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CORRECT SOLVABILITY OF A GENERAL DIFFERENTIAL EQUATION OF THE FIRST ORDER IN THE SPACE $L_p(\mathbb{R})$

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Abstract. We consider the equation

$$-r(x)y'(x)+q(x)y(x)=f(x)\,,\quad x\in\mathbb{R} \tag{1}$$

where $f\in L_p(\mathbb{R})$, $p\in[1,\infty]$ ($L_\infty(\mathbb{R}):=C(\mathbb{R})$) and

$$0<r\in C^{\mathrm{loc}}(\mathbb{R})\,,\quad 0\le q\in L_1^{\mathrm{loc}}(\mathbb{R})\,. \tag{2}$$

We obtain minimal requirements to the functions r and q, in addition to (2), under which equation (1) is correctly solvable in $L_p(\mathbb{R})$, $p\in[1,\infty]$.

1. Introduction

In the present paper, we continue the investigations of [1, 2, 3, 5]. We consider the equation

$$-r(x)y'(x)+q(x)y(x)=f(x)\,,\quad x\in\mathbb{R} \tag{1.1}$$

where $f\in L_p$, ($L_p(\mathbb{R}):=L_p$), $p\in[1,\infty)$, ($L_\infty(\mathbb{R}):=C(\mathbb{R})$), and

$$0<r\in C^{\mathrm{loc}}(\mathbb{R})\,,\qquad 0\le q\in L_1^{\mathrm{loc}}(\mathbb{R})\,. \tag{1.2}$$

By a solution of (1.1) we mean any absolutely continuous function y satisfying (1.1) almost everywhere. In addition, we say that for a given $p\in[1,\infty]$ equation (1.1) is correctly solvable in L_p if

I) for any function $f\in L_p$, equation (1.1) has a unique solution $y\in L_p$;

II) there is an absolute constant $c(p)\in(0,\infty)$ such that the solution $y\in L_p$ of (1.1) satisfies the inequality

$$\|y\|_p\le c(p)\|f\|_p\,,\qquad \forall f\in L_p\quad (\|f\|_p:=\|f\|_{L_p})\,. \tag{1.3}$$

For brevity, below we say "question on I)–II)" and "problem I)–II)". We use the letters c, $c(\cdot)$ to denote absolute positive constants which are not essential for exposition and may differ even within a single chain of computations.

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Note that in [1, 2, 3] we studied a boundary value problem consisting of (1.1) and the boundary conditions

$$\lim_{|x|\to\infty} y(x) = 0\,. \tag{1.4}$$

Here conditions (1.4) were useful in the study of the boundary value problem because its solution y, if it exists, is of the form (see [2]):

$$y(x) = (Gf)(x) = \mu(x)\int_x^\infty \theta(\xi)f(\xi)d\xi\,, \qquad f\in L_p,\ x\in\mathbb{R}\,. \tag{1.5}$$

Here

$$\mu(x) = \exp\Big(\int_0^x \frac{q(\xi)}{r(\xi)}d\xi\Big)\,,\quad \theta(x) = \frac{1}{r(x)}\exp\Big(-\int_0^x \frac{q(\xi)}{r(\xi)}d\xi\Big)\,,\quad x\in\mathbb{R}\,. \tag{1.6}$$

In I)–II), the solution $y\in L_p$ of (1.1) is also of form (1.5), but in the absence of the a priori property (1.4) it is more difficult to prove (1.5). In any case, whenever formula (1.5) is proven, the conditions for I)–II) to hold clearly coincide with the conditions for the integral operator $G\colon L_p\to L_p$ to be bounded (for a fixed $p\in[1,\infty]$). We study these conditions and obtain a precise answer to the question on correct invertibility in L_p of the simplest differential operation

$$Ly = -ry' + qy$$

with coefficients of constant sign (see (1.2)). We thus propose three unconditional criteria for I)–II) to hold: Theorems 3.1, 3.3 and 3.5 (see §3; the cases $p=1$, $p\in(1,\infty)$ and $p=\infty$ are considered separately, according to theorems on norms of integral operators). Note that to apply these criteria to particular equations (1.1), one needs sharp by order estimates of improper integrals (see §3). This implies that in their original form, these criteria are applicable to particular equations only under strong additional requirements to r and q which guarantee all required estimates. On the other hand, we use Theorems 3.1, 3.3 and 3.5 as a starting point for obtaining less general but much more efficient conditions for practical checking of conditions I)–II). See §§5–6 for some simple examples.

Likewise, but using special techniques due to M. Otelbaev (see [2, 3]), one can obtain efficient necessary (and similar sufficient) conditions for the validity of I)–II), expressed as local requirements to r and q. The proofs of such conditions are more complicated than simple argument of §§5–6 and need a separate exposition. We will present them in a forthcoming paper.

2. Preliminaries

Throughout this section (and also in §§3, 4), conditions (1.2) are assumed to be valid and are not mentioned in our statements.

We need some facts from [2]. Assume that

$$S_1 = \infty\,,\qquad S_1 \overset{\text{def}}{=} \int_{-\infty}^0 \frac{q(t)}{r(t)}\,dt\,. \tag{2.1}$$

Then for every $x \in \mathbb{R}$, one can define the function (see [1, 2]):

$$d(x) = \inf_{d>0}\{d : \Phi(x,d) = 2\}, \qquad \Phi(x,d) \stackrel{\text{def}}{=} \int_{x-d}^{x+d} \frac{q(t)}{r(t)}\, dt\,. \tag{2.2}$$

Functions of such a form were introduced by Otelbaev (see [6]).

Consider the boundary value problem (see §1):

$$-r(x)y'(x) + q(x)y(x) = f(x)\,, \qquad x \in \mathbb{R}\,, \tag{2.3}$$

$$\lim_{|x|\to\infty} y(x) = 0\,. \tag{2.4}$$

Definition 2.1 ([2])**.** Let $p \in [1,\infty]$. We say that the boundary value problem (2.3)–(2.4) is correctly solvable in L_p if equation (2.3) is correctly solvable, and, regardless of $f \in L_p$, the solution $y \in L_p$ of (2.3) satisfies (2.4).

We now give the conditions for correct solvability of (2.3)–(2.4).

Theorem 2.2 ([2])**.** *Let* $p \in (1,\infty)$, $p' = p(p-1)^{-1}$. *Problem* (2.3)–(2.4) *is correctly solvable in* L_p *if and only if the following conditions hold:*

1) $M_p < \infty$ *where* $M_p = \sup_{x\in\mathbb{R}} M_p(x)$,

$$M_p(x) = \Big[\int_{-\infty}^{x} \exp\Big(-p\int_t^x \frac{q(\xi)}{r(\xi)}\, d\xi\Big)\, dt\Big]^{1/p}$$

$$\times \Big[\int_x^{\infty} \frac{1}{r(t)^{p'}} \exp\Big(-p'\int_x^t \frac{q(\xi)}{r(\xi)}\, d\xi\Big)\, dt\Big]^{1/p'}; \tag{2.5}$$

2) $S_1 = \infty$ *(see* (2.1)*);*

3) $A_{p'} < \infty$. *Here*

$$A_{p'} = \sup_{x\in\mathbb{R}} A_{p'}(x) \tag{2.6}$$

$$A_{p'}(x) = \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)^{p'}}\,, \qquad x \in \mathbb{R}\,. \tag{2.7}$$

Theorem 2.3 ([2])**.** *Problem* (2.3)–(2.4) *is correctly solvable in* L_1 *if and only if the following conditions hold:*

1) $S_1 = \infty$ *(see* (2.1)*);*

2)

$$r_0 > 0, \qquad r_0 \stackrel{\text{def}}{=} \inf_{x\in\mathbb{R}} r(x)\,; \tag{2.8}$$

3) $M_1 < \infty$, $M_1 = \sup_{x\in\mathbb{R}} M_1(x)$ *where*

$$M_1(x) = \frac{1}{r(x)} \int_{-\infty}^{x} \exp\Big(-\int_t^x \frac{q(\xi)}{r(\xi)}\, d\xi\Big)\, dt\,, \qquad x \in \mathbb{R}\,. \tag{2.9}$$

Theorem 2.4 ([2])**.** *Problem* (2.3)–(2.4) *is correctly solvable in* $C(\mathbb{R})$ *if and only if* $A_0 = 0$ *where*

$$A_0 = \lim_{|x|\to\infty} A(x)\,, \quad A(x) = \int_x^\infty \frac{1}{r(t)} \exp\Big(-\int_x^t \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\,, \quad x \in \mathbb{R}\,. \tag{2.10}$$

Our proofs (see §4, §6) are based on the following theorems.

Theorem 2.5 ([2])**.** *Let* $p \in (1,\infty)$*, let* μ*,* θ *be continuous positive functions defined on* $\mathbb{R}$*, and let*

$$(\mathcal{K}f)(x) = \mu(x)\int_x^\infty \theta(\xi)f(\xi)\,d\xi\,, \qquad x \in \mathbb{R}\,. \tag{2.11}$$

Then the operator $\mathcal{K}\colon L_p \to L_p$ *is bounded if and only if* $H_p < \infty$*. Here* $H_p = \sup_{x\in\mathbb{R}} H_p(x)$*,*

$$H_p(x) = \Big[\int_{-\infty}^x \mu(t)^p\,dt\Big]^{1/p}\Big[\int_x^\infty \theta(\xi)^{p'}\,d\xi\Big]^{1/p'}\,, \tag{2.12}$$

$$p' = \frac{p}{p-1}\,, \qquad x \in \mathbb{R}\,.$$

In addition, we have the inequalities

$$H_p \le \|\mathcal{K}\|_{p\to p} \le (p)^{1/p}\cdot(p')^{1/p'}H_p\,. \tag{2.13}$$

Remark 2.6. Theorem 2.5 follows from Hardy's inequality (see [7]). See, e.g., [2] for such a proof. In [8], an original direct proof of this theorem (under weaker requirements on μ and θ) are given.

Theorem 2.7 ([4])**.** *Let* $\mathcal{K}$ *be the operator* (2.11)*. Then*

$$\|\mathcal{K}\|_{C(\mathbb{R})\to C(\mathbb{R})} = \sup_{x\in\mathbb{R}} \mu(x)\int_x^\infty \theta(t)\,dt\,. \tag{2.14}$$

Further, let $-\infty \le a < b \le \infty$*, suppose that the functions* $\mu(x)$ *and* $\theta(x)$ *are continuous for* $x \in (a,b)$ *and let* $\tilde{\mathcal{K}}$ *be the integral operator*

$$(\tilde{\mathcal{K}}f)(x) = \mu(x)\int_x^b \theta(\xi)f(\xi)\,d\xi\,, \qquad x \in (a,b)\,. \tag{2.15}$$

Then

$$\|\tilde{\mathcal{K}}f\|_{L_1(a,b)\to L_1(a,b)} = \sup_{x\in(a,b)} \theta(x)\int_a^x \mu(t)\,dt\,. \tag{2.16}$$

Note that some technical assertions are given in §5.

3. Results

Below, we present only unconditional criteria for the validity of I)–II). Some consequences requiring additional requirements to r and q are given in §5.

Theorem 3.1. *Let* $p \in (1,\infty)$*. Equation* (1.1) *is correctly solvable in* L_p *if and only if* $M_p < \infty$ *(see Theorem 2.2).*

Corollary 3.2. *Let $p \in (1, \infty)$. Equation* (1.1) *is correctly solvable in L_p if and only if the operator $G\colon L_p \to L_p$ (see* (1.5)*) is bounded. In the latter case, $S_1 = \infty$ (see* (2.1)*), and the solution $y \in L_p$ of* (1.1) *is of the form $y = Gf$. In addition,*

$$c^{-1}(p)M_p \le \|G\|_{p\to p} \le c(p)M_p\,. \tag{3.1}$$

Thus, the only difference between Theorem 2.2 and Theorem 3.1 is the condition $A_{p'} < \infty$ (see (2.7)). This condition is a minimal requirement, in addition to the conditions of Theorem 3.1, which guarantees (2.4). In §7, we give an example of equation (1.1) for which $S_1 = \infty$, $M_p < \infty$ and $A_{p'} = \infty$.

Theorem 3.3. *Equation* (1.1) *is correctly solvable in L_1 if and only if $M_1 < \infty$ (see Theorem 2.3).*

Corollary 3.4. *Equation* (1.1) *is correctly solvable in L_1 if and only if the operator $G\colon L_1 \to L_1$ is bounded (see* (1.5)*). In the latter case, $S_1 = \infty$ (see* (2.1)*) and*

$$\|G\|_{1\to 1} = M_1\,. \tag{3.2}$$

Thus the only difference between Theorem 2.3 and Theorem 3.3 is the condition $r_0 > 0$ (see (2.8)). This condition is a minimal requirement, in addition to the conditions of Theorem 3.3, which guarantees (2.4). In §7, we give an example of equation (1.1) for which $S_1 = \infty$, $M_1 < \infty$ and $r_0 = 0$.

Theorem 3.5. *Equation* (1.1) *is correctly solvable in $C(\mathbb{R})$ if and only if the following conditions hold:*

1)

$$S_2 = \infty, \qquad S_2 = \int_0^\infty \frac{q(t)}{r(t)}\,dt\,, \tag{3.3}$$

2) $A < \infty$, $\qquad A = \sup_{x\in\mathbb{R}} A(x)$, *where (see* (1.6)*)*

$$A(x) = \mu(x)\int_x^\infty \theta(t)\,dt\,, \qquad x \in \mathbb{R}\,. \tag{3.4}$$

Thus, the only difference between Theorem 2.4 and Theorem 3.5 is the condition $A_0 = 0$ (see Theorem 2.4. Thus equality is a minimal requirement, in addition to 1)–2), which guarantees (2.4). In §7, we give an equation (1.1) for which $S_2 = \infty$, $A < \infty$, $A_0 > 0$.

4. Proofs

Proof of Theorem 3.1. *Necessity.*

The following obvious remark is stated as a separate assertion.

Lemma 4.1. *Let y be a solution of* (1.1)*, let μ and θ be defined by* (1.6)*, and let $-\infty < x \le t < \infty$. Then we have the following equality:*

$$\frac{y(t)}{\mu(t)} = \frac{y(x)}{\mu(x)} - \int_x^t \theta(\xi)f(\xi)\,d\xi\,. \tag{4.1}$$

Fix numbers $-\infty < t_1 < t_2 < \infty$ and introduce the function

$$f_0(\xi) = \begin{cases} \theta(\xi)^{p'-1}, & \xi \in [t_1, t_2], \\ 0, & \xi \notin [t_1, t_2], \end{cases} \qquad p' = \frac{p}{p-1}. \tag{4.2}$$

Then

$$\|f_0\|_p = \Big[\int_{t_1}^{t_2} |f_0(\xi)|^p d\xi\Big]^{1/p} = \Big[\int_{t_1}^{t_2} \theta(\xi)^{p'} d\xi\Big]^{1/p} < \infty. \tag{4.3}$$

In (1.1), set $f = f_0$, and denote by $y_0 \in L_p$ the solution of such an equation (1.1). In (4.1), set $f := f_0$, $y := y_0$, $x := t_2$ and let $t \ge x = t_2$. Then (see (4.2))

$$y_0(t) = y_0(t_2)\frac{\mu(t)}{\mu(t_2)}, \qquad t \ge t_2. \tag{4.4}$$

From (4.4) it follows that $y_0(t_2) = 0$ since otherwise (see (1.6)):

$$\begin{aligned} \infty > \|y_0\|_p^p &\ge \int_{t_2}^{\infty} |y_0(t)|^p\, dt = |y_0(t_2)|^p \int_{t_2}^{\infty} \Big(\frac{\mu(t)}{\mu(t_2)}\Big)^p dt \\ &\ge |y_0(t_2)|^p \int_{t_2}^{\infty} 1\, dt = \infty \quad \Rightarrow \quad y_0(t_2) = 0 \quad \Rightarrow \quad \text{(see (4.4))} \end{aligned}$$

$$y_0(t) = 0, \qquad t \ge t_2. \tag{4.5}$$

Hence, by (4.1) and (4.5), we have

$$y_0(x) = \mu(x) \int_x^{\infty} \theta(\xi) f_0(\xi)\, d\xi, \qquad x \in \mathbb{R}. \tag{4.6}$$

Let us now estimate $\|y_0\|_p$ from below (see (4.6), (4.1), (4.2)):

$$\begin{aligned} \|y_0\|_p^p &= \int_{-\infty}^{\infty} \mu(t)^p \Big| \int_t^{\infty} \theta(\xi) f_0(\xi)\, d\xi\Big|^p dt \\ &\ge \int_{-\infty}^{t_1} \mu(t)^p \Big(\int_t^{\infty} \theta(\xi) f_0(\xi)\, d\xi\Big)^p dt \\ &\ge \int_{-\infty}^{t_1} \mu(t)^p dt \Big(\int_{t_1}^{t_2} \theta(\xi) f_0(\xi)\, d\xi\Big)^p \quad \Rightarrow \end{aligned}$$

$$\|y_0\|_p \ge \Big(\int_{-\infty}^{t_1} \mu(t)^p dt\Big)^{1/p} \Big(\int_{t_1}^{t_2} \theta(\xi)^{p'}\, d\xi\Big). \tag{4.7}$$

Therefore, by (1.3), (4.3) and (4.7), we get

$$\begin{aligned} \Big(\int_{-\infty}^{t_1} \mu(t)^p dt\Big)^{1/p} \Big(\int_{t_1}^{t_2} \theta(\xi)^{p'}\, d\xi\Big) &\le \|y_0\|_p \le c(p)\|f_0\|_p \\ &= c(p)\Big(\int_{t_1}^{t_2} \theta(\xi)^{p'} d\xi\Big)^{1/p} \quad \Rightarrow \end{aligned}$$

$$\Big(\int_{-\infty}^{t_1} \mu(t)^p dt\Big)^{1/p} \Big(\int_{t_1}^{t_2} \theta(\xi)^{p'} d\xi\Big)^{1/p'} \le c(p) < \infty, \qquad t_1 \le t_2. \tag{4.8}$$

Since in (4.8) the numbers t_1 and t_2 ($t_1 \le t_2$) are arbitrary, from (4.8) for $x \in \mathbb{R}$ we get

$$\Big[\int_{-\infty}^{x} \exp\Big(p\int_0^t \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big]^{1/p}\Big[\int_x^\infty \frac{1}{r(\xi)^{p'}}\exp\Big(-p'\int_0^\xi \frac{q(s)}{r(s)}\,ds\Big)\,d\xi\Big]^{1/p'} \le c(p) < \infty\,.$$

Using the relations

$$\int_0^t \frac{q(\xi)}{r(\xi)}\,d\xi = \int_0^x \frac{q(\xi)}{r(\xi)}\,d\xi - \int_t^x \frac{q(\xi)}{r(\xi)}\,d\xi\,, \qquad x \ge t\,,$$

$$\int_0^\xi \frac{q(s)}{r(s)}\,ds = \int_0^x \frac{q(s)}{r(s)}\,ds + \int_x^\xi \frac{q(s)}{r(s)}\,ds\,, \qquad x \le \xi\,,$$

we easily bring the second inequality to the form (2.5), which immediately give the inequality $M_p < \infty$ (see Theorem 2.2).

Proof of Theorem 3.1. *Sufficiency.*

Let $p \in (1,\infty)$ and $M_p < \infty$. Set $y = Gf$ (see (1.5), (1.6)). Then from Hölder's inequality, it follows that

$$\frac{|y(x)|}{\mu(x)} \le \Big(\int_x^\infty \theta(\xi)^{p'}\,d\xi\Big)^{1/p'}\|f\|_p\,, \qquad x \in R\,. \tag{4.9}$$

It remains to show that (1.5) presents the unique solution of (1.1) that belongs to L_p. Since $M_p < \infty$ then due to (4.9) we obtain that the integral (2.5) converges for $x \in \mathbb{R}$, the function $y = Gf$ is defined, absolutely continuous and satisfies (1.1) almost everywhere on $\mathbb{R}$. Moreover, (1.3) holds (see Theorem 2.5). Finally, the solution z of the homogeneous equation

$$-r(x)z'(x) + q(x)z(x) = 0\,, \qquad x \in \mathbb{R} \tag{4.10}$$

is of the form

$$z(x) = c\exp\Big(\int_0^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,, \qquad x \in \mathbb{R},\ c = \text{const}$$

and belongs to L_p only for $c = 0$ because otherwise we would have

$$\infty > \|z\|_p^p \ge \|z\|_{L_p(0,\infty)}^p = |c|^p\int_0^\infty \exp\Big(p\int_0^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dx$$
$$\ge |c|^p\int_0^\infty 1\,dx = \infty \quad \text{contradiction} \quad \Rightarrow \quad c = 0\,,$$

i.e., $z \equiv 0$. The theorem is proved. □

Proof of Corollary 3.2. Let $p \in (1,\infty)$ and $M_p < \infty$. Then $M_p(x)\big|_{x=0} < \infty$ and therefore

$$\int_{-\infty}^0 \exp\Big(-p\int_t^0 \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt < \infty\,.$$

Assume now that $S_1 < \infty$. This implies that

$$\infty > \int_{-\infty}^0 \exp\Big(-p\int_t^0 \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt \ge \int_{-\infty}^0 \exp(-pS_1)\,dt = \infty\,.$$

Contradiction. Hence $S_1 = \infty$. The remaining assertions of the corollary are immediate consequences of Theorems 3.1 and 2.5. $\square$

Proof of Theorem 3.3 (*Necessity*) **and Corollary 3.4.** Let $f \in L_1$, $n \geq 1$ and

$$f_n(x) = \begin{cases} f(x), & \text{if} \quad x \in [-n, n] \\ 0, & \text{if} \quad x \notin [-n, n] \end{cases} \tag{4.11}$$

Denote by y_n the solution of (1.1) with the right-hand side f_n which lies in L_1. Then, repeating the initial part of the proof of necessity of the conditions of Theorem 3.1, we obtain that $y(n) = 0$ and therefore (see (4.1), (1.5))

$$y_n(x) = (Gf_n)(x) = \mu(x) \int_x^n \theta(\xi) f_n(\xi)\, d\xi, \qquad x \leq n. \tag{4.12}$$

Now from (4.11), (4.12) and (1.3), it follows that

$$\|y_n\|_{L_1(-n,n)} \leq \|y_n\|_{L_1} \leq c\|f_n\|_{L_1} = c(1)\|f_n\|_{L_1(-n,n)}. \tag{4.13}$$

Since f_n is an arbitrary function from the class $L_1(-n, n)$, from (4.13) we obtain that the operator $G_n \colon L_1(-n, n) \to L_1(-n, n)$ where

$$(G_n \tilde{f})(x) = \mu(x) \int_x^n \theta(\xi) \tilde{f}(\xi)\, d\xi, \qquad \tilde{f} \in L_1(-n, n), \qquad |x| \leq n \tag{4.14}$$

is bounded, and (see Theorem 2.7)

$$\sup_{|x| \leq n} \theta(x) \int_{-n}^x \mu(t)\, dt = \|G\|_{L_1(-n,n) \to L_1(-n,n)} \leq c(1) < \infty.$$

The latter bound holds for all $n \geq 1$, and therefore $M_1 < \infty$ and (3.2) holds. The equality $S_1 = \infty$ can be checked in the same way as Corollary 3.2.

Proof of Theorem 3.3 (*Sufficiency*) **and Corollary 3.4.** Let $M_1 < \infty$. Set $y = Gf$ for $f \in L_1$ (see (1.5), (1.6)). Since $\|G\|_{1 \to 1} < \infty$ (see (2.9), (2.16)), we have $y \in L_1$. In addition, from (1.5), (1.6) for $t \geq x$ we get

$$\frac{y(x)}{\mu(x)} = \int_x^t \theta(\xi) f(\xi)\, d\xi + \int_t^\infty \theta(\xi) f(\xi)\, d\xi = \int_x^t \theta(\xi) f(\xi)\, d\xi + \frac{y(t)}{\mu(t)},$$

i.e., (4.1) holds. Further, since the function $\theta(\xi)$ is continuous for $\xi \in \mathbb{R}$, from (4.1) and absolute continuity of the integral, it follows that the function $y(t)$, $t \in \mathbb{R}$ is absolutely continuous too. Now from (1.5), (1.6) it is easy to obtain (1.1), i.e., y is a solution of (1.1), $y \in L_1$ and (see above) (1.3) holds with $c(1) = \|G\|_{1 \to 1} = M_1$. Finally, ro prove that (1.1) has a unique solution in the class L_1, one can use the same argument as in Theorem 3.1. $\square$

Proof of Theorem 3.5. *Necessity.* Suppose that equation (1.1) is correctly solvable in $C(\mathbb{R})$, and let z be the function from (4.10) with $c = 1$. If $S_2 < \infty$, then setting $c = 1$ in (4.10) we obtain that $z \in C(\mathbb{R})$, and z is a solution of (1.1) with $f \equiv 0$. Then by (1.3) we have

$$0 < \|z\|_{C(\mathbb{R})} = \exp(S_2) \leq c(\infty)\|0\|_{C(\mathbb{R})} = 0.$$

Contradiction. Hence $S_2 = \infty$. Now set $f \equiv 1$ in (1.1), and let y be a solution of this equation such that $y \in C(\mathbb{R})$. Since $S_2 = \infty$ and $y \in C(\mathbb{R})$, from (4.1) it follows that $\frac{y(t)}{\mu(t)} \to 0$ as $t \to \infty$, and therefore (1.5) holds with $f \equiv 1$. Together with (1.3), this implies that

$$A = \sup_{x\in\mathbb{R}} \int_x^\infty \frac{1}{r(t)} \exp\Big(- \int_x^t \frac{q(\xi)}{r(\xi)}\, d\xi\Big)\, dt = \sup_{x\in\mathbb{R}} |y(x)|$$
$$= \|y\|_{C(\mathbb{R})} \le c(\infty)\|f\|_{C(\mathbb{R})} = c(\infty) < \infty\,.$$

Proof of Theorem 3.5. *Sufficiency.*

For $f \in C(\mathbb{R})$ set $y = Gf$ (see (1.5), (1.6)). Clearly, $y \in C(\mathbb{R})$ since

$$\|y\|_{C(\mathbb{R})} = \sup_{x\in\mathbb{R}} \mu(x)\Big| \int_x^\infty \theta(\xi) f(\xi)\, d\xi\Big| \le \sup_{x\in\mathbb{R}} \Big(\mu(x) \int_x^\infty \theta(\xi)\, d\xi\Big) \|f\|_{C(\mathbb{R})}$$
$$= A\|f\|_{C(\mathbb{R})} < \infty\,,$$

and this also gives (1.3). In addition, y is a unique solution of (1.1) in the class $C(\mathbb{R})$ because the function z from (4.10) for $c \neq 0$ does not belong to $C(\mathbb{R})$ in view of (3.3). □

5. Additional assertions

In this section, we show that one can extract from result of §3 certain assertions which are efficient and convenient for investigating concrete equations. Several such theorems are presented below; see §6 for their proofs.

Theorem 5.1. *If equation* (1.1) *is correctly solvable in* L_p, $p \in [1, \infty]$, *then* $B_p < \infty$. *Here (see* (2.2)*)*

$$B_p = \begin{cases} \sup_{x\in\mathbb{R}} \frac{d(x)}{r(x)}, & \textit{for} \quad p = 1 \\ \sup_{x\in\mathbb{R}} d(x)^{1/p} \Big(\int_x^{x+d(x)} \frac{dt}{r(t)^{p'}} \Big)^{1/p'}, \ p' = \frac{p}{p-1}\,, & \textit{for} \quad p \in (1,\infty) \\ \sup_{x\in\mathbb{R}} \int_{x-d(x)}^{x+d(x)} \frac{dt}{r(t)}\,, & \textit{for} \quad p = \infty\,. \end{cases} \tag{5.1}$$

The main difference between the necessary condition (5.1) and those of the theorems of §3 is the fact that the condition $B_p < \infty$ is usually local because for particular equations one has, as a rule, the inequality $d_0 < \infty$ where

$$d_0 = \sup_{x\in\mathbb{R}} d(x)\,. \tag{5.2}$$

Even if $d_0 = \infty$, when studying the condition $B_p < \infty$, we nevertheless can rely on powerful of local analysis since $d(x) < \infty$ for any $x \in \mathbb{R}$ (see [2]). In particular, below we give convenient tools for establishing two-sided, sharp by order estimates of the function d, which guarantee all our assertions in most cases.

Note that under some non-rigid additional requirement to the function d, the condition $B_p < \infty$ becomes also sufficient for the correct solvability of equation (1.1) in L_p for all $p \in [1, \infty]$. Thus (under this additional condition on d), when investigating I)–II), we can study, instead of hard global conditions of §3, the significantly and conceptually easier local requirement $B_p < \infty$. However, to prove the latter fact, we need separate considerations, which will be given in a forthcoming paper.

Now we present our results. The following assertion serves as a main tool for obtaining estimates for the function d (see Theorem 5.3 below).

Theorem 5.2. [2] *Let $S_1 = \infty$ (see (2.1)). For $x \in \mathbb{R}$, the inequality $\eta \geq d(x)$ $(0 \leq \eta \leq d(x)$ holds if and only if $F(\eta) \geq 2$ $(F(\eta) \leq 2)$. Here*

$$F(\eta) \overset{\text{def}}{=} \int_{x-\eta}^{x+\eta} \frac{q(t)}{r(t)}\, dt\,.$$

Theorem 5.3. *Suppose that the following conditions hold:*

1)

(5.3) $$0 < r \in C^{\text{loc}}(\mathbb{R}), \qquad 0 \leq q \in L_1^{\text{loc}}(\mathbb{R});$$

2) *there exist functions q_1 and q_2 such that*

(5.4) $$q = q_1 + q_2$$

(5.5) $$0 < q_1 \in C^{\text{loc}}(\mathbb{R}), \qquad q_2 \in L_1^{\text{loc}}(\mathbb{R}), \qquad \frac{q_1}{r} \in AC^{(1)}_{\text{loc}(\mathbb{R})};$$

3)

(5.6) $$\varkappa_1(x) \to 0\,, \qquad \varkappa_2(x) \to 0 \qquad \text{as} \qquad |x| \to \infty\,,$$

where

(5.7) $$\varkappa_1(x) = \Big(\frac{r(x)}{q_1(x)}\Big)^2 \sup_{|\xi| \leq 2\frac{r(x)}{q_1(x)}} \Big| \int_{x-\xi}^{x+\xi} \Big(\frac{q_1(s)}{r(s)}\Big)'' ds \Big|,$$

(5.8) $$\varkappa_2(x) = \sup_{|\xi| \leq 2\frac{r(x)}{q_1(x)}} \Big| \int_{x-\xi}^{x+\xi} \frac{q_2(s)}{r(s)}\, ds \Big|.$$

Then the following relations hold:

(5.9) $$d(x) = \frac{r(x)}{q_1(x)}(1 + \delta(x))\,, \qquad |\delta(x)| \leq c(\varkappa_1(x) + \varkappa_2(x))\,, \qquad |x| \gg 1\,,$$

(5.10) $$c^{-1}\frac{r(x)}{q_1(x)} \leq d(x) \leq c\frac{r(x)}{q_1(x)}\,, \qquad x \in \mathbb{R}\,.$$

We note another useful property of the function d.

Theorem 5.4 ([2])**.** *Let $S_1 = \infty$ (see (2.1)). Then $d_0 < \infty$ (see (5.2)) if and only if there exists $a \in (0, \infty)$ such that $m(a) > 0$. Here*

(5.11) $$m(a) = \inf_{x \in \mathbb{R}} \int_{x-a}^{x+a} \frac{q(t)}{r(t)}\, dt\,.$$

In the following theorems, we give sufficient conditions for correct solvability of equation (1.1) in L_p, $p \in [1, \infty]$.

Theorem 5.5. *Let* $m(a) > 0$ *(see* (5.11)*),* $\ell(a) < \infty$ *for some* $a \in (0, \infty)$. *Here*

$$\ell(a) = \sup_{x \in \mathbb{R}} \int_{x-a}^{x+a} \frac{dt}{r(t)}. \tag{5.12}$$

Suppose that there exists $\alpha \in [0, \infty)$ *such that the following inequality holds:*

$$\frac{r(t)}{r(x)} + \frac{r(x)}{r(t)} \leq c \exp\left(\alpha \Big| \int_x^t \frac{q(\xi)}{r(\xi)} \, d\xi \Big|\right), \qquad x, t \in \mathbb{R}\,. \tag{5.13}$$

Then equation (1.1) *is correctly solvable in* L_p *for all* $p > \alpha$.

The next assertion gives us a way for checking (5.13).

Theorem 5.6. *Suppose that conditions* (1.2) *hold and, in addition.*

1)

$$r \in AC^{\mathrm{loc}}(\mathbb{R})\,; \tag{5.14}$$

2) *there exist functions* q_1 *and* q_2 *such that*

$$q = q_1 + q_2 \tag{5.15}$$

$$0 < q_1 \in L_1^{\mathrm{loc}}(\mathbb{R})\,, \qquad q_2 \in L_1^{\mathrm{loc}}(\mathbb{R})\,; \tag{5.16}$$

3) *there exists* $\alpha > 0$ *such that*

$$|r'(x)| \leq \alpha q_1(x)\,, \qquad x \in \mathbb{R}\,; \tag{5.17}$$

4) *the following inequality holds:*

$$C_0 = \sup_{x,t \in \mathbb{R}} \Big| \int_x^t \frac{q_2(\xi)}{r(\xi)} \, d\xi \Big| < \infty\,. \tag{5.18}$$

Then the estimate (5.13) *holds.*

Theorem 5.7. *Suppose that the following conditions hold.*

1) $m(a) > 0$ *for some* $a \in (0, \infty)$ *(see* (5.11)*);*

2)

$$r_0 > 0\,, \qquad r_0 \overset{\mathrm{def}}{=} \inf_{x \in \mathbb{R}} r(x)\,. \tag{5.19}$$

Then (1.1) *is correctly solvable in* L_p *for all* $p \in [1, \infty]$.

Theorem 5.8. *Let* $m(a) > 0$ *and* $\ell(a) < \infty$ *for some* $a \in (0, \infty)$. *Then equation* (1.1) *is correctly solvable in* $C(\mathbb{R})$.

6. Proofs of additional assertions

Proof of Theorem 5.1. Suppose that for a given $p \in [1,\infty]$ conditions I)–II) hold. Consider the cases 1) $p=1$; 2) $p \in (1,\infty)$; 3) $p=\infty$.

Then we have

1) $S_1=\infty$ (see (2.1)) by Corollary 3.4 and $M_1<\infty$ (see (2.9)) by Theorem 3.3. We now use (2.2):

$$\begin{aligned}\infty > M_1 &= \sup_{x\in\mathbb{R}} \frac{1}{r(x)}\int_{-\infty}^{x} \exp\Big(-\int_t^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\\ &\ge \sup_{x\in\mathbb{R}} \frac{1}{r(x)}\int_{x-d(x)}^{x} \exp\Big(-\int_t^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\\ &\ge \sup_{x\in\mathbb{R}} \frac{d(x)}{r(x)} \exp\Big(-\int_{x-d(x)}^{x+d(x)} \frac{q(\xi)}{r(\xi)}\,d\xi\Big) = e^{-2}\sup_{x\in\mathbb{R}}\frac{d(x)}{r(x)} = c^{-1}B_1\,;\end{aligned}$$

2) $S_1=\infty$ (see (2.1)) by Corollary 3.2 and $M_p<\infty$ (see (2.5)) by Theorem 3.1. We now use (2.2):

$$\begin{aligned}\infty > M_p &= \sup_{x\in\mathbb{R}}\Big(\int_{-\infty}^{x} \exp\Big(-p\int_t^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big)^{1/p}\Big(\int_x^\infty \frac{1}{r(t)^{p'}}\exp\Big(-p'\int_x^t \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big)^{1/p'}\\
&\ge \sup_{x\in\mathbb{R}}\Big(\int_{x-d(x)}^{x} \exp\Big(-p\int_t^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big)^{1/p}\Big(\int_x^{x+d(x)} \frac{1}{r(t)^{p'}}\exp\Big(-p'\int_x^t \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big)^{1/p'}\\
&\ge \sup_{x\in\mathbb{R}}\Big(\int_{x-d(x)}^{x} \exp\Big(-p\int_{x-d(x)}^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big)^{1/p}\\
&\quad\times\Big(\int_x^{x+d(x)} \frac{1}{r(t)^{p'}}\exp\Big(-p'\int_x^{x+d(x)} \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big)^{1/p'}\\
&= \sup_{x\in\mathbb{R}}\exp\Big(-\int_{x-d(x)}^{x+d(x)} \frac{q(\xi)}{r(\xi)}\,d\xi\Big)d(x)^{1/p}\Big(\int_x^{x+d(x)}\frac{dt}{r(t)^{p'}}\Big)^{1/p'} = e^{-2}B_p\,.\end{aligned}$$

3) $S_2=\infty$ (see (3.3)). Then the function d (see (2.2)) is defined on $\mathbb{R}$. This can be proved in the same way as in the case $S_1=\infty$ (see [2]). In addition, it is known (see [2]) that the function $d(x)$ is continuous for $x\in\mathbb{R}$, and $x-d(x)\to\infty$ as $x\to\infty$. Further, $A<\infty$ (see (3.4)). We now use (2.2):

$$\begin{aligned}\infty > A &= \sup_{x\in\mathbb{R}}\int_{x-d(x)}^{\infty}\frac{1}{r(t)}\exp\Big(-\int_{x-d(x)}^{t}\frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\\ &\ge \sup_{x\in\mathbb{R}}\int_{x-d(x)}^{x+d(x)}\frac{1}{r(t)}\exp\Big(-\int_{x-d(x)}^{x+d(x)}\frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt = e^{-2}B_\infty.\end{aligned}$$

$\square$

Proof of Theorem 5.3. We need the following assertion.

Lemma 6.1. *Let $f \in AC^{(1)}_{\mathrm{loc}}(\mathbb{R})$, $x \in \mathbb{R}$, $\eta \geq 0$. Then*

$$\int_{x-\eta}^{x+\eta} f(t)\,dt = 2\eta f(x) + \int_0^\eta \int_0^t \int_{x-\xi}^{x+\xi} f''(s)\,ds\,d\xi\,dt\,. \tag{6.1}$$

Proof. The following relations are obvious:

$$\begin{aligned}
\int_{x-\eta}^{x+\eta} f(t)\,dt &= \int_{x-\eta}^{x} f(t)\,dt + \int_{x}^{x+\eta} f(t)\,dt = \int_0^\eta [f(x+t)+f(x-t)]\,dt \\
&= 2\eta f(x) + \int_0^\eta [f(x+t)-f(x)]\,dt - \int_0^\eta [f(x)-f(x-t)]\,dt \\
&= 2\eta f(x) + \int_0^\eta \int_x^{x+t} f'(\xi)\,d\xi\,dt - \int_0^\eta \int_{x-t}^{x} f'(\xi)\,d\xi\,dt \\
&= 2\eta f(x) + \int_0^\eta \int_0^t [f(x+s)]'\,ds\,dt - \int_0^\eta \int_0^t [f(x-s)]'\,ds\,dt \\
&= 2\eta f(x) + \int_0^\eta \int_0^t \int_{x-s}^{x+s} f''(\tau)\,d\tau\,ds\,dt\,.
\end{aligned}$$

□

Let $x \in \mathbb{R}$, $\eta \geq 0$. Then (see (5.4), (6.1)):

$$\begin{aligned}
F(\eta) &= \int_{x-\eta}^{x+\eta} \frac{q(t)}{r(t)}\,dt = \int_{x-\eta}^{x+\eta} \frac{q_1(t)}{r(t)}\,dt + \int_{x-\eta}^{x+\eta} \frac{q_2(t)}{r(t)}\,dt \\
&= 2\eta \frac{q_1(x)}{r(x)} + \int_0^\eta \int_0^t \int_{x-\xi}^{x+\xi} \Big(\frac{q_1(s)}{r(s)}\Big)''\,ds\,d\xi\,dt + \int_{x-\eta}^{x+\eta} \frac{q_2(t)}{r(t)}\,dt\,.
\end{aligned} \tag{6.2}$$

Set

$$\eta = \eta_1(x) = \frac{3}{2}\frac{r(x)}{q_1(x)}\,, \qquad |x| \gg 1\,. \tag{6.3}$$

In the following relations we use (6.2), (6.3), (5.7), (5.8) and (5.6):

$$\begin{aligned}
F(\eta_1(x)) &\geq 3 - \frac{9}{8}\Big(\frac{r(x)}{q_1(x)}\Big)^2 \sup_{|\xi| \leq 2\frac{r(x)}{q_1(x)}} \Big| \int_{x-\xi}^{x+\xi} \Big(\frac{q_1(s)}{r(s)}\Big)''\,ds\Big| \\
&\quad - \sup_{|\xi| \leq 2\frac{r(x)}{q_1(x)}} \Big| \int_{x-\xi}^{x+\xi} \frac{q_2(s)}{r(s)}\,ds\Big| = 3 - \frac{9}{8}\varkappa_1(x) - \varkappa_2(x) \geq 2\,.
\end{aligned} \tag{6.4}$$

From (6.4) and Theorem 5.2 it follows that $\eta \geq d(x)$, $|x| \gg 1$. Set

$$\eta = \eta_2(x) = \frac{1}{2}\frac{r(x)}{q_1(x)}\,, \qquad |x| \gg 1\,.$$

Then we obtain in a similar way that

$$F(\eta_2(x)) \le 1 + \frac{1}{8}\Big(\frac{r(x)}{q_1(x)}\Big)^2 \sup_{|\xi|\le 2\frac{r(x)}{q_1(x)}} \Big| \int_{x-\xi}^{x+\xi} \Big(\frac{q_1(s)}{r(s)}\Big)'' ds\Big|$$

$$+ \sup_{|\xi|\le \frac{2r(x)}{q_1(x)}} \Big| \int_{x-\xi}^{x+\xi} \frac{q_2(s)}{r(s)}\, ds\Big| \le 1 + \frac{\varkappa_1(x)}{8} + \varkappa_2(x) \le 2\,. \qquad (6.5) \qquad \square$$

From (6.5) and Theorem 5.2 it follows that $\eta_2(x) \le d(x)$, $|x| \gg 1$. Thus we obtained (5.9). Let

$$\varphi(x) = \frac{d(x)q_1(x)}{r(x)}\,, \qquad x \in \mathbb{R}\,. \qquad (6.6)$$

Since the function $d(x)$ is positive and continuous for $x \in \mathbb{R}$, so is the function φ (see (6.6)), (5.3), (5.5)). In addition, $\varphi(x) \in \big[\frac{1}{2}, \frac{3}{2}\big]$ for $|x| \ge |x_0| \gg 1$ (see (5.9)). The function φ is positive and continuous on $[-x_0, x_0]$ and hence is separated from zero and bounded, and we obtain (5.10).

Proof of Theorem 5.5. We need the following assertion.

Lemma 6.2. *Let $\delta > 0$ and $m(a) > 0$, $\ell(a) < \infty$ for some $a \in (0,\infty)$ (see* (5.11) *and* (5.12)*). Then*

$$\sup_{x\in\mathbb{R}} \int_{-\infty}^{x} \frac{1}{r(t)} \exp\Big(-\delta \int_t^x \frac{q(\xi)}{r(\xi)}\, d\xi\Big)\, dt = c(\delta) < \infty\,, \qquad (6.7)$$

$$\sup_{x\in\mathbb{R}} \int_{x}^{\infty} \frac{1}{r(t)} \exp\Big(-\delta \int_x^t \frac{q(\xi)}{r(\xi)}\, d\xi\Big)\, dt = c(\delta) < \infty\,. \qquad (6.8)$$

Proof. Estimates (6.7) and (6.8) are proved in the same way; therefore, we only consider (6.8). Let us introduce segments $\{\Delta_n\}_{n=1}^{\infty}$:

$$\Delta_n = [\Delta_n^-, \Delta_n^+]\,, \qquad \Delta_n^{\pm} = x_n \pm a\,, \qquad x_n = x + (2n-1)a\,, \qquad \Delta_1^- = x\,. \qquad (6.9)$$

We have the equality

$$\Delta_{n+1}^- = \Delta_n^+\,, \qquad n \ge 1\,. \qquad (6.10)$$

Below we use (6.9) and (6.10):

$$\int_x^\infty \frac{1}{r(t)} \exp\Big(-\delta \int_x^t \frac{q(\xi)}{r(\xi)}\, d\xi\Big)\, dt = \sum_{n=1}^{\infty} \int_{\Delta_n} \frac{1}{r(t)} \exp\Big(-\delta \int_{\Delta_1^-}^t \frac{q(\xi)}{r(\xi)}\, d\xi\Big)\, dt$$

$$\le \sum_{n=1}^{\infty} \Big(\int_{\Delta_n} \frac{dt}{r(t)}\Big) \exp\Big(-\delta \int_{\Delta_1^-}^{\Delta_n^-} \frac{q(\xi)}{r(\xi)}\, d\xi\Big)$$

$$\le \ell(a) \sum_{n=1}^{\infty} \exp(-\delta m(a)(n-1)) = c(\delta) < \infty\,.$$

Let us now go over to the main assertion for $p = 1$. By Theorem 3.3, we have to prove that $M_1 < \infty$ (see (2.9)).

Below we use the inequalities (5.13), $m(a) > 0$, $\ell(a) < \infty$ and (6.7):

$$\begin{aligned} M_1 &= \sup_{x\in\mathbb{R}} \frac{1}{r(x)} \int_{-\infty}^{x} \exp\Big(-\int_t^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt \\ &= \sup_{x\in\mathbb{R}} \int_{-\infty}^{x} \frac{r(t)}{r(x)}\frac{1}{r(t)} \exp\Big(-\int_t^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt \\ &\le \sup_{x\in\mathbb{R}} \int_{-\infty}^{x} \frac{1}{r(t)} \exp\Big(-(1-\alpha)\int_t^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt < \infty\,. \end{aligned}$$

For $p \in (1,\infty)$, by Theorem 3.1, we have to show that $M_p < \infty$ (see (2.5)). Below, for $p \in (1,\infty)$, we use the same tools as in the case $p = 1$ (see above):

$$\begin{aligned} M_p &= \sup_{x\in\mathbb{R}} \Big[\int_{-\infty}^{x} \exp\Big(-p\int_t^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big]^{1/p} \Big[\int_x^\infty \frac{1}{r(t)^{p'}} \exp\Big(-p'\int_x^t \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big]^{1/p'} \\ &= \sup_{x\in\mathbb{R}} \Big[\int_{-\infty}^{x} \frac{r(t)}{r(x)}\frac{1}{r(t)} \exp\Big(-p\int_t^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big]^{1/p} \\ &\quad\times \Big[\int_x^\infty \Big(\frac{r(x)}{r(t)}\Big)^{p'-1} \frac{1}{r(t)} \exp\Big(-p'\int_x^t \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\Big]^{1/p'} \\ &\le \sup_{x\in\mathbb{R}} \Big[\int_{-\infty}^{x} \frac{1}{r(t)} \exp\Big(-(p-\alpha)\int_t^x \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big]^{1/p'} \\ &\quad\times \Big[\int_x^\infty \frac{1}{r(t)} \exp\Big(-(p'-\alpha(p'-1))\int_x^t \frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big]^{1/p'} \\ &\le c\sup_{x\in\mathbb{R}} \Big[\int_x^\infty \frac{1}{r(t)} \exp\Big(-p'\Big(1-\frac{\alpha}{p}\Big)\int_x^t \frac{q(\xi)}{r(\xi)}\Big)\,dt\Big]^{1/p'} < \infty\,. \end{aligned}$$

□

Proof of Theorem 5.6. The assumptions of the theorem allow one to deduce the following obvious implications:

$$-\alpha q_1(\xi) \le r'(\xi) \le \alpha q_1(\xi), \qquad \xi\in\mathbb{R} \quad \Rightarrow$$

$$-\alpha\frac{q_1(\xi)}{r(\xi)} \le \frac{r'(\xi)}{r(\xi)} \le \alpha\frac{q_1(\xi)}{r(\xi)}\,, \qquad \xi\in\mathbb{R}\,. \tag{6.11}$$

Let, say, $x \le t$ (the case $x \ge t$ can be considered in the same way). From (6.11) it follows that

$$-\alpha\int_x^t \frac{q_1(\xi)}{r(\xi)}d\xi \le \ln\frac{r(t)}{r(x)} \le \alpha\int_x^t \frac{q_1(\xi)}{r(\xi)}\,d\xi \quad \Rightarrow$$

$$\begin{aligned} \ln\frac{r(t)}{r(x)} &\le \alpha\int_x^t \frac{q(\xi)-q_2(\xi)}{r(\xi)}\,d\xi = \alpha\int_x^t \frac{q(\xi)}{r(\xi)}\,d\xi - \alpha\int_x^t \frac{q_2(\xi)}{r(\xi)}\,d\xi \\ &\le \alpha\Big|\int_x^t \frac{q(\xi)}{r(\xi)}\,d\xi\Big| + \alpha\Big|\int_x^t \frac{q_2(\xi)}{r(\xi)}\,d\xi\Big| \le \alpha\Big|\int_x^t \frac{q(t)}{r(\xi)}\,d\xi\Big| + c\,, \end{aligned}$$

$$\begin{aligned}\ln\frac{r(t)}{r(x)} &\ge -\alpha\int_x^t \frac{q_1(\xi)}{r(\xi)}\,d\xi = -\alpha\int_x^t\frac{q(\xi)-q_2(\xi)}{r(\xi)}\,d\xi\\ &= -\alpha\int_x^t\frac{q(\xi)}{r(\xi)}d\xi + \alpha\int_x^t\frac{q_2(\xi)d\xi}{r(\xi)}\\ &\ge -\alpha\Big|\int_x^t\frac{q(\xi)d\xi}{r(\xi)}\Big| - \alpha\Big|\int_x^t\frac{q_2(\xi)d\xi}{r(\xi)}\Big| \ge -\alpha\Big|\int_x^t\frac{q(\xi)d\xi}{r(\xi)}\Big| - c\,.\end{aligned}$$

This implies (5.13). □

Proof of Theorem 5.7. We need the following assertion.

Lemma 6.3. *Suppose that the conditions of the theorem hold and $\delta > 0$. Then one has the inequalities*

$$\sup_{x\in\mathbb{R}}\int_{-\infty}^x \exp\Big(-\delta\int_t^x\frac{q(\xi)}{r(\xi)}d\xi\Big)\,dt = c < \infty\,, \tag{6.12}$$

$$\sup_{x\in\mathbb{R}}\int_x^\infty \exp\Big(-\delta\int_x^t\frac{q(\xi)}{r(\xi)}d\xi\Big)\,dt = c < \infty\,. \tag{6.13}$$

Proof. Below we use the notation of the proof of Lemma 6.2.

$$\begin{aligned}\sup_{x\in\mathbb{R}}\int_x^\infty \exp\Big(-\delta\int_x^t\frac{q(\xi)}{r(\xi)}\,d\xi\Big) &= \sum_{n=1}^\infty\int_{\Delta_n}\exp\Big(-\delta\int_{\Delta_1^-}^t\frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\\ &\le \sum_{n=1}^\infty 2a\exp\Big(-\delta\int_{\Delta_1^-}^{\Delta_n^-}\frac{q(\xi)}{r(\xi)}\,d\xi\Big) \le 2a\sum_{n=1}^\infty\exp(-\delta m(a)(n-1)) = c<\infty\,.\end{aligned}$$

Inequality (6.12) is checked in the same way. □

To finish the proof, we use Lemma 6.3 and Theorems 3.1 and 3.3. For $p = 1$ and $p\in(1,\infty)$, we get, respectively,

$$\begin{aligned}M_1 &= \sup_{x\in\mathbb{R}}\frac{1}{r(x)}\int_{-\infty}^x\exp\Big(-\int_t^x\frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\\ &\le \frac{1}{r_0}\sup_{x\in\mathbb{R}}\int_{-\infty}^x\exp\Big(-\int_t^x\frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt = c<\infty\,,\end{aligned}$$

$$\begin{aligned}M_p &= \sup_{x\in\mathbb{R}}\Big(\int_{-\infty}^x\exp\Big(-p\int_t^x\frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big)^{1/p}\\ &\quad\times\Big(\int_x^\infty\frac{1}{r(t)}p'\exp\Big(-p'\int_x^t\frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\Big)^{1/p'}\\ &\le\frac{c}{r_0}\sup_{x\in\mathbb{R}}\Big(\int_x^\infty\exp\Big(-p'\int_x^t\frac{q(\xi)}{r(\xi)}\,ds\Big)\,dt\Big)^{1/p'} = c<\infty\,.\end{aligned}$$

□

Proof of Theorem 5.8. This is an immediate consequence of Lemma 6.2 and Theorem 3.5. □

7. Examples

As an example, we have the following assertion.

Theorem 7.1. [5] *Suppose that in* (1.2) *we have* $r \equiv 1$ *and* $p \in [1, \infty]$. *Then equation* (1.1) *is correctly solvable in* L_p *if and only if* $q_0(a) > 0$ *for some* $a \in (0, \infty)$. *Here*

$$q_0(a) = \inf_{x \in \mathbb{R}} \int_{x-a}^{x+a} q(t)\, dt\,. \tag{7.1}$$

Proof of Theorem 7.1. *Necessity.*
From Corollaries 3.2, 3.4 and Theorem 3.5, we obtain

$$\int_{-\infty}^{\infty} q(t)\, dt = \infty \tag{7.2}$$

and therefore the function $d(x)$, $x \in \mathbb{R}$ (see (2.2)), is defined. Further, from Theorem 5.1 and the equality $r \equiv 1$ it follows that

$$\infty > B_p \geq \sup_{x \in \mathbb{R}} d(x) := d_0, \qquad p \in [1, \infty]. \tag{7.3}$$

From (7.3) we now get (see (7.1) and (2.2))

$$q_0(d_0) = \inf_{x \in \mathbb{R}} \int_{x-d_0}^{x+d_0} q(t)\, dt \geq \inf_{x \in \mathbb{R}} \int_{x-d(x)}^{x+d(x)} q(t)\, dt = 2\,.$$

Proof of Theorem 7.1. *Sufficiency.*
Since $d_0 < \infty$ (see (7.3) and Theorem 5.4), we have (see (5.11) and (7.1))

$$m(d_0) = q_0(d_0) \geq 1\,, \qquad r_0 = 1$$

and it remains to refer to Theorem 5.7. □

Below we give an example of an equation which is correctly solvable in L_p, $p \in [1, \infty]$, for which the boundary problem (2.3)–(2.4) is not correctly solvable. See [2, 3] for various situations related to (2.3)–(2.4).
Consider equation (1.1) with

$$r(x) = \frac{1}{2(1+x^2)}\,, \qquad q(x) = 1\,, \qquad x \in \mathbb{R}\,. \tag{7.4}$$

We need the following assertion.

Lemma 7.2. *Suppose that* (7.4) *holds and we are given numbers* α, β, γ *satisfying the following conditions:*

$$\alpha \geq 1\,, \qquad |\gamma - 1| < 2\beta\,. \tag{7.5}$$

Then we have inequalities

$$J(x, \alpha) \leq c r(x)\,, \qquad I(x, \beta, \gamma) \leq c r(x)^{1-\gamma}\,, \qquad x \in R\,. \tag{7.6}$$

Here

$$J(x,\alpha)=\int_{-\infty}^{x}\exp\Big(-\alpha\int_t^x\frac{q(\xi)}{r(\xi)}\Big)\,dt\,,\qquad x\in\mathbb{R}\,,\tag{7.7}$$

$$I(x,\beta,\gamma)=\int_x^{\infty}\frac{1}{r(t)^{\gamma}}\exp\Big(-\beta\int_x^t\frac{q(\xi)}{r(\xi)}\,d\xi\Big)\,dt\,,\qquad x\in\mathbb{R}.\tag{7.8}$$

Proof. Inequalities (7.7) and (7.8) are proved in the same way, by integrating by parts. Therefore we only prove (7.8):

$$\begin{aligned}I(x,\beta,\gamma)&=\int_x^{\infty}\frac{1}{r(t)^{\gamma}}\exp\Big(-\beta\int_x^t\frac{d\xi}{r(\xi)}\Big)\,dt\\&=-\frac{1}{\beta}\int_x^{\infty}\frac{1}{r(t)^{\gamma-1}}d\Big(\exp\Big(-\beta\int_x^t\frac{d\xi}{r(\xi)}\Big)\Big)\\&=\frac{1}{\beta r(x)^{\gamma-1}}-\frac{\gamma-1}{\beta}\int_x^{\infty}\frac{r'(t)}{r(t)^{\gamma}}\exp\Big(-\beta\int_x^t\frac{d\xi}{r(\xi)}\Big)\,dt\\&\le\frac{1}{\beta r(x)^{\gamma-1}}+\frac{|\gamma-1|}{2\beta}I(x,\beta,\gamma)\quad\Rightarrow\quad(7.8).\end{aligned}$$

□

Let $p=1$. By Lemma 7.2, we obtain $M_1<\infty$ and then by Theorem 3.3, equation (1.1) in the case (7.4) is correctly solvable in L_1. Since here we have $r_0=0$ (see (2.8)), by Theorem 2.3, the boundary problem (2.3)–(2.4) is not correctly solvable in L_1. Let $p\in(1,\infty)$. By Lemma 7.2, for $\alpha=p$, $\gamma=\beta=p'$, we obtain $M_p<\infty$; therefore, by Theorem 3.1, equation (1.1) in the case (7.4) is correctly solvable in L_p. Let us estimate $A_{p'}$ (see (2.7)). We apply Theorem 5.3 to the pair of functions (7.4) and easily obtain

$$\frac{c^{-1}}{1+x^2}\le d(x)\le\frac{c}{1+x^2}\,,\qquad x\in\mathbb{R}\,.\tag{7.9}$$

Since $p'>1$, from (7.9), we easily get the following relations (see (2.7)):

$$A_{p'}=\sup_{x\in\mathbb{R}}A_{p'}(x)=\sup_{x\in\mathbb{R}}\int_{x-d(x)}^{x+d(x)}\frac{dt}{r(t)^{p'}}\ge c^{-1}\sup_{x\in\mathbb{R}}(1+x^2)^{p'-1}=\infty.$$

Hence, problem (2.3)–(2.4) is not correctly solvable in L_p for $p\in(1,\infty)$, due to Theorem 2.2.

Let $p=\infty$. In this case, $A<\infty$ by Lemma 7.2, and by Theorem 3.5 equation (1.1) in the case (7.4) is correctly solvable in $C(\mathbb{R})$. Further, we establish the equality $A_0=1$ (see (2.10)) in a straightforward way; therefore, problem (2.3)–(2.4) is not correctly solvable in $C(\mathbb{R})$ by Theorem 2.4.

References

[1] Chernyavskaya, N., *Conditions for correct solvability of a simplest singular boundary value problem*, Math. Nachr. **243** (2002), 5–18.

[2] Chernyavskaya, N., Shuster, L., *Conditions for correct solvability of a simplest singular boundary value problem of general form. I*, Z. Anal. Anwendungen **25** (2006), 205–235.

[3] Chernyavskaya, N., Shuster, L., *Conditions for correct solvability of a simplest singular boundary value problem of general form. II*, Z. Anal. Anwendungen **26** (2007), 439–458.

[4] Kantorovich, L.W., Akilov, G.P., *Functional Analysis*, Nauka, Moscow, 1977.

[5] Lukachev, M., Shuster, L., *On uniqueness of soltuion of a linear differential equation without boundary conditions*, Funct. Differ. Equ. **14** (2007), 337–346.

[6] Mynbaev, K., Otelbaev, M., *Weighted Function Spaces and the Spectrum of Differential Operators*, Nauka, Moscow, 1988.

[7] Opic, B., Kufner, A., *Hardy Type Inequalities*, Pitman Research Notes in Mathematics Series, vol. 219, Harlow, Longman, 1990.

[8] Otelbaev, M., *Estimates of the Spectrum of the Sturm-Liouville Operator*, Alma-Ata, Gilim, 1990, in Russian.

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