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ON SOME CONSEQUENCES OF A GENERALIZED CONTINUITY

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Abstract. In normed linear space settings, modifying the sequential definition of continuity of an operator by replacing the usual limit "lim" with arbitrary linear regular summability methods $\mathbf{G}$ we consider the notion of a generalized continuity ($(\mathbf{G_1}, \mathbf{G_2})$-continuity) and examine some of its consequences in respect of usual continuity and linearity of the operators between two normed linear spaces.

1. Introduction

One of the most fundamental and useful properties of a linear mapping on a normed linear space is that the continuity of the mapping on the whole space follows from the continuity of the mapping at one point. Generally non-linear mappings do not have this property. In this note we show that the above result can be actually obtained for non-linear mappings under certain conditions if we modify the sequential definition of continuity. For this purpose we introduce the notion of a generalized continuity (called $(\mathbf{G_1}, \mathbf{G_2})$-continuity), whose inspiration comes from [10]. We also investigate when $(\mathbf{G_1}, \mathbf{G_2})$-continuity implies linearity of the mapping.

n classical analysis, such an approach started from an American Mathematical Monthly problem by Robbins [16] in 1946. As an answer, Buck [5] in 1948 showed that if one replaces the usual convergence in the sequential definition of real functions by Cesaro summability, then the function (called Cesaro continuous) has to be linear, i.e. of the form $ax + b$ and so is also continuous on the whole of $\mathbb{R}$. The linearity and continuity actually follows if the function is Cesaro continuous at one point only. Subsequently general results in this direction were obtained in [1]–[4, 12, 17]–[19] where impacts on continuity and linearity of real functions were studied by replacing usual convergence by different summability methods like convergence in arithmatic mean, A-summability (convergence by a regular summability matrix A), strong matrix summability, almost convergence (using the notion of Banach limits), statistical convergence, μ-statistical convergence. The

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whole spectrum of definitions and results and many relevant references can be seen from the beautifully written paper by Connor and Grosse-Erdmann [10]. In particular in [10], the authors introduced a very general approach by replacing usual convergence by an arbitrary linear functional **G** defined on a suitable class of sequences and named the corresponding notion of continuity as **G**-continuity. However it should be noted that all the above mentioned work was done for real functions and results were proved by using the intrinsic properties of real number system. Very recently a lot of work has been done using an arbitrary linear functional **G** in [6]–[8] where applications to continuity, compactness and connectedness were studied.

2. Main results

Before we can begin, it is necessary to introduce some definitions and notations. By X we will mean a normed linear space. We will use boldface letters $\mathbf{x}, \mathbf{y}, \mathbf{z} \dots$ for sequences $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}} \dots$ in X. If $f\colon X \to X$ is a mapping then we define $f(\mathbf{x}) = \{f(x_n)\}_{n\in\mathbb{N}}$. By a method of sequential convergence, or briefly a method, we mean a linear mapping $\mathbf{G}$ from a set $\mathbf{c_G}$ of X-valued sequences to X. A sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ is said to be $\mathbf{G}$-convergent to $u \in X$ if $\mathbf{x} \in \mathbf{c_G}$ and $\mathbf{G}(\mathbf{x}) = u$. In particular, lim denotes the limit mapping $\lim \mathbf{x} = \lim_n x_n$ on the set $\mathbf{c}$ of all convergent sequences. A method $\mathbf{G}$ is called regular if every convergent sequence $\mathbf{x}$ in X is also $\mathbf{G}$-convergent with $\mathbf{G}(\mathbf{x}) = \lim \mathbf{x}$. It should be noted that all the summability methods stated above including statistical convergence, ideal convergence and Cesaro summability are regular sequential methods. We are now ready to introduce our main definition.

Definition 2.1. Let X and Y be two normed linear spaces and let $f\colon X \to Y$ be a mapping. Let $\mathbf{G_1}$ and $\mathbf{G_2}$ be regular linear summability methods on X and Y respectively. Then f will be called $(\mathbf{G_1}, \mathbf{G_2})$-continuous at $u \in X$provided that whenever any sequence $\mathbf{x}= \{x_n\}_{n\in\mathbb{N}}$ in X is $\mathbf{G_1}$-convergent to $u \in X$, $f(\mathbf{x})= \{f(x_n)\}_{n\in\mathbb{N}}$ is $\mathbf{G_2}$-convergent to $f(u)$ (which we briefly express as $\mathbf{G_2}(f(\mathbf{x}))= f(\mathbf{G_1}(\mathbf{x}))$.

Clearly any linear mapping $f\colon X \to Y$ is $(\mathbf{G_1}, \mathbf{G_2})$-continuous on whole X if it is so at a point.

We first construct an example to show that $(\mathbf{G_1}, \mathbf{G_2})$-continuity is independent of the notion of continuity as well as linearity.

Example 2.2. Let X be a normed space. There exists a regular method $\mathbf{G}$ and a nonlinear mapping $f\colon X \to X$ which is $(\mathbf{G}, \mathbf{G})$-continuous at θ but not continuous at θ where θ is the null element of X.

Proof. This example is modeled after Example 3 in [10]. Choose a non-zero element v_0 of X with $\|v_0\| = 1$. Now construct a sequence $\{v_n\}_{n\in\mathbb{N}}$ where $v_n = (n+1)v_{n-1}$ $\forall n = 1, 2, 3, \dots$. Then $\|v_n\| \geq n+1 \forall n$ and so $\|v_n\| \to \infty$ as $n \to \infty$. We now define $\mathbf{c_G}$ and $\mathbf{G}$. Let Y be the set of sequences $\mathbf{y} = \{y_n\}_{n\in\mathbb{N}}$ with $y_n \in \{\theta\} \cup \{v_k : k \in \mathbb{N}\}$ for each n such that the non-zero y_n's have $\|y_n\| \to \infty$ as $n \to \infty$. Let W be the linear span of Y. We now define $\mathbf{c_G} = \mathbf{c} + W$ where $\mathbf{c}$ is the set of all convergent sequences in X and $\mathbf{G}\colon \mathbf{c_G} \to \mathbf{X}$ by $\mathbf{G}(\mathbf{x}) = \lim \mathbf{z}$ if $\mathbf{x} = \mathbf{z} + \mathbf{w}$ with $\mathbf{z} \in \mathbf{c}$ and $\mathbf{w} \in \mathbf{W}$.

We first need to show that $\mathbf{G}$ is well-defined. For this it is sufficient to prove that $\mathbf{c_G}$ is the direct sum of $\mathbf{c}$ and W which will immediately follow if we can show that $\mathbf{c_G} \cap W = \{\{\theta\}_{n\in\mathbb{N}}\}$ sequence consisting of the zero element θ only. For this we now show that if $\mathbf{w} \in \mathbf{W}$ does not consist of θ only then the norm of the non-zero elements of $\mathbf{w}$ tends to infinity. Let $\mathbf{w} = \sum\limits_{p=1}^{M} a_p \mathbf{y^p} \neq \{\theta\}_{n\in\mathbb{N}}$ with $a_p \in R$ and $\mathbf{y^p} \in \mathbf{Y}$. Let $B = \max\limits_p |a_p|$ and let

$$b = \min\left\{\Big|\sum_{p\in H} a_p\Big| : H \subset \{1, 2, \dots, M\}, \sum_{p\in H} a_p \neq 0\right\}.$$

Clearly $b > 0$. Now from the construction of Y, we can choose a sequence K_n of positive integers with $K_n \to \infty$ such that for $p = 1, 2, \dots, M$ and $n \in \mathbb{N}$,

$$y_n^p = \theta \quad \text{or} \quad \|y_n^p\| \geq \|v_{K_n}\|\,.$$

Now for $n \in \mathbb{N}$, we can write

$$w_n = \sum_{j=1}^{\infty}\Big(\sum_{y_n^p = v_j} a_p\Big) v_j$$

where only finitely many terms are non-zero. If $w_n \neq \theta$, we must have

$$\sum_{y_n^p = v_j} a_p \neq 0$$

for some j. Then if $R = R(n)$ be the largest such j clearly we must have $R \geq K_n$. Hence

$$\begin{aligned}
\|w_n\| &= \Big\|\Big(\sum_{y_n^p = v_R} a_p\Big) v_R - \sum_{y_n^p \neq v_R} a_p y_n^p\Big\| \\
&\geq b \cdot \|v_R\| - B \cdot M \cdot \|v_{R-1}\| \\
&= \Big(b \cdot \frac{\|v_R\|}{\|v_{R-1}\|} - B \cdot M\Big)\|v_{R-1}\| \\
&= (b \cdot (R+1) - B \cdot M)\|v_{R-1}\| \\
&\geq (b \cdot (K_n+1) - B \cdot M)\|v_{K_n - 1}\| \quad \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty\,.
\end{aligned}$$

From above we can conclude that $\mathbf{c_G} = \mathbf{c} + W$ is the direct sum of $\mathbf{c}$ and W and so $\mathbf{G}$ is well-defined. Evidently $\mathbf{G}$ is also regular and linear.

Now we define $f\colon X \to X$ by

$$f(u) = v_k \quad \text{if} \quad u = \frac{v_0}{k} \quad = \theta \quad \text{otherwise.}$$

Since $\frac{v_0}{k} \to \theta$ whereas $f\big(\frac{v_0}{k}\big)$ is not a convergent sequence (since it is unbounded), so f can not be continuous at θ. We show that f is $(\mathbf{G}, \mathbf{G})$-continuous at θ. For this let $\mathbf{x} = \mathbf{z} + \mathbf{w} \in c_G$ with $\mathbf{G}(\mathbf{x}) = \lim \mathbf{z} = \theta$. Now $f(x_n) \in \{\theta\} \cup \{v_k : k \in \mathbb{N}\} \forall n$.

Further $f(x_n) \neq \theta$ only if $x_n = z_n + w_n \in \left\{\frac{v_0}{k} : k \in \mathbb{N}\right\}$ which for sufficiently large n implies that $w_n = \theta$ and $z_n \in \left\{\frac{v_0}{k} : k \in \mathbb{N}\right\}$. Hence $f(\mathbf{x})$ either has finitely many non-zero elements or if it has infinitely many non-zero elements their norm tend to infinity. In any case we have $f(\mathbf{x}) \in \mathbf{c_G}$ and $\mathbf{G}(\mathbf{x}) = \theta$. This shows that f is $(\mathbf{G}, \mathbf{G})$-continuous at θ. It is easy to verify that the mapping f defined above is not linear. □

In the remaining part of this note we try to establish certain sufficient conditions under which $(\mathbf{G_1}, \mathbf{G_2})$-continuity of a function at a point will ensure its continuity on the whole space and even its linearity.

We now introduce the following property of a method $\mathbf{G}$ which will again be needed for linearity.

(NL1) There is a sequence $\mathbf{t} = \{t_n\}_{n\in\mathbb{N}}$ consisting of 0 and 2 such that for any $u \in X$ the sequence $\mathbf{t}u = \{t_n u\}_{n\in\mathbb{N}}$ is $\mathbf{G}$-convergent with $\mathbf{G}(\mathbf{t}u) = u$.

We also assume that $f\colon X \to Y$ is a mapping such that

$(*)$ for every point u_0 of discontinuity of f, there is a $\alpha \neq \theta$ and two sequences $\mathbf{x}$ and $\mathbf{y}$ with $x_n \to u_0$, $y_n \to u_0$ and $f(x_n) - f(y_n) = \alpha \forall n$.

Theorem 2.3. *Let $\mathbf{G_1}$ and $\mathbf{G_2}$ be regular linear methods satisfying the property* (NL1) *with respect to the same sequence $\mathbf{t} = \{t_n\}_{n\in\mathbb{N}}$ on X and Y respectively. Then every mapping $f\colon X \to Y$ satisfying the above property $(*)$ and for which $f(2u) = 2f(u) \forall u \in X$, $(\mathbf{G_1}, \mathbf{G_2})$-continuity at one point of X implies continuity in the usual sense on whole of X.*

Proof. Suppose that f is $(\mathbf{G_1}, \mathbf{G_2})$-continuous at $v_0 \in X$. Let $u_0 \in X$ be any arbitrary point. We primarily show that for all sequences $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ with $x_n \to u_0$ the sequences $f(\mathbf{x})$ are $\mathbf{G_2}$-convergent to a fixed element of Y. By (NL1) there is a sequence $\mathbf{t} = \{t_n\}_{n\in\mathbb{N}}$ consisting of 0 and 2 such that for any $u \in X$ the sequence $\mathbf{t}u = \{t_n u\}_{n\in\mathbb{N}}$ is $\mathbf{G_1}$-convergent with $\mathbf{G_1}(\mathbf{t}u) = u$. We construct the sequences $\mathbf{y}$, $\mathbf{y}'$ as follows.

$$y_n = t_n a_0 + (2 - t_n) x_n$$

and

$$y'_n = t_n x_n + (2 - t_n) b_0\,,$$

where $a_0 = b_0 = (v_0 - u_0)$. Now observe that $\{t_n x_n - t_n u_0\}_{n\in\mathbb{N}}$ is actually convergent to θ. Hence by linearity and regularity of $\mathbf{G_1}$ this sequence is also $\mathbf{G_1}$-convergent to θ which consequently implies that $\mathbf{G_1}(t_n x_n) = \mathbf{G_1}(\mathbf{t}u_0) = u_0$. So we can conclude that $\mathbf{y}$ and $\mathbf{y}'$ are also $\mathbf{G_1}$-convergent with

$$\mathbf{G_1}(\mathbf{y}) = a_0 + 2u_0 - u_0 = v_0$$

$$\mathbf{G_1}(\mathbf{y}') = u_0 + 2b_0 - b_0 = v_0\,.$$

Let us define the sequences $\mathbf{w}$ and $\mathbf{w}'$ by

$$w_n = t_n f(a_0) + (2 - t_n) f(x_n)\,,$$

and

$$w'_n = t_n f(x_n) + (2 - t_n) f(b_0)\,.$$

Note that $f(y_n) = f(2a_0) = 2f(a_0) = w_n$ when $t_n = 2$ and also $f(y_n) = f(2x_n) = 2f(x_n) = w_n$ when $t_n = 0$. Thus $w_n = f(y_n)$ for all $n \in \mathbb{N}$. Similar arguments show that $w'_n = f(y'_n)$ for all $n \in \mathbb{N}$. Since both $\mathbf{y}$ and $\mathbf{y}'$ are $\mathbf{G_1}$-convergent to v_0 so $(\mathbf{G_1}, \mathbf{G_2})$-continuity of f at v_0 implies that both $\mathbf{w}$ and $\mathbf{w}'$ are $\mathbf{G_2}$-convergent to $f(v_0)$. Hence the sequence $\mathbf{w} + \mathbf{w}'$ is $\mathbf{G_2}$-convergent to $2f(v_0)$ by the regularity of $\mathbf{G_2}$. But

$$\begin{aligned} w_n + w'_n &= 2f(x_n) + t_n f(a_0) + (2 - t_n) f(b_0) \\ &= 2f(x_n) + y''_n \qquad \text{(say).} \end{aligned}$$

Observe that the sequence $\mathbf{y}'' = \{y''_n\}_{n\in\mathbb{N}}$ where $y''_n = t_n f(a_0) + (2 - t_n) f(b_0)$ is $\mathbf{G_2}$-convergent to $f(a_0)+f(b_0)$. Hence $2f(\mathbf{x}) = (2f(x_n))$ must also be $\mathbf{G_2}$-convergent with

$$\mathbf{G_2}(2f(\mathbf{x})) = 2f(v_0) - f(a_0) - f(b_0)\,. \tag{1}$$

Note that this value is same for all sequences $\mathbf{x}$ with $x_n \to u_0$. Further it should be noted that linearity of the method $\mathbf{G_2}$ ensures that $\mathbf{G_2}(f(\mathbf{x})) = \frac{1}{2}\mathbf{G_2}(\mathbf{2f}(\mathbf{x}))$ exists.

Now if f is not continuous at u_0 then u_0 is a point of discontinuity of f and so by $(*)$ there exists a $\alpha \neq \theta$ and two sequences $\mathbf{x}$ and $\mathbf{y}$ with $x_n \to u_0$, $y_n \to u_0$ and $f(x_n) - f(y_n) = \alpha \forall n$. But using the regularity and linearity of $\mathbf{G_2}$, we can conclude that $\mathbf{G_2}(2f(\mathbf{x})) - \mathbf{G_2}(2f(bfy)) = 2\alpha \neq \theta$ which contradicts (1). Thus f can not have any discontinuity and so f must be continuous on whole of X. $\square$

Remark 2.4. Note that in the above theorem f was not assumed to be linear. It is easy to give examples of methods $\mathbf{G}$ satisfying the condition (NL1). For example if $\mathbf{G}$ is the Cesaro summability then the sequence $(2, 0, 2, 0, 2, 0, \dots)$ is the required sequence $\mathbf{t} = \{t_n\}_{n\in\mathbb{N}}$ with which $\mathbf{G}$ satisfies (NL1). Also one may check that taking a more general version of the condition (NL1) where $\mathbf{G}(\mathbf{t}u) = u$ is replaced by $\mathbf{G}(\mathbf{t}u) = \beta u$, $0 < \beta < 2$, one can prove the result in the same way by taking $a_0 = (v_0 - (2 - \beta)u_0)/\beta$ and $b_0 = (v_0 - \beta u_0)/(2 - \beta)$.

Remark 2.5. We would also like to point out that in the definition of the condition (NL1) if the number 2 is replaced by a positive integer k then it seems that Theorem 2.3 will work for mappings $f\colon X \to Y$ satisfying the condition that $f(ku) = kf(u)$ for all $u \in X$. In particular for $k = 1$ the condition (NL1) can be rewritten as

(NL1) There is a sequence $\mathbf{t} = \{t_n\}_{n\in\mathbb{N}}$ consisting of 0 and 1 such that for any $u \in X$ the sequence $\mathbf{t}u = \{t_n u\}_{n\in\mathbb{N}}$ is $\mathbf{G}$-convergent with $\mathbf{G}(\mathbf{t}u) = \beta u$, $0 < \beta < 1$ which is same as condition (L1) of [10] and one can easily verify that Theorem 2.3 will work for any mapping $f\colon X \to Y$ without any additional assumption.

Theorem 2.6. *Let $\mathbf{G_1}$ and $\mathbf{G_2}$ be regular linear methods satisfying the property* (NL1) *with respect to the same sequence $\mathbf{t} = \{t_n\}_{n\in\mathbb{N}}$ on X and Y respectively. Then every homogeneous mapping $f\colon X \to Y$ that is $(\mathbf{G_1}, \mathbf{G_2})$-continuous is additive and so linear.*

Proof. Let $f\colon X \to Y$ be a homogeneous $(\mathbf{G_1}, \mathbf{G_2})$-continuous mapping and let $u, v \in X$. By (NL1) there is a sequence $\mathbf{t} = \{t_n\}_{n\in\mathbb{N}}$ consisting of 0 and 2 such that for any $x \in X$ the sequence $\mathbf{t}x = \{t_n x\}_{n\in\mathbb{N}}$ is $\mathbf{G_1}$-convergent with $\mathbf{G_1}(\mathbf{t}x) = x$. We define $\mathbf{w} = \{w_n\}_{n\in\mathbb{N}}$ where

$$w_n = t_n u + (2 - t_n)v\,,$$

which then is either $2u$ or $2v$ according as $t_n = 2$ or 0. Since $\mathbf{G_1}$ is linear and regular, so the sequence $\{w_n\}_{n\in\mathbb{N}}$ is $\mathbf{G_1}$-convergent with $\mathbf{G_1}(\mathbf{w}) = u + v$. Again $f(w_n)$ is equal to $f(2u) = 2f(u)$ if $t_n = 2$ and equal to $f(2v) = 2f(v)$ if $t_n = 0$. Hence $f(w_n) = t_n f(u) + (2 - t_n)f(v)$ and so we have

$$\mathbf{G_2}\big(f(\mathbf{w})\big) = f(u) + f(v)\,.$$

Since f is $(\mathbf{G_1}, \mathbf{G_2})$-continuous, we must have

$$f(u+v) = f\big(\mathbf{G_1}(\mathbf{w})\big) = \mathbf{G_2}\big(f(\mathbf{w})\big) = f(u) + f(v)\,.$$

This proves that f is additive and so is linear. □

Example 2.7. We now show that there exist methods $\mathbf{G}$ which do not satisfy the condition (NL1) but for which homogeneous $(\mathbf{G}, \mathbf{G})$-continuous mappings $f\colon X \to X$ are additive and so linear.

Let $\mathbf{t} = (1, 0, -1, 1, 0, -1, \dots)$ and let $\mathbf{c_G} = \mathbf{c} + \{\mathbf{t}u : u \in X\}$ where we define $\mathbf{G}(\mathbf{x}) = \lim_n x_{3n+2}$ for any $\mathbf{x} \in \mathbf{c_G}$. Then it is easy to verify that $\mathbf{G}$ is a linear regular method which does not satisfy the condition (NL1).

Let $f\colon X \to X$ be a homogeneous $(\mathbf{G}, \mathbf{G})$-continuous mapping. Let $u, v \in X$. Construct a sequence of the form $\mathbf{x} = \mathbf{y} + \mathbf{t}\frac{u-v}{2}$ where $\mathbf{y} = \{y_n\}_{n\in\mathbb{N}}$ is such that $\lim_n y_n = u + v$. Now proceeding as in Example 2 in [10] we can prove that $f(u + v) = f(u) + f(v)$.

In the next result we show that if we strengthen the condition (NL1) further then we can obtain the linearity of homogeneous mappings from the $(\mathbf{G_1}, \mathbf{G_2})$-continuity on the whole space. For this we now introduce the following condition.

(NL2) There are disjoint sequences $\mathbf{t} = \{t_n\}_{n\in\mathbb{N}}$ and $\mathbf{l} = \{l_n\}_{n\in\mathbb{N}}$ consisting of 0 and 2 (i.e. their supports are disjoint) such that for any $z \in X$ the sequence $\mathbf{tz} = \{t_n z\}_{n\in\mathbb{N}}$ is $\mathbf{G}$-convergent with $\mathbf{G}(\mathbf{tz}) = z$ and the sequence $\mathbf{lz} = \{l_n z\}_{n\in\mathbb{N}}$ is $\mathbf{G}$-convergent with $\mathbf{G}(\mathbf{lz}) = \beta z$ where $0 < \beta$ and $1 + \beta \neq 2$.

Theorem 2.8. *Let $\mathbf{G_1}$ and $\mathbf{G_2}$ be regular linear methods satisfying property (NL2) with respect to same sequences $\mathbf{t} = \{t_n\}_{n\in\mathbb{N}}$ and $\mathbf{l} = \{l_n\}_{n\in\mathbb{N}}$ on X and Y respectively. Then every homogeneous mapping $f\colon X \to Y$ that is $(\mathbf{G_1}, \mathbf{G_2})$-continuous at θ is linear.*

Proof. Since f is homogeneous, $f(\theta) = \theta$. By (NL2) there are disjoint sequences $\mathbf{t} = \{t_n\}_{n\in\mathbb{N}}$ and $\mathbf{l} = \{l_n\}_{n\in\mathbb{N}}$ consisting of 0 and 2 with the above property. Define $m_n = 2 - t_n - l_n$. Then $\mathbf{m} = \{m_n\}_{n\in\mathbb{N}}$ is a sequence of 0 and 2 such that for any $z \in X$ the sequence $\mathbf{mz} = \{m_n z\}_{n\in\mathbb{N}}$ is $\mathbf{G_1}$-convergent with $\mathbf{G_1}(\mathbf{mz}) = (2 - \beta - 1)z = \sigma z$ (say). Clearly $\mathbf{t}, \mathbf{l}$ and $\mathbf{m}$ have disjoint supports and $\sigma \neq 0$. Let $u, v \in X$. Now construct the sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$,

$$x_n = -t_n \sigma u - l_n(\sigma/\beta)v + m_n(u + v)\,.$$

Then $\mathbf{x}$ is $\mathbf{G_1}$-convergent with $\mathbf{G_1}(\mathbf{x}) = \theta$. By $(\mathbf{G_1}, \mathbf{G_2})$-continuity of f at θ we then must have $\mathbf{G_2}(f(\mathbf{x})) = f(\theta) = \theta$. As $f(x_n) = t_n f(-\sigma u) + l_n f((\sigma/\beta)v) + m_n f(u+v)$, it follows that

$$\theta = \mathbf{G_2}\big(f(\mathbf{x})\big) = f(-\sigma u) + \beta f\big((-\sigma/\beta)v\big) + \sigma f(u + v)\,. \tag{3}$$

Putting $u = \theta$ and then $v = \theta$ we get from (3), $f(-\sigma u) = -\sigma f(u)$ and $\beta f((-\sigma/\beta)v) = -\sigma f(v)$. Substituting these values in (3) and dividing by σ we finally obtain

$$f(u + v) = f(u) + f(v)\,.$$

This completes the proof of the theorem. □

Remark 2.9. It is easy to verify that there exist methods $\mathbf{G}$ satisfying condition (NL2). For example if $\mathbf{G}$ is the method of Cesaro summability then $(2, 0, 2, 0, 2, 0, \dots)$ and $(0, 2, 0, 0, 0, 2, 0, 0, 0, 2, \dots)$ are two such sequences $\mathbf{t}$ and $\mathbf{l}$ with which $\mathbf{G}$ satisfies condition (NL2). Also it can be noted that the condition (NL2) can be generalized in the same way as observed in Remark 2.4. Also as pointed out in Remark 2.5, the definition of condition (NL2) can also be given in terms of any positive integer k with necessary modifications and Theorem 2.8 can be proved similarly.

Concluding Remark and an Open Problem. It can be noted that summability methods like Cesaro summability, statistical (ideal) convergence (see [14], [13]) and more can be defined in topological spaces and so in general normed linear spaces. As most of these summability methods give rise to regular methods considered here, so the above modification of the notion of continuity to generalized continuity seems very natural. It seems to be a very interesting open problem as to find the necessary and sufficient condition (or at least the least possible condition) under which non-linear mappings will be continuous on the whole space if they are continuous at one point.

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References

[1] Antoni, J., *On the A-continuity of real functions II*, Math. Slovaca **36** (1986), no. 3, 283–287.

[2] Antoni, J., Salat, T., *On the A-continuity of real functions*, Acta Math. Univ. Comenian **39** (1980), 159–164.

[3] Boos, J., *Classical and Modern Methods in Summability*, Oxford Univ. Press, Oxford, 2000.

[4] Borsik, J., Salat, T., *On F -continuity of real functions*, Tatra Mt. Math. Publ. **2** (1993), 37–42.

[5] Buck, R.C., *Solution of problem 4216*, Amer. Math. Monthly **55** (1948), 36.

[6] Cakalli, H., *Sequential definitions of compactness*, Appl.Math. Lett. **21** (2008), no. 6, 594–598.

[7] Cakalli, H., *On G-continuity*, Comput. Math. Appl. **61** (2011), 313–318.

[8] Cakalli, H., *Sequential definitions of connectedness*, Appl. Math. Lett. **25** (2012), 461–465.

[9] Cakalli, H., Das, P., *Fuzzy compactness via summability*, Appl. Math. Lett. **22** (2009), no. 11, 1665–1669.

[10] Connor, J., Grosse-Erdmann, K.-G., *Sequential definitions of continuity for real functions*, Rocky Mountain J. Math. **33** (2003), no. 1, 93–121.

[11] Dik, M., Canak, I., *New types of continuities*, Abstr. Appl. Anal. **2010** (2010), p.6. DOI: 10.1155/2010/258980

[12] Iwinski, T.B., *Some remarks on Toeplitz methods and continuity*, Comment. Math. Prace Mat. **17** (1972), 37–43.

[13] Lahiri, B.K., Das, P., *I and I^* convergence in topological spaces*, Math. Bohemica **130** (2005), no. 2, 153–160.

[14] Maio, G.D., Kocinac, Lj.D.R., *Statistical convergence in topology*, Topology Appl. **156** (2008), 28–45.

[15] Posner, E.C., *Summability preserving functions*, Proc. Amer. Math. Soc. **112** (1961), 73–76.

[16] Robbins, H., *Problem 4216*, Amer. Math. Monthly **53** (1946), 470–471.

[17] Savas, E., Das, G., *On the A-continuity of real functions*, İstanbulÜniv. Fen Fak. Mat. Derg. **53** (1994), 61–66.

[18] Spigel, E., Krupnik, N., *On the A-continuity of real functions*, J. Anal. **2** (1994), 145–155.

[19] Srinivasan, V.K., *An equivalent condition for the continuity of a function*, Texas J. Sci. **32** (1980), 176–177.

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