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UNIQUENESS OF THE STEREOGRAPHIC EMBEDDING

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Abstract. The standard conformal compactification of Euclidean space is the round sphere. We use conformal geodesics to give an elementary proof that this is the only possible conformal compactification.

1. Introduction

At the 34th Winter School on Geometry and Physics, held as always in Srní in the Czech Republic, Charles Francès gave a plenary series of talks on 'Conformal boundaries in pseudo-Riemannian geometry.' He discussed the results contained in [3] and, in particular, the notions of *conformal embedding*

$$(M,[g]) \hookrightarrow^{\mathrm{open}} (N,[h])$$

and *conformal maximality* (where there is no non-trivial conformal embedding). The prototype conformal embedding is

$$(\mathbb{R}^n, [\text{Euclidean metric}]) \hookrightarrow (S^n, [\text{round metric}])$$

where the image is the complement of a single point, which we shall write as ∞, and the mapping in the other direction $S^n \setminus \{\infty\} \to \mathbb{R}^n$ is the standard stereographic projection, which is well-known to be conformal. It is proved in [3] that this is the only non-trivial conformal embedding of Euclidean space. In particular, it is the only possible *conformal compactification* of $\mathbb{R}^n$ as follows.

Theorem 1. *Suppose $n \geq 3$ and (M,g) is a compact Riemannian n-manifold and $\iota\colon \mathbb{R}^n \hookrightarrow M$ realises $\mathbb{R}^n$ as an open subset of M such that $\iota^* g$ is conformal to the standard Euclidean metric on $\mathbb{R}^n$. Then M must be the n-sphere S^n, the metric g must be conformal to the standard round metric, and the embedding must be inverse to the standard stereographic projection.*

Of course, this is the expected result and in [3] two proofs are presented. One proof [3, Theorem 1.4] is via a removable singularities theorem based on the complement of $\mathbb{R}^n \hookrightarrow S^n$ having small Hausdorff dimension. The other proof [3, Proposition 2.3] is based on B. Schmidt's notion of *conformal boundary* as further developed by Francès.

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This article gives a proof of Theorem 1 using *conformal geodesics*. Apart from the elementary nature of this proof, it seems to be a natural one and a good application of the theory conformal geodesics, which are perhaps under-used in conformal differential geometry (unfortunately, as lamented in [6], their behaviour is not so well understood in general).

Although Theorem 1 also holds when $n = 2$, the result is of a different nature (global complex analysis rather than local to global differential geometry). Also note that the corresponding theorem is false for projective differential geometry: there are two distinct projective compactifications of $\mathbb{R}^n$, namely $\mathbb{R}^n \hookrightarrow \mathbb{RP}_n$ as a standard affine coördinate patch and $\mathbb{R}^n \hookrightarrow S^n$ onto an open hemisphere by inverse central projection.

This article starts with an exposition of conformal geodesics, which is more than is strictly needed for the proof of Theorem 1. Hopefully, this exposition will be useful for their further application in conformal geometry.

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2. Conformal Geodesics

Also known as *conformal circles* [1], these special curves may be defined as follows. For any metric g_{ab} in the conformal class, a curve $\gamma \hookrightarrow M$ with parameterization $\tau\colon \gamma \to \mathbb{R}$ is said to be a *conformal geodesic* if and only if

$$\partial A^a = 3\frac{V\cdot A}{V\cdot V}A^a - \frac{3A\cdot A}{2V\cdot V}V^a + V\cdot V \mathrm{P}_b{}^a V^b - 2\mathrm{P}_{bc}V^bV^cV^a\,, \tag{1}$$

where

- V^a is tangent along γ so that $V^a\nabla_a\tau = 1$, the 'velocity vector,'
- $\partial = \partial/\partial\tau \equiv V^a\nabla_a$ acting on any tensor field along γ,
- ∇_a is the Levi-Civita connection for g_{ab},
- $A^a \equiv \partial V^a$, the 'acceleration vector',
- P_{ab} is the conformal Schouten tensor
$$\mathrm{P}_{ab} = \frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)}Rg_{ab}\right)\,,$$
- R_{ab} is the Ricci tensor and R the scalar curvature,
- $X\cdot Y$ is the inner product X^aY_a of two fields X^a and Y^a.

The equation (1) is an ordinary differential equation for both the curve γ and its parameterization $\tau\colon \gamma \to \mathbb{R}$. Alternatively, if one uses

$$B^a \equiv \frac{1}{V\cdot V}A^a - 2\frac{V\cdot A}{(V\cdot V)^2}V^a$$

instead of the naïve acceleration, then (1) becomes

$$\partial B^a = V\cdot B B^a - \tfrac{1}{2}B\cdot B V^a + \mathrm{P}_b{}^a V^b \tag{2}$$

(which is equation (2) from [6]). Conformal geodesics were introduced in [7] and one can verify, as is done explicitly in [1], that (1) (equivalently (2) as is done in [6]), is conformally invariant. Unlike ordinary geodesics, however, the parameter τ is determined only up to projective freedom $\tau \mapsto (a\tau+b)/(c\tau+d)$.

Although dependent on a choice of metric in the conformal class, one can choose instead to use ordinary arc-length $t\colon \gamma \to \mathbb{R}$ as a parameter. Besides, as noted in [2] via the Cartan connection and directly verified in [1], any smooth curve on a conformal manifold inherits a preferred family of parameterizations defined up to projective freedom so the location of conformal geodesics and how they are parameterized can be regarded as separate issues. As noted in [6], an advantage of using arc-length is that it avoids the spurious feature that a projective parameterization τ might blow up for no good reason. Thus, for a choice of metric in the conformal class, one can insist on the unit vector field $U^a \equiv V^a/\sqrt{V\cdot V}$ along γ with corresponding parameter $t\colon \gamma \to \mathbb{R}$ such that

- $\delta t = 1$ where $\delta = \partial/\partial t \equiv U^a\nabla_a$ acting on any tensor field,
- $C^a = \delta U^a$ is the acceleration field along γ,

and then it is readily verified that

$$\delta C^a = \mathrm{P}_b{}^a U^b - (C.C + \mathrm{P}_{bc}U^bU^c)U^a\,. \tag{3}$$

This is the form of the conformal geodesic equation used in [6] to conclude that

- conformal geodesics always exist for short time,
- conformal geodesics are uniquely determined by initial conditions:
  - an arbitrary unit velocity vector $U^a|_{t=0}$,
  - an arbitrary acceleration vector $C^a|_{t=0}$ orthogonal to $U^a|_{t=0}$,
- a conformal geodesic can always be continued as long as C^a is finite.

Indeed, as observed in [6, Theorem 1.1], the proof is immediate from Picard's Theorem and one can also add that, provided the acceleration remains bounded and that solutions exist for $0 \le t \le 1$,

- the end point $\gamma^{-1}(1)$ depends smoothly on the initial conditions.

These conclusions are also valid for conformal geodesics formulated according to (1) provided that the acceleration but also the preferred parameter τ both remain bounded. (Observe that if our chosen metric g_{ab} is conformally rescaled $\hat{g}_{ab} = \Omega^2 g_{ab}$, then our various notions of acceleration change:

- $\hat{A}^a = A^a - V\cdot V\Upsilon^b + 2(V\cdot\Upsilon)V^b$,
- $\hat{B}_a = B_a - \Upsilon_a$,
- $\hat{C}_a = C_a - \Upsilon_a + (U\cdot\Upsilon)U_a$,

where $\Upsilon_a = \nabla_a \log\Omega$ but whether any of these notions remain finite is clearly independent of choice of metric in the conformal class.)

Notice that if $\gamma \hookrightarrow M$ is a conformal geodesic passing through $a, b, \in M$ with preferred parameterization $\tau : \gamma \to \mathbb{R}$ so that $\tau(a) = 0$ and $\tau(b) = 1$, then the same curve can be also be parameterized by $\hat{\tau} = 1 - \tau$. More specifically, the defining equation (1) holds with τ replaced by $\hat{\tau}$ but now $\hat{\tau}(b) = 0$ and $\hat{\tau}(a) = 1$. We shall refer to this manœuvre as *reversing the parameterization*.

In Euclidean space, we can solve (1) explicitly. In fact, if one restricts (1) to a Euclidean plane $\mathbb{R}^2 \hookrightarrow \mathbb{R}^n$, then one may readily verify that the curve

$$\tau \mapsto \big(x(\tau), y(\tau)\big) = \frac{2}{(2-\alpha\tau)^2 + \beta^2\tau^2}\big((2-\alpha\tau)\tau, \beta\tau^2\big) \tag{4}$$

satisfies

$$\frac{d}{d\tau}A^a = 3\frac{V\cdot A}{V\cdot V}A^a - \frac{3A\cdot A}{2V\cdot V}V^a\,, \tag{5}$$

where

$$V^a = (dx/d\tau, dy/d\tau) \quad\text{and}\quad A^a = (d^2x/d\tau^2, d^2y/d\tau^2)\,.$$

In case of any difficulty, one can ask Maple [5] to verify that (5) holds:

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x:=2*(2-alpha*t)*t/((2-alpha*t)^2+beta^2*t^2);
y:=2*beta*t^2/((2-alpha*t)^2+beta^2*t^2);
Vx:=diff(x,t): Vy:=diff(y,t): Ax:=diff(Vx,t): Ay:=diff(Vy,t):
VV:=Vx^2+Vy^2: VA:=Vx*Ax+Vy*Ay: AA:=Ax^2+Ay^2:
checkx:=simplify(VV*diff(Ax,t)-3*VA*Ax+3/2*AA*Vx);
checky:=simplify(VV*diff(Ay,t)-3*VA*Ay+3/2*AA*Vy);
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Alternatively, one can start by solving (5) with initial conditions

$$(x,y,\ldots)|_{\tau=0} = (0,0,\ldots,0) \qquad (dx/d\tau, dy/d\tau,\ldots)|_{\tau=0} = (1,0,\ldots,0)$$
$$(d^2x/d\tau^2, d^2y/d\tau^2,\ldots)|_{\tau=0} = (0,0,0,\ldots,0)$$

since, by symmetry, such a solution must have the form $\tau \mapsto (f(\tau),0,\ldots,0)$ for some smooth function $f(\tau)$. Then (5) turns into the ODE

$$f''' = 3\frac{f'f''}{(f')^2}f'' - \frac{3}{2}\frac{(f'')^2}{(f')^2}f' \quad\text{or}\quad 3(f'')^2 - 2f'f''' = 0\,,$$

which is the Schwarzian differential equation. As usual, this is solved by writing

$$3(f'')^2 - 2f'f''' = 4\sqrt{f'}(f')^2\Big(\frac{1}{\sqrt{f'}}\Big)''$$

from which one sees firstly that $f'(\tau) = 1\ \forall\tau$ and then that $f(\tau) = \tau$. Having found one solution, conformal invariance of the conformal geodesic equation (1) ensures that applying any Möbius transformation gives another. In particular, composing with the planar transformation

$$(x,y) \mapsto \frac{(x,y) - \frac{1}{2}(x^2+y^2)(\alpha,-\beta)}{1 - \alpha x + \beta y + \frac{1}{4}(\alpha^2+\beta^2)(x^2+y^2)}$$

gives (4), as required.

In any case, we have found in (4) the unique conformal circle in $\mathbb{R}^n$ with initial conditions

$$(x,y,\ldots)|_{\tau=0} = (0,0,\ldots,0) \qquad (dx/d\tau, dy/d\tau,\ldots)|_{\tau=0} = (1,0,\ldots,0)$$
$$(d^2x/d\tau^2, d^2y/d\tau^2,\ldots)|_{\tau=0} = (\alpha,\beta,0,\ldots,0)$$

(and one may also verify that for $\beta \neq 0$, the trajectory is, indeed, a round circle in $\mathbb{R}^2$ (centred at $(0,1/\beta)$) whilst if $\beta = 0$, the trajectory is the x-axis with projective parameterization $x = 2\tau/(2-\alpha\tau)$). Notice that the curve (4) is defined for all τ provided $\beta \neq 0$, has a limit at

$$\frac{2}{\alpha^2+\beta^2}(-\alpha,\beta)$$

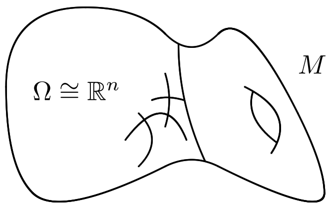
as $\tau \to \pm\infty$, and passes through

$$\frac{2}{(2-\alpha)^2+\beta^2}\bigl(2-\alpha,\beta\bigr)$$

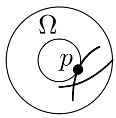
when $\tau = 1$.

3. The Uniqueness Theorem

In this section we prove Theorem 1. For convenience, let us denote by Ω the open subset $\iota(\mathbb{R}^n) \subset (M, g)$ that is supposed conformal to $\mathbb{R}^n$. Initially, we know very little about $\Omega \subset M$:



In particular, the boundary $\partial\Omega \subset M$ may be very badly behaved. Nevertheless, by surrounding each point of $\partial\Omega$ by a coördinate patch U in which one considers Euclidean spheres centred on points in $U \cap \Omega$ (cf. [4, Lemme 14]), it follows that in $\partial\Omega$ there is a dense set of *highly accessible* points p,



namely those for which all smooth curves $\gamma(s)$ emanating from p with velocity on one side of a suitable hyperplane in T_pM remain inside Ω for $0 < s < \epsilon$ for some $\epsilon > 0$. In particular, this property is valid for conformal circles emanating from p with velocity pointing into Ω in this sense. Let us fix a highly accessible point in $\partial\Omega$ and, at the risk of putting the cart before the horse, denote it by ∞. We aim to show that $\partial\Omega = \{\infty\}$. Indeed, if we can show this, then Theorem 1 follows easily because then the conformal Weyl curvature, in case $n \geq 4$, or the Cotton-York tensor, in case $n = 3$, vanishes on M by continuity. This implies that M is conformally flat and Theorem 1 follows from Liouville's theorem.

Let us fix a conformal geodesic $\gamma \hookrightarrow M$ with preferred parameterization $\tau\colon \gamma \to \mathbb{R}$ emanating from ∞. By rescaling the parameterization we may arrange that γ is contained inside Ω for $0 < \tau \leq 1$. Finally, by reversing the parameterization, we obtain a conformal geodesic starting at a point inside Ω when $\tau = 0$ and ending up at a point $\infty \in \partial\Omega$ when $\tau = 1$.

But recall that conformal geodesics and their preferred parameterizations depend only on the conformal structure and that Ω is supposed to be conformal to the standard Euclidean metric on $\mathbb{R}^n$. At the end of the previous section we determined all the conformal circles in $\mathbb{R}^n$ to be straight lines or round circles. A round circle is bounded and so we are left with having constructed what, from the Euclidean viewpoint, is simply a straight line. After a Euclidean motion, this straight line may as well be the x-axis with parameterization starting at the origin and to have the parameter τ run from 0 to 1 at ∞ completely fixes the curve as

$$\tau \mapsto \Big(\frac{\tau}{1-\tau}, 0, 0, \ldots, 0\Big). \tag{6}$$

In other words, it is the case $(\alpha, \beta) = (2, 0)$ in the discussion above. Let us compare this curve with the conformal geodesic in $\mathbb{R}^n$ having initial conditions

$$\big(x, y, \ldots\big)|_{\tau=0} = \big(0, 0, \ldots, 0\big) \qquad \big(dx/d\tau, dy/d\tau, \ldots\big)|_{\tau=0} = \big(1, 0, \ldots, 0\big)$$
$$\big(d^2x/d\tau^2, d^2y/d\tau^2, \ldots\big)|_{\tau=0} = \big(\alpha, \sigma(2-\alpha), 0, \ldots, 0\big)$$

for some fixed σ and for some $0 \leq \alpha < 2$. According to our previous discussion, it is the curve (4) with $\beta = \sigma(2-\alpha)$, namely

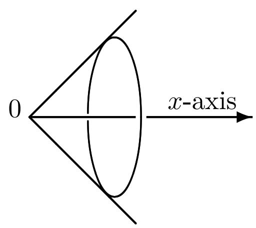
$$\tau \mapsto \frac{2}{(2-\alpha\tau)^2 + \sigma^2(2-\alpha)^2\tau^2}\big((2-\alpha\tau)\tau, \sigma(2-\alpha)\tau^2, 0, \ldots, 0\big).$$

When $\tau = 1$ this conformal geodesic passes through the point

$$\frac{2}{2-\alpha}\frac{1}{1+\sigma^2}\big(1, \sigma, 0, \ldots, 0\big). \tag{7}$$

Now let $\alpha \uparrow 2$. We see that the endpoint (7) moves monotonely out along the straight line in the direction $(1, \sigma, 0, \ldots, 0)$. At least, this is what we see in $\Omega \cong \mathbb{R}^n$. Viewed in $M \supset \Omega$, there is no problem with the limit curve with initial conditions $(\alpha, \beta) = (2, 0)$. By construction, it is the curve (6) joining the origin $0 \in \mathbb{R}^n \cong \Omega \hookrightarrow M$ to $\infty \in M \setminus \Omega$. We conclude that the endpoint (7) tends to $\infty \in M$ along the straight line in the direction $(1, \sigma, 0, \ldots, 0)$. At this point, there are several ways to complete the proof, one of which is as follows.

Even if we restrict to $|\sigma| \leq 1$ to force uniform convergence to ∞, we are free to rotate around the x-axis to obtain an entire cone



all the rays of which end up at $\infty \in M$. But now we may repeat this argument starting with any ray from this cone, rather than the x-axis. In this way we obtain a finite collection of cones that cover all rays emanating from the origin. We conclude that ∞ is the only boundary point of $\Omega \subset M$. Q.E.D.

The discussion in this article was confined to the Riemannian setting. For indefinite signature the conformal geodesic equation (1) evidently breaks down if the velocity vector V^a is null. The natural curves in indefinite signature then fall into two classes: conformal geodesics and null geodesics. These classes have a different nature and, in particular, the null geodesics are controlled by a second order ordinary differential equation with only the velocity needed as initial condition and only an affine freedom allowed in their parameterization. Although, as detailed in [6], equation (2) can be used simultaneously to describe both conformal and null geodesics, it is their initial conditions that do not fit so well together. It would be interesting to try to adapt the Riemannian reasoning as above to indefinite signature, bearing in mind the cautionary examples in [6].

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