpha+1}}\frac{((\rho'(t))_+)^{\alpha+1}}{(\rho(t)\beta_1(\tau(t),t_1)\tau'(t))^{\alpha}}$$
$$+\frac{\frac{{p_0}^{\alpha}}{\delta_0}}{(\alpha+1)^{\alpha+1}}\frac{((\rho'(t))_+)^{\alpha+1}}{(\rho(t)\beta_1(\tau(t),t_1)\tau'(t))^{\alpha}}\,. \tag{2.20}$$

Integrating (2.20) from t_2 ($t_2\ge t_1$) to t, we get

$$\int_{t_2}^{t}\Big[\rho(s)\frac{Q(s)}{2^{\alpha-1}}-\frac{1}{(\alpha+1)^{\alpha+1}}\Big(1+\frac{{p_0}^{\alpha}}{\delta_0}\Big)\frac{((\rho'(s))_+)^{\alpha+1}}{(\rho(s)\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\Big]\,\mathrm{d}s$$
$$\le\omega(t_2)+\frac{{p_0}^{\alpha}}{\delta_0}\nu(t_2)\,,$$

which contradicts (2.10). The proof is complete. □

By Lemma 2, similar to the proof of Theorem 1, we obtain the following result.

Theorem 2. *Let* $0<\alpha\le1$, $\tau(t)\in C^1([t_0,\infty))$ *and* $\tau'(t)>0$. *Assume that* (2.3) *holds and* $\tau(t)\le\delta(t)$. *Further, assume that there exists a function* $\rho\in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1\ge t_0$, *there is a* $t_2>t_1$ *such that*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\Big[\rho(s)Q(s)-\frac{(1+\frac{{p_0}^{\alpha}}{\delta_0})}{(\alpha+1)^{\alpha+1}}\frac{((\rho'(s))_+)^{\alpha+1}}{(\rho(s)\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\Big]\,\mathrm{d}s=\infty\,, \tag{2.21}$$

where $(\rho'(t))_+:=\max\{0,\rho'(t)\}$. *Then* (E) *is almost oscillatory.*

Theorem 3. *Let $\alpha \ge 1$, $\tau(t) \in C^1([t_0,\infty))$ and $\tau'(t) > 0$. Assume that* (2.3) *holds and $\tau(t) \le \delta(t)$. Furthermore, assume that there exists a function $\rho \in C^1([t_0,\infty),(0,\infty))$, for all sufficiently large $t_1 \ge t_0$, there is a $t_2 > t_1$ such that*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \Big[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{(1+\frac{p_0{}^\alpha}{\delta_0})}{4\alpha} \frac{((\rho'(s))_+)^2}{\rho(s)\left(\beta_2(\tau(s),t_1)\right)^{\alpha-1}\beta_1(\tau(s),t_1)\tau'(s)}\Big]\,\mathrm{d}s = \infty\,, \tag{2.22}$$

where $(\rho'(t))_+ := \max\{0,\rho'(t)\}$. Then (E) *is almost oscillatory.*

Proof. Assume that x is a positive solution of (E), which does not tend to zero asymptotically. By the proof of Lemma 4, we obtain (2.5) and (2.7). Then, from Lemma 5, we get (2.8) and (2.9).

Define the function ω and ν by (2.11) and (2.14), respectively. Proceeding as in the proof of Theorem 1, we obtain (2.12) and (2.15). It follows from (2.12) that

$$\begin{aligned}\omega'(t) = \rho'(t)\frac{a(t)(z''(t))^\alpha}{z^\alpha(\tau(t))} &+ \frac{\rho(t)\left[a(t)(z''(t))^\alpha\right]'}{z^\alpha(\tau(t))}\\ &- \frac{\alpha\left[\rho(t)a(t)(z''(t))^\alpha\right]^2\tau'(t)}{z^{2\alpha}(\tau(t))}\frac{z^{\alpha-1}(\tau(t))z'(\tau(t))}{\rho(t)a(t)(z''(t))^\alpha}\,.\end{aligned} \tag{2.23}$$

In view of (2.5), (2.8), (2.9) and $\tau(t) \le t$, we see that

$$\begin{aligned}\frac{z^{\alpha-1}(\tau(t))z'(\tau(t))}{a(t)(z''(t))^\alpha} &= \frac{z^{\alpha-1}(\tau(t))z'(\tau(t))}{a(t)(z''(t))^\alpha}\\ &\ge \frac{\left(a^{1/\alpha}(\tau(t))z''(\tau(t))\right)^\alpha}{a(t)(z''(t))^\alpha}\left(\beta_2(\tau(t),t_1)\right)^{\alpha-1}\beta_1(\tau(t),t_1)\\ &\ge \left(\beta_2(\tau(t),t_1)\right)^{\alpha-1}\beta_1(\tau(t),t_1)\,.\end{aligned} \tag{2.24}$$

Substituting (2.24) into (2.23), and using (2.12), we get

$$\begin{aligned}\omega'(t) \le \frac{\rho(t)\left[a(t)(z''(t))^\alpha\right]'}{z^\alpha(\tau(t))} &+ \frac{(\rho'(t))_+}{\rho(t)}\omega(t)\\ &- \frac{\alpha\tau'(t)\left(\beta_2(\tau(t),t_1)\right)^{\alpha-1}\beta_1(\tau(t),t_1)}{\rho(t)}\omega^2(t)\,.\end{aligned} \tag{2.25}$$

On the other hand, from (2.15), we have

$$\begin{aligned}\nu'(t) = \rho'(t)\frac{a(\delta(t))(z''(\delta(t)))^\alpha}{z^\alpha(\tau(t))} &+ \frac{\rho(t)\left[a(\delta(t))(z''(\delta(t)))^\alpha\right]'}{z^\alpha(\tau(t))}\\ &- \frac{\alpha\left[\rho(t)a(\delta(t))(z''(\delta(t)))^\alpha\right]^2\tau'(t)}{z^{2\alpha}(\tau(t))}\frac{z^{\alpha-1}(\tau(t))z'(\tau(t))}{\rho(t)a(\delta(t))(z''(\delta(t)))^\alpha}\,.\end{aligned} \tag{2.26}$$

By (2.5), (2.8), (2.9) and $\tau(t) \le \delta(t)$, we see that

$$\begin{aligned}\frac{z^{\alpha-1}(\tau(t))z'(\tau(t))}{a(\delta(t))(z''(\delta(t)))^{\alpha}} &= \frac{z^{\alpha-1}(\tau(t))z'(\tau(t))}{a(\delta(t))(z''(\delta(t)))^{\alpha}} \\ &\ge \frac{\left(a^{1/\alpha}(\tau(t))z''(\tau(t))\right)^{\alpha}}{a(\delta(t))(z''(\delta(t)))^{\alpha}} \left(\beta_2(\tau(t),t_1)\right)^{\alpha-1} \beta_1(\tau(t),t_1) \\ &\ge \left(\beta_2(\tau(t),t_1)\right)^{\alpha-1} \beta_1(\tau(t),t_1)\,. \qquad (2.27)\end{aligned}$$

Substituting (2.27) into (2.26), and applying (2.15), we get

$$\begin{aligned}\nu'(t) \le{}& \frac{\rho(t)\left[a(\delta(t))(z''(\delta(t)))^{\alpha}\right]'}{z^{\alpha}(\tau(t))} + \frac{(\rho'(t))_+}{\rho(t)}\nu(t) \\ &- \frac{\alpha\tau'(t)\left(\beta_2(\tau(t),t_1)\right)^{\alpha-1}\beta_1(\tau(t),t_1)}{\rho(t)}\nu^2(t)\,. \qquad (2.28)\end{aligned}$$

Using (2.25) and (2.28), we have

$$\begin{aligned}\omega'(t) + \frac{{p_0}^{\alpha}}{\delta_0}\nu'(t) \le{}& \rho(t)\frac{\left[a(t)(z''(t))^{\alpha}\right]' + \frac{{p_0}^{\alpha}}{\delta_0}\left[a(\delta(t))(z''(\delta(t)))^{\alpha}\right]'}{z^{\alpha}(\tau(t))} \\ &+ \frac{(\rho'(t))_+}{\rho(t)}\omega(t) - \frac{\alpha\left(\beta_2(\tau(t),t_1)\right)^{\alpha-1}\beta_1(\tau(t),t_1)\tau'(t)}{\rho(t)}\omega^2(t) \\ &+ \frac{{p_0}^{\alpha}}{\delta_0}\Big[\frac{(\rho'(t))_+}{\rho(t)}\nu(t) - \frac{\alpha\left(\beta_2(\tau(t),t_1)\right)^{\alpha-1}\beta_1(\tau(t),t_1)\tau'(t)}{\rho(t)}\nu^2(t)\Big]\,. \qquad (2.29)\end{aligned}$$

Applying (2.7), (2.29) and the inequality

$$Bu - Au^2 \le \frac{B^2}{4A}\,, \qquad A > 0\,,$$

we have

$$\begin{aligned}\omega'(t) + \frac{{p_0}^{\alpha}}{\delta_0}\nu'(t) \le{}& -\rho(t)\frac{Q(t)}{2^{\alpha-1}} \\ &+ \frac{(1+\frac{{p_0}^{\alpha}}{\delta_0})}{4\alpha}\frac{((\rho'(t))_+)^2}{\rho(t)\left(\beta_2(\tau(t),t_1)\right)^{\alpha-1}\beta_1(\tau(t),t_1)\tau'(t)}\,. \qquad (2.30)\end{aligned}$$

Integrating (2.30) from t_2 ($t_2 \ge t_1$) to t, we obtain

$$\begin{aligned}\int_{t_2}^{t}\Big[\rho(s)\frac{Q(s)}{2^{\alpha-1}} - \frac{(1+\frac{{p_0}^{\alpha}}{\delta_0})}{4\alpha}\frac{((\rho'(s))_+)^2}{\rho(s)\left(\beta_2(\tau(s),t_1)\right)^{\alpha-1}\beta_1(\tau(s),t_1)\tau'(s)}\Big]\,\mathrm{d}s \\ \le \omega(t_2) + \frac{{p_0}^{\alpha}}{\delta_0}\nu(t_2)\,,\end{aligned}$$

which contradicts (2.22). The proof is complete. □

From Lemma 2, similar to the proof of Theorem 3, we obtain the following result.

Theorem 4. *Let* $0 < \alpha \le 1$, $\tau(t) \in C^1([t_0,\infty))$ *and* $\tau'(t) > 0$. *Assume that* (2.3) *holds and* $\tau(t) \le \delta(t)$. *Furthermore, assume that there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 \ge t_0$, *there is a* $t_2 > t_1$ *such that*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\Big[\rho(s)Q(s)-\frac{(1+\frac{p_0{}^{\alpha}}{\delta_0})}{4\alpha}\frac{((\rho'(s))_+)^2}{\rho(s)\left(\beta_2(\tau(s),t_1)\right)^{\alpha-1}\beta_1(\tau(s),t_1)\tau'(s)}\Big]\,\mathrm{d}s = \infty\,, \tag{2.31}$$

where $(\rho'(t))_+ := \max\{0,\rho'(t)\}$. *Then* (E) *is almost oscillatory.*

Now we shall establish some criteria for the oscillation of (E) for the case when $\tau(t) \ge \delta(t)$.

Theorem 5. *Assume that* (2.3) *holds,* $\alpha \ge 1$ *and* $\tau(t) \ge \delta(t)$. *Moreover, assume that there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 \ge t_0$, *there is a* $t_2 > t_1$ *such that*

$$\limsup_{t\to\infty}\int_{t_2}^{t}\Big[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\frac{(1+\frac{p_0{}^{\alpha}}{\delta_0})}{(\alpha+1)^{\alpha+1}}\frac{((\rho'(s))_+)^{\alpha+1}}{(\rho(s)\beta_1(\delta(s),t_1)\delta'(s))^{\alpha}}\Big]\mathrm{d}s = \infty\,, \tag{2.32}$$

where $(\rho'(t))_+ := \max\{0,\rho'(t)\}$. *Then* (E) *is almost oscillatory.*

Proof. Assume that x is a positive solution of (E), which does not tend to zero asymptotically. From the proof of Lemma 4, we obtain (2.5) and (2.7). Hence by Lemma 5, we get (2.8).

Define the function ω by

$$\omega(t) = \rho(t)\frac{a(t)(z''(t))^{\alpha}}{z^{\alpha}(\delta(t))}\,. \tag{2.33}$$

Then $\omega(t) > 0$ due to Lemma 4, and

$$\begin{aligned}\omega'(t) &= \rho'(t)\frac{a(t)(z''(t))^{\alpha}}{z^{\alpha}(\delta(t))} + \rho(t)\Big(\frac{a(t)(z''(t))^{\alpha}}{z^{\alpha}(\delta(t))}\Big)'\\ &= \rho'(t)\frac{a(t)(z''(t))^{\alpha}}{z^{\alpha}(\delta(t))} + \frac{\rho(t)\,[a(t)(z''(t))^{\alpha}]'}{z^{\alpha}(\delta(t))}\\ &\quad - \frac{\alpha\rho(t)a(t)(z''(t))^{\alpha}z^{\alpha-1}(\delta(t))z'(\delta(t))\delta'(t)}{z^{2\alpha}(\delta(t))}\,.\end{aligned} \tag{2.34}$$

By (2.5), (2.8) and $\delta(t) \le t$, we have

$$z'(\delta(t)) \ge \Big(a^{1/\alpha}(\delta(t))z''(\delta(t))\Big)\,\beta_1(\delta(t),t_1) \ge \Big(a^{1/\alpha}(t)z''(t)\Big)\,\beta_1(\delta(t),t_1)\,.$$

It follows from (2.33) and (2.34) that

$$\omega'(t) \le \frac{\rho(t)\,[a(t)(z''(t))^{\alpha}]'}{z^{\alpha}(\delta(t))} + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\alpha\beta_1(\delta(t),t_1)\delta'(t)}{\rho^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t)\,. \tag{2.35}$$

Similarly, define another function ν by

$$\nu(t) = \rho(t)\frac{a(\delta(t))(z''(\delta(t)))^{\alpha}}{z^{\alpha}(\delta(t))}\,. \tag{2.36}$$

Then $\nu(t) > 0$ due to Lemma 4, and

$$\begin{aligned}\nu'(t) &= \rho'(t)\frac{a(\delta(t))(z''(\delta(t)))^{\alpha}}{z^{\alpha}(\delta(t))} + \rho(t)\Big(\frac{a(\delta(t))(z''(\delta(t)))^{\alpha}}{z^{\alpha}(\delta(t))}\Big)' \\ &= \rho'(t)\frac{a(\delta(t))(z''(\delta(t)))^{\alpha}}{z^{\alpha}(\delta(t))} + \frac{\rho(t)\left[a(\delta(t))(z''(\delta(t)))^{\alpha}\right]'}{z^{\alpha}(\delta(t))} \\ &\quad - \frac{\alpha\rho(t)a(\delta(t))(z''(\delta(t)))^{\alpha}z^{\alpha-1}(\delta(t))z'(\delta(t))\delta'(t)}{z^{2\alpha}(\delta(t))}\,. \end{aligned} \tag{2.37}$$

From (2.5) and (2.8), we have

$$z'(\delta(t)) \geq \big(a^{1/\alpha}(\delta(t))z''(\delta(t))\big)\beta_1\big(\delta(t), t_1\big)\,,$$

which follows from (2.36) and (2.37) that

$$\begin{aligned}\nu'(t) &\leq \frac{\rho(t)\left[a(\delta(t))(z''(\delta(t)))^{\alpha}\right]'}{z^{\alpha}(\delta(t))} + \frac{\rho'(t)}{\rho(t)}\nu(t) \\ &\quad - \frac{\alpha\beta_1(\delta(t), t_1)\delta'(t)}{\rho^{1/\alpha}(t)}\nu^{(\alpha+1)/\alpha}(t)\,. \end{aligned} \tag{2.38}$$

Using (2.35) and (2.38), we get

$$\begin{aligned}\omega'(t) + \frac{{p_0}^{\alpha}}{\delta_0}\nu'(t) &\leq \rho(t)\frac{\left[a(t)(z''(t))^{\alpha}\right]' + \frac{{p_0}^{\alpha}}{\delta_0}\left[a(\delta(t))(z''(\delta(t)))^{\alpha}\right]'}{z^{\alpha}(\delta(t))} \\ &\quad + \frac{(\rho'(t))_+}{\rho(t)}\omega(t) - \frac{\alpha\beta_1(\delta(t), t_1)\delta'(t)}{\rho^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t) \\ &\quad + \frac{{p_0}^{\alpha}}{\delta_0}\Big[\frac{(\rho'(t))_+}{\rho(t)}\nu(t) - \frac{\alpha\beta_1(\delta(t), t_1)\delta'(t)}{\rho^{1/\alpha}(t)}\nu^{(\alpha+1)/\alpha}(t)\Big]\,. \end{aligned} \tag{2.39}$$

By (2.5), (2.7), (2.39) and $\tau(t) \geq \delta(t)$, we obtain

$$\begin{aligned}\omega'(t) + \frac{{p_0}^{\alpha}}{\delta_0}\nu'(t) &\leq -\,\rho(t)\frac{Q(t)}{2^{\alpha-1}} + \frac{(\rho'(t))_+}{\rho(t)}\omega(t) - \frac{\alpha\beta_1(\delta(t), t_1)\delta'(t)}{\rho^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t) \\ &\quad + \frac{{p_0}^{\alpha}}{\delta_0}\Big[\frac{(\rho'(t))_+}{\rho(t)}\nu(t) - \frac{\alpha\beta_1(\delta(t), t_1)\delta'(t)}{\rho^{1/\alpha}(t)}\nu^{(\alpha+1)/\alpha}(t)\Big]\,. \end{aligned} \tag{2.40}$$

Using (2.40) and the inequality (2.19), we have

$$\begin{aligned}\omega'(t) + \frac{{p_0}^{\alpha}}{\delta_0}\nu'(t) &\leq -\,\rho(t)\frac{Q(t)}{2^{\alpha-1}} + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{((\rho'(t))_+)^{\alpha+1}}{(\rho(t)\beta_1(\delta(t), t_1)\delta'(t))^{\alpha}} \\ &\quad + \frac{\frac{{p_0}^{\alpha}}{\delta_0}}{(\alpha+1)^{\alpha+1}}\frac{((\rho'(t))_+)^{\alpha+1}}{(\rho(t)\beta_1(\delta(t), t_1)\delta'(t))^{\alpha}}\,. \end{aligned} \tag{2.41}$$

Integrating (2.41) from t_2 ($t_2 \geq t_1$) to t, we get

$$\int_{t_2}^{t}\Big[\rho(s)\frac{Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}}\Big(1 + \frac{{p_0}^{\alpha}}{\delta_0}\Big)\frac{((\rho'(s))_+)^{\alpha+1}}{(\rho(s)\beta_1(\delta(s), t_1)\delta'(s))^{\alpha}}\Big]\,\mathrm{d}s$$

$$\leq \omega(t_2) + \frac{{p_0}^{\alpha}}{\delta_0}\nu(t_2)\,,$$

which contradicts (2.32). The proof is complete. □

From Lemma 2, similar to the proof of Theorem 5, we obtain the following result.

Theorem 6. *Assume that* (2.3) *holds,* $0 < \alpha \le 1$ *and* $\tau(t) \ge \delta(t)$*. Moreover, assume that there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$*, for all sufficiently large* $t_1 \ge t_0$*, there is a* $t_2 > t_1$ *such that*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \Big[\rho(s)Q(s) - \frac{(1+\frac{p_0{}^{\alpha}}{\delta_0})}{(\alpha+1)^{\alpha+1}} \frac{((\rho'(s))_+)^{\alpha+1}}{(\rho(s)\beta_1(\delta(s),t_1)\delta'(s))^{\alpha}}\Big]\,\mathrm{d}s = \infty\,, \tag{2.42}$$

where $(\rho'(t))_+ := \max\{0,\rho'(t)\}$*. Then* (E) *is almost oscillatory.*

By using (2.34) and (2.37), similar to the proof of Theorem 3, we obtain the following result.

Theorem 7. *Assume that* (2.3) *holds,* $\alpha \ge 1$ *and* $\tau(t) \ge \delta(t)$*. Furthermore, assume that there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$*, for all sufficiently large* $t_1 \ge t_0$*, there is a* $t_2 > t_1$ *such that*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \Big[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{(1+\frac{p_0{}^{\alpha}}{\delta_0})}{4\alpha} \frac{((\rho'(s))_+)^{2}}{\rho(s)\left(\beta_2(\delta(s),t_1)\right)^{\alpha-1}\beta_1(\delta(s),t_1)\delta'(s)}\Big]\,\mathrm{d}s = \infty\,, \tag{2.43}$$

where $(\rho'(t))_+ := \max\{0,\rho'(t)\}$*. Then* (E) *is almost oscillatory.*

From Lemma 2 and Theorem 7, similar to the proof of Theorem 3, we establish the following result.

Theorem 8. *Assume that* (2.3) *holds,* $0 < \alpha \le 1$ *and* $\tau(t) \ge \delta(t)$*. Furthermore, assume that there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$*, for all sufficiently large* $t_1 \ge t_0$*, there is a* $t_2 > t_1$ *such that*

$$\limsup_{t\to\infty} \int_{t_2}^{t} \Big[\rho(s)Q(s) - \frac{(1+\frac{p_0{}^{\alpha}}{\delta_0})}{4\alpha} \frac{((\rho'(s))_+)^{2}}{\rho(s)\left(\beta_2(\delta(s),t_1)\right)^{\alpha-1}\beta_1(\delta(s),t_1)\delta'(s)}\Big]\,\mathrm{d}s = \infty\,, \tag{2.44}$$

where $(\rho'(t))_+ := \max\{0,\rho'(t)\}$*. Then* (E) *is almost oscillatory.*

Remark 3. From Theorems 1–8, we can get some oscillation criteria for (E) with different choices of ρ.

3. Further results

In this section, we will establish some Philos-type oscillation results for (E).

Theorem 9. *Let* $\alpha \ge 1$*,* $\tau(t) \in C^1([t_0,\infty))$ *and* $\tau'(t) > 0$*. Assume that* (2.3) *holds and* $\tau(t) \le \delta(t)$*. Moreover, assume that* $H \in C(\mathbb{D},\mathbb{R})$ *has the property* P *and there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$*, for all sufficiently large* $t_1 \ge t_0$*, there is a* $t_2 > t_1$ *such that*

$$-\frac{\partial}{\partial s}H(t,s) - \frac{\rho'(s)}{\rho(s)}H(t,s) = \frac{h(t,s)(H(t,s))^{\alpha/(\alpha+1)}}{\rho(s)}\,, \quad (t,s) \in \mathbb{D}_0\,, \tag{3.1}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^{t} G_1(t,s)\,\mathrm{d}s = \infty\,, \tag{3.2}$$

where

$$G_1(t,s) := H(t,s)\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{(1+\frac{p_0{}^{\alpha}}{\delta_0})}{(\alpha+1)^{\alpha+1}} \frac{(h_-(t,s))^{\alpha+1}}{(\rho(s)\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\,,$$

$h_-(t,s) := \max\{0,-h(t,s)\}$. *Then* (E) *is almost oscillatory.*

Proof. Assume that x is a positive solution of (E), which does not tend to zero asymptotically. We define ω and ν as in Theorem 1. Then, we obtain (2.18). From (2.18) with $(\rho'(t))_+$ replaced by $\rho'(t)$, we get

$$\begin{aligned}\rho(t)\frac{Q(t)}{2^{\alpha-1}} \le & -\omega'(t) - \frac{p_0{}^{\alpha}}{\delta_0}\nu'(t) + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{\alpha\beta_1(\tau(t),t_1)\tau'(t)}{\rho^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t)\\ & + \frac{p_0{}^{\alpha}}{\delta_0}\Big[\frac{\rho'(t)}{\rho(t)}\nu(t) - \frac{\alpha\beta_1(\tau(t),t_1)\tau'(t)}{\rho^{1/\alpha}(t)}\nu^{(\alpha+1)/\alpha}(t)\Big]\,. \end{aligned} \tag{3.3}$$

In (3.3), replace t by s and multiply both sides by $H(t,s)$, integrate with respect to s from t_2 ($t_2 \ge t_1$) to t, we have

$$\begin{aligned}\int_{t_2}^{t} H(t,s)\rho(s)\frac{Q(s)}{2^{\alpha-1}}\mathrm{d}s \le & -\int_{t_2}^{t} H(t,s)\omega'(s)\,\mathrm{d}s + \int_{t_2}^{t} H(t,s)\frac{\rho'(s)}{\rho(s)}\omega(s)\,\mathrm{d}s\\ & - \int_{t_2}^{t} H(t,s)\frac{\alpha\beta_1(\tau(s),t_1)\tau'(s)}{\rho^{1/\alpha}(s)}\omega^{(\alpha+1)/\alpha}(s)\,\mathrm{d}s\\ & - \frac{p_0{}^{\alpha}}{\delta_0}\int_{t_2}^{t} H(t,s)\nu'(s)\,\mathrm{d}s + \frac{p_0{}^{\alpha}}{\delta_0}\int_{t_2}^{t} H(t,s)\frac{\rho'(s)}{\rho(s)}\nu(s)\,\mathrm{d}s\\ & - \frac{p_0{}^{\alpha}}{\delta_0}\int_{t_2}^{t} H(t,s)\frac{\alpha\beta_1(\tau(s),t_1)\tau'(s)}{\rho^{1/\alpha}(s)}\nu^{(\alpha+1)/\alpha}(s)\,\mathrm{d}s\,.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\int_{t_2}^{t} H(t,s)\rho(s)\frac{Q(s)}{2^{\alpha-1}}\mathrm{d}s \le\ & H(t,t_2)\omega(t_2) - \int_{t_2}^{t}\Big[-\frac{\partial}{\partial s}H(t,s) - \frac{\rho'(s)}{\rho(s)}H(t,s)\Big]\omega(s)\,\mathrm{d}s\\ & - \int_{t_2}^{t} H(t,s)\frac{\alpha\beta_1(\tau(s),t_1)\tau'(s)}{\rho^{1/\alpha}(s)}\omega^{(\alpha+1)/\alpha}(s)\,\mathrm{d}s + \frac{p_0{}^{\alpha}}{\delta_0}H(t,t_2)\nu(t_2)\\ & - \frac{p_0{}^{\alpha}}{\delta_0}\int_{t_2}^{t}\Big[-\frac{\partial}{\partial s}H(t,s) - \frac{\rho'(s)}{\rho(s)}H(t,s)\Big]\nu(s)\,\mathrm{d}s\\ & - \frac{p_0{}^{\alpha}}{\delta_0}\int_{t_2}^{t} H(t,s)\frac{\alpha\beta_1(\tau(s),t_1)\tau'(s)}{\rho^{1/\alpha}(s)}\nu^{(\alpha+1)/\alpha}(s)\,\mathrm{d}s\,.\end{aligned}$$

Then

$$\int_{t_2}^{t} H(t,s)\rho(s)\frac{Q(s)}{2^{\alpha-1}}\mathrm{d}s \le H(t,t_2)\omega(t_2) + \frac{{p_0}^{\alpha}}{\delta_0}H(t,t_2)\nu(t_2)$$

$$+\int_{t_2}^{t}\Big[\frac{h_-(t,s)(H(t,s))^{\alpha/(\alpha+1)}}{\rho(s)}\omega(s) - H(t,s)\frac{\alpha\beta_1(\tau(s),t_1)\tau'(s)}{\rho^{1/\alpha}(s)}\omega^{(\alpha+1)/\alpha}(s)\Big]\,\mathrm{d}s$$

$$+\frac{{p_0}^{\alpha}}{\delta_0}\int_{t_2}^{t}\Big[\frac{h_-(t,s)(H(t,s))^{\alpha/(\alpha+1)}}{\rho(s)}\nu(s) - H(t,s)\frac{\alpha\beta_1(\tau(s),t_1)\tau'(s)}{\rho^{1/\alpha}(s)}\nu^{(\alpha+1)/\alpha}(s)\Big]\,\mathrm{d}s\,.$$

Using the above inequality and the inequality (2.19), we get

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t}\Big[H(t,s)\rho(s)\frac{Q(s)}{2^{\alpha-1}} - \frac{(1+\frac{{p_0}^{\alpha}}{\delta_0})}{(\alpha+1)^{\alpha+1}}\frac{(h_-(t,s))^{\alpha+1}}{(\rho(s)\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\Big]\mathrm{d}s$$
$$\le \omega(t_2) + \frac{{p_0}^{\alpha}}{\delta_0}\nu(t_2)\,,$$

which contradicts (3.2). The proof is complete. □

From Theorem 2, similar to the proof of Theorem 9, we derive the following result.

Theorem 10. *Let* $0 < \alpha \le 1$, $\tau(t) \in C^1([t_0,\infty))$ *and* $\tau'(t) > 0$. *Assume that* (2.3) *holds and* $\tau(t) \le \delta(t)$. *Moreover, assume that* $H \in C(\mathbb{D},\mathbb{R})$ *has the property* P *and there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 \ge t_0$, *there is a* $t_2 > t_1$ *such that* (3.1) *holds and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t}F_1(t,s)\,\mathrm{d}s = \infty\,, \tag{3.4}$$

where

$$F_1(t,s) := H(t,s)\rho(s)Q(s) - \frac{(1+\frac{{p_0}^{\alpha}}{\delta_0})}{(\alpha+1)^{\alpha+1}}\frac{(h_-(t,s))^{\alpha+1}}{(\rho(s)\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\,,$$

$h_-(t,s) := \max\{0,-h(t,s)\}$. *Then* (E) *is almost oscillatory.*

From (2.7) and (2.29) in Theorem 3, similar to the proof of Theorem 9, we obtain the following criterion.

Theorem 11. *Let* $\alpha \ge 1$, $\tau(t) \in C^1([t_0,\infty))$ *and* $\tau'(t) > 0$. *Assume that* (2.3) *holds and* $\tau(t) \le \delta(t)$. *Furthermore, assume that* $H \in C(\mathbb{D},\mathbb{R})$ *has the property* P *and there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 \ge t_0$, *there is a* $t_2 > t_1$ *such that*

$$-\frac{\partial}{\partial s}H(t,s) - \frac{\rho'(s)}{\rho(s)}H(t,s) = \frac{h(t,s)(H(t,s))^{1/2}}{\rho(s)}, \quad (t,s) \in \mathbb{D}_0\,, \tag{3.5}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t}G_2(t,s)\,\mathrm{d}s = \infty\,, \tag{3.6}$$

where

$$G_2(t,s) := H(t,s)\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{(1+\frac{p_0{}^{\alpha}}{\delta_0})}{4\alpha}\frac{(h_-(t,s))^2}{\rho(s)\left(\beta_2(\tau(s),t_1)\right)^{\alpha-1}\beta_1(\tau(s),t_1)\tau'(s)}\,,$$

$h_-(t,s) := \max\{0, -h(t,s)\}$. *Then* (E) *is almost oscillatory.*

By Theorem 4, similar to the proof of Theorem 9, we obtain the following criterion.

Theorem 12. *Let* $0 < \alpha \le 1$, $\tau(t) \in C^1([t_0,\infty))$ *and* $\tau'(t) > 0$. *Assume that* (2.3) *holds and* $\tau(t) \le \delta(t)$. *Furthermore, assume that* $H \in C(\mathbb{D},\mathbb{R})$ *has the property* P *and there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 \ge t_0$, *there is a* $t_2 > t_1$ *such that* (3.5) *holds and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t} F_2(t,s)\,\mathrm{d}s = \infty\,, \tag{3.7}$$

where

$$F_2(t,s) := H(t,s)\rho(s)Q(s) - \frac{(1+\frac{p_0{}^{\alpha}}{\delta_0})}{4\alpha}\frac{(h_-(t,s))^2}{\rho(s)\left(\beta_2(\tau(s),t_1)\right)^{\alpha-1}\beta_1(\tau(s),t_1)\tau'(s)}\,,$$

$h_-(t,s) := \max\{0, -h(t,s)\}$. *Then* (E) *is almost oscillatory.*

By (2.40) in Theorem 5, similar to the proof of that of Theorem 9, we establish the following result.

Theorem 13. *Assume that* (2.3) *holds,* $\alpha \ge 1$ *and* $\tau(t) \ge \delta(t)$. *Moreover, assume that* $H \in C(\mathbb{D},\mathbb{R})$ *has the property* P *and there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 \ge t_0$, *there is a* $t_2 > t_1$ *such that* (3.1) *holds and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t} G_3(t,s)\,\mathrm{d}s = \infty\,, \tag{3.8}$$

where

$$G_3(t,s) := H(t,s)\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{(1+\frac{p_0{}^{\alpha}}{\delta_0})}{(\alpha+1)^{\alpha+1}}\frac{(h_-(t,s))^{\alpha+1}}{(\rho(s)\beta_1(\delta(s),t_1)\delta'(s))^{\alpha}}\,,$$

$h_-(t,s) := \max\{0, -h(t,s)\}$. *Then* (E) *is almost oscillatory.*

By Theorem 6, similar to the proof of that of Theorem 9, we establish the following result.

Theorem 14. *Assume that* (2.3) *holds,* $0 < \alpha \le 1$ *and* $\tau(t) \ge \delta(t)$. *Moreover, assume that* $H \in C(\mathbb{D},\mathbb{R})$ *has the property* P *and there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 \ge t_0$, *there is a* $t_2 > t_1$ *such that* (3.1) *holds and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_2)}\int_{t_2}^{t} F_3(t,s)\,\mathrm{d}s = \infty\,, \tag{3.9}$$

where

$$F_3(t,s) := H(t,s)\rho(s)Q(s) - \frac{(1+\frac{p_0{}^\alpha}{\delta_0})}{(\alpha+1)^{\alpha+1}} \frac{(h_-(t,s))^{\alpha+1}}{(\rho(s)\beta_1(\delta(s),t_1)\delta'(s))^\alpha},$$

$h_-(t,s) := \max\{0, -h(t,s)\}$. *Then* (E) *is almost oscillatory.*

From Theorem 7, similar to the proof of that of Theorem 9, we derive the following result.

Theorem 15. *Assume that* (2.3) *holds,* $\alpha \geq 1$ *and* $\tau(t) \geq \delta(t)$. *Furthermore, assume that* $H \in C(\mathbb{D}, \mathbb{R})$ *has the property* P *and there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 \geq t_0$, *there is a* $t_2 > t_1$ *such that* (3.5) *holds and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^{t} G_4(t,s)\,\mathrm{d}s = \infty, \tag{3.10}$$

where

$$G_4(t,s) := H(t,s)\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{(1+\frac{p_0{}^\alpha}{\delta_0})}{4\alpha} \frac{(h_-(t,s))^2}{\rho(s)\left(\beta_2(\delta(s),t_1)\right)^{\alpha-1}\beta_1(\delta(s),t_1)\delta'(s)},$$

$h_-(t,s) := \max\{0, -h(t,s)\}$. *Then* (E) *is almost oscillatory.*

From Theorem 8, similar to the proof of that of Theorem 9, we give the following result.

Theorem 16. *Assume that* (2.3) *holds,* $0 < \alpha \leq 1$ *and* $\tau(t) \geq \delta(t)$. *Furthermore, assume that* $H \in C(\mathbb{D}, \mathbb{R})$ *has the property* P *and there exists a function* $\rho \in C^1([t_0,\infty),(0,\infty))$, *for all sufficiently large* $t_1 \geq t_0$, *there is a* $t_2 > t_1$ *such that* (3.5) *holds and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_2)} \int_{t_2}^{t} F_4(t,s)\,\mathrm{d}s = \infty, \tag{3.11}$$

where

$$F_4(t,s) := H(t,s)\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{(1+\frac{p_0{}^\alpha}{\delta_0})}{4\alpha} \frac{(h_-(t,s))^2}{\rho(s)\left(\beta_2(\delta(s),t_1)\right)^{\alpha-1}\beta_1(\delta(s),t_1)\delta'(s)},$$

$h_-(t,s) := \max\{0, -h(t,s)\}$. *Then* (E) *is almost oscillatory.*

Remark 4. From Theorems 9-16, we can obtain some oscillation criteria for (E) with different choices of ρ and H.

4. Examples

In this section, we will give two examples to illustrate our main results.

Example 1. Consider the third-order quasi-linear neutral differential equation

$$\left[t\left(\left[x(t)+p_0x\left(\frac{t}{2}\right)\right]''\right)^3\right]' + \frac{\lambda}{t^6}x^3\left(\frac{t}{2}\right) = 0, \quad \lambda > 0. \tag{4.1}$$

Let $a(t) = t$, $p(t) = p_0 > 0$, $\tau(t) = \delta(t) = t/2$, $\alpha = 3$, $q(t) = \lambda/t^6$ and $\delta_0 = 1/2$. Then, we have $Q(t) = q(t) = \lambda/t^6$ and

$$\beta_1(t,t_1) := \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)}\,\mathrm{d}s = \int_{t_1}^{t} \frac{1}{s^{1/3}}\,\mathrm{d}s \geq t^{2/3}\,,$$

for t sufficiently large. It is easy to see that (2.3) holds. Set $\rho(t) = t^5$. We obtain

$$\limsup_{t\to\infty} \int_{t_2}^{t} \Big[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{(1+\frac{{p_0}^{\alpha}}{\delta_0})}{(\alpha+1)^{\alpha+1}} \frac{((\rho'(s))_+)^{\alpha+1}}{(\rho(s)\beta_1(\tau(s),t_1)\tau'(s))^{\alpha}}\Big]\,\mathrm{d}s$$
$$\geq \Big[\frac{\lambda}{4} - \frac{5^4\cdot 2^5\cdot(1+2{p_0}^3)}{4^4}\Big] \limsup_{t\to\infty} \int_{t_2}^{t} \frac{1}{s}\,\mathrm{d}s = \infty\,,$$

if

$$\lambda > \frac{5^4\cdot 2^5\cdot(1+2{p_0}^3)}{4^3}\,.$$

Hence by Theorem 1, (4.1) is almost oscillatory when

$$\lambda > \frac{5^4\cdot 2^5\cdot(1+2{p_0}^3)}{4^3}\,.$$

When $p_0 = 1/3$,

$$\frac{5^4\cdot 2^5\cdot(1+2{p_0}^3)}{4^3} < \frac{9^3}{2}\,.$$

Thus, our results improve results of [2]; see [2, Example 1].

Also, since $\beta_2(t,t_1) \geq \frac{2}{5}t^{\frac{5}{3}}$, we have

$$\limsup_{t\to\infty} \int_{t_2}^{t} \Big[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{(1+\frac{{p_0}^{\alpha}}{\delta_0})}{4\alpha} \frac{((\rho'(s))_+)^{2}}{(\rho(s)\beta_2(\tau(s),t_1))^{\alpha-1}\beta_1(\tau(s),t_1)\tau'(s)}\Big]\,\mathrm{d}s$$
$$\geq \Big[\frac{\lambda}{4} - \frac{5^4\cdot 2^5(1+2p_0^3)}{3\cdot 4^2}\Big] \lim_{t\to\infty} \int_{t_1}^{t} \frac{1}{s}\,\mathrm{d}s = \infty$$

if $\lambda > \frac{5^4\cdot 2^5(1+2p_0^3)}{12}$. Hence by Theorem 3, equation(4.1) is almost oscillatory when $\lambda > \frac{5^4\cdot 2^5(1+2p_0^3)}{12}$. However for this example Theorem 1 is better than Theorem 3.

Example 2. Consider the third-order quasi-linear neutral differential equation

$$\Big[t\Big(\Big[x(t) + p_0 x\Big(\frac{t}{2}\Big)\Big]''\Big)^3\Big]' + \frac{\lambda}{t^6}x^3\Big(\frac{3t}{2}\Big) = 0\,, \quad \lambda > 0\,. \tag{4.2}$$

Let $a(t) = t$, $p(t) = p_0 > 0$, $\tau(t) = 3t/2$, $\delta(t) = t/2$, $\alpha = 3$ and $q(t) = \lambda/t^6$. Then, we have $\delta'(t) = \delta_0 = 1/2$, $Q(t) = q(t) = \lambda/t^6$,

$$\beta_1(t,t_1) := \int_{t_1}^{t} \frac{1}{a^{1/\alpha}(s)}\,\mathrm{d}s = \int_{t_1}^{t} \frac{1}{s^{1/3}}\,\mathrm{d}s \geq t^{2/3}\,,$$

and

$$\beta_2(t,t_1) := \int_{t_1}^{t}\int_{t_1}^{s} \frac{1}{a^{1/\alpha}(u)}\,\mathrm{d}u\,\mathrm{d}s \geq \frac{2}{5}t^{5/3}\,,$$

for t sufficiently large. It is easy to verify that (2.3) holds. Set $\rho(t) = t^5$. We get

$$\limsup_{t\to\infty}\int_{t_2}^{t}\left[\rho(s)\frac{Q(s)}{2^{\alpha-1}}-\frac{(1+\frac{p_0{}^{\alpha}}{\delta_0})}{4\alpha}\frac{((\rho'(s))_+)^2}{\rho(s)\left(\beta_2(\delta(s),t_1)\right)^{\alpha-1}\beta_1(\delta(s),t_1)\delta'(s)}\right]\mathrm{d}s$$
$$\geq\left[\frac{\lambda}{4}-\frac{\left(\frac{25}{2}\right)^2\cdot 2^5\cdot(1+2p_0{}^3)}{12}\right]\limsup_{t\to\infty}\int_{t_2}^{t}\frac{1}{s}\,\mathrm{d}s=\infty\,,$$

if

$$\lambda>\frac{\left(\frac{25}{2}\right)^2\cdot 2^5\cdot(1+2p_0{}^3)}{3}\,.$$

Hence by Theorem 7, (4.2) is almost oscillatory when

$$\lambda>\frac{\left(\frac{25}{2}\right)^2\cdot 2^5\cdot(1+2p_0{}^3)}{3}\,.$$

Also, we have

$$\lim_{t\to\infty}\sup\int_{t_2}^{t}\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\frac{(1+\frac{p_0{}^{\alpha}}{\delta_0})}{(\alpha+1)^{\alpha+1}}\frac{((\rho'(s))_+)^{\alpha+1}}{(\rho(s)\beta_1(\delta(s),t_1)\delta'(s))^{\alpha}}\right]\mathrm{d}s$$
$$\geq\left[\frac{\lambda}{4}-\frac{5^4\cdot 2^5(1+2p_0^3)}{4^4}\right]\lim_{t\to\infty}\int_{t_2}^{t}\frac{1}{s}\,\mathrm{d}s=\infty$$

if $\lambda > \frac{5^4\cdot 2^5(1+2p_0^3)}{4^3}$. Hence by Theorem 5, equation(4.2) is almost oscillatory when $\lambda > \frac{5^4\cdot 2^5(1+2p_0^3)}{4^3}$. For this example Theorem 5 is better than Theorem 7.

5. Conclusions

In this paper, we have established some new oscillation theorem for (E) for the case when $0 \leq p(t) \leq p_0 < \infty$. Our results can be applied when $\tau(t) \geq t$; see Theorems 5, 6, 7, 8, 13, 14, 15, 16. It would be interesting to study (E) under the cases when $p(t) < -1$, $\lim_{t\to\infty} p(t) = \infty$ or $p(t)$ is an oscillatory function. Moreover, it is interesting to find another method to remove the restrictions $\tau\circ\delta = \delta\circ\tau$ and $\delta(t) \leq t$.

References

[1] Baculíková, B., Džurina, J., *Oscillation of third-order functional differential equations*, Electron. J. Qual. Theory Differ. Equ. **43** (2010), 1–10.

[2] Baculíková, B., Džurina, J., *Oscillation of third-order neutral differential equations*, Math. Comput. Modelling **52** (2010), 215–226.

[3] Baculíková, B., Džurina, J., *Oscillation of third-order nonlinear differential equations*, Appl. Math. Lett. **24** (2011), 466–470.

[4] Baculíková, B., Elabbasy, E. M., Saker, S. H., Džurina, J., *Oscillation criteria for third-order nonlinear differential equations*, Math. Slovaca **58** (2008), 1–20.

[5] Erbe, L., *Existence of oscillatory solutions and asymptotic behavior for a class of third order linear differential equations*, Pacific J. Math. **64** (1976), 369–385.

[6] Grace, S. R., Agarwal, R. P., Pavani, R., Thandapani, E., *On the oscillation of certain thir d order nonlinear functional differential equations*, Appl. Math. Comput. **202** (2008), 102–112.

[7] Han, Z., Li, T., Zhang, C., Sun, S., *An oscillation criterion for third order neutral delay differential equations*, J. Appl. Anal., to appear.

[8] Hanan, M., *Oscillation criteria for third order differential equations*, Pacific J. Math. **11** (1961), 919–944.

[9] Hartman, P., Winter, A., *Linear differential and difference equations with monotone solutions*, Amer. J. Math. **75** (1953), 731–743.

[10] Karpuz, B., Öcalan, Ö., Öztürk, S., *Comparison theorems on the oscillation and asymptotic behavior of higher-order neutral differential equations*, Glasgow Math. J. **52** (2010), 107–114.

[11] Philos, Ch. G., *Oscillation theorems for linear differential equations of second order*, Arch. Math. **53** (1989), 482–492.

[12] Saker, S. H., Džurina, J., *On the oscillation of certain class of third-order nonlinear delay differential equations*, Math. Bohem. **135** (2010), 225–237.

[13] Zhong, J., Ouyang, Z., Zou, S., *Oscillation criteria for a class of third-order nonlinear neutral differential equations*, J. Appl. Anal. (2010), to appear.

Ramanujan Institute for Advanced Study in Mathematics,
University of Madras,
Chennai, India
E-mail: ethandapani@yahoo.co.in

School of Control Science and Engineering,
Shandong University,
Jinan, Shandong 250061, P. R. China
E-mail: litongx2007@163.com