

**CIRCULAR UNITS OF REAL ABELIAN FIELDS  
WITH FOUR RAMIFIED PRIMES**

VLADIMÍR SEDLÁČEK

**ABSTRACT.** In this paper we study the groups of circular numbers and circular units in Sinnott's sense in real abelian fields with exactly four ramified primes under certain conditions. More specifically, we construct  $\mathbb{Z}$ -bases for them in five special infinite families of cases. We also derive some results about the corresponding module of relations (in one family of cases, we show that the module of Ennola relations is cyclic). The paper is based upon the thesis [6], which builds upon the results of the paper [2].

1. INTRODUCTION

Circular units appear in many situations in algebraic number theory because in some sense, for a given abelian field, they form a good approximation of the full group of units, which is usually very hard to describe explicitly. The index of the group of circular units in the full group of units is closely related to the class number of the maximal real subfield of the respective field, which was already known to E. Kummer in the case of a prime-power cyclotomic field and which was generalized by W. Sinnott to any abelian field. Circular units can be also used for a construction of annihilators of ideal class group of a given real abelian field, which was discovered by F. Thaine and generalized by K. Rubin (see [8] and [4]).

In contrast to the full group of units, the Sinnott group of circular units is given by explicit generators, nevertheless a  $\mathbb{Z}$ -basis of this group was described only in a few very special cases, for example when the abelian field is cyclotomic, has at most two ramified primes, or has three ramified primes and satisfies some other conditions. More details can be found in [1] and [2].

The aim of this paper is to present new results in the case of a real abelian field having four ramified primes under some other assumptions. Additionally, we will also explore the structure of the module of all relations (among the generators of the group of circular numbers) modulo the norm relations.

Except for some parts of Section 7, all results in this self-contained paper come from the author's thesis [6], where they are usually explained in greater detail (and

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the notation used there is exactly the same as here). In particular, the complete proofs of the theorems in Section 6 can be found there.

2. BASIC DEFINITIONS AND RESULTS ABOUT CIRCULAR NUMBERS AND UNITS

For the remainder of this section, let  $k \neq \mathbb{Q}$  be a real abelian field,  $K$  be its genus field in the narrow sense,  $P$  be the set of ramified primes of  $k/\mathbb{Q}$  and  $K_p$  be the maximal subfield of  $K$  ramified over  $\mathbb{Q}$  only at  $p \in P$ . Since  $\text{Gal}(K/\mathbb{Q})$  has a natural action on  $K$  (given by evaluating an automorphism on an element), this makes  $K$  and  $K^\times$  into  $\mathbb{Z}[\text{Gal}(K/\mathbb{Q})]$ -modules.

**Definition 2.1.** The group  $D(k)$  of circular numbers of  $k$  is given as

$$D(k) := \langle \{-1\} \cup \{\eta_I \mid \emptyset \subsetneq I \subseteq P\} \rangle_{\mathbb{Z}[\text{Gal}(K/\mathbb{Q})]},$$

where  $\langle \dots \rangle_{\mathbb{Z}[\text{Gal}(K/\mathbb{Q})]}$  means “generated as a  $\mathbb{Z}[\text{Gal}(K/\mathbb{Q})]$ -submodule of  $K^\times$ ” and

$$\eta_I := N_{\mathbb{Q}} \left( \zeta_{\text{cond} \left( \prod_{i \in I} K_i \right)} \right) / \left( \prod_{i \in I} K_i \right) \cap k \left( 1 - \zeta_{\text{cond} \left( \prod_{i \in I} K_i \right)} \right),$$

where  $N$  denotes the norm operator,  $\text{cond}(L)$  is the conductor of an abelian field  $L$ , and the product of fields denotes their compositum. The subgroup of totally positive elements of  $D(k)$  will be denoted by  $D^+(k)$ .

**Definition 2.2.** The group  $C(k)$  of circular numbers of  $k$  is  $E(k) \cap D(k)$ , where  $E(k)$  is the group of units of the ring of algebraic integers of  $k$ . The subgroup of totally positive elements of  $C(k)$  will be denoted by  $C^+(k)$ .

In [3], it is proven that the above definition of  $C(k)$  gives the same group as Sinnott’s original definition in [7].

Here are a few well known facts about circular units:

**Lemma 2.3.** Let  $\emptyset \subsetneq I \subseteq P$ .

- (1) For  $|I| > 1$ , we have  $\eta_I \in E(k)$ .
- (2) For  $|I| = 1$ , we have  $\eta_I \notin E(k)$ , but  $\eta_I^{1-\sigma} \in E(k)$  for any  $\sigma \in \text{Gal}(K/\mathbb{Q})$ .

**Corollary 2.4.** We have

$$C(k) = \langle \{-1\} \cup \{\eta_I \mid I \subseteq P, |I| \geq 2\} \cup \{\eta_{\{p\}}^{1-\sigma} \mid p \in P, \sigma \in \text{Gal}(K/\mathbb{Q})\} \rangle_{\mathbb{Z}[\text{Gal}(K/\mathbb{Q})]}$$

and

$$C^+(k) = \langle \{\eta_I \mid I \subseteq P, |I| \geq 2\} \cup \{\eta_{\{p\}}^{1-\sigma} \mid p \in P, \sigma \in \text{Gal}(K/\mathbb{Q})\} \rangle_{\mathbb{Z}[\text{Gal}(K/\mathbb{Q})]}.$$

**Proposition 2.5.** The  $\mathbb{Z}$ -rank of  $D^+(k)$  is  $[k : \mathbb{Q}] + |P| - 1$  and the  $\mathbb{Z}$ -rank of  $C^+(k)$  is  $[k : \mathbb{Q}] - 1$ .

**Lemma 2.6.** If  $L' \subseteq L$  are abelian fields, then for any  $\epsilon \in C(L)$  (or  $C^+(L)$ ) we have  $N_{L/L'}(\epsilon) \in C(L')$  (or  $C^+(L')$ ), respectively.

3. THE SPECIAL CASE OF FOUR RAMIFIED PRIMES

In the remainder of the paper, we will fix  $k$  to be a real abelian field with exactly four ramified primes  $p_1, p_2, p_3, p_4$  and we will abbreviate  $D(k), D^+(k), C(k), C^+(k)$  simply as  $D, D^+, C, C^+$ . We will also use the convention that whenever any of the indices  $i, j, l, h$  appear on the same line, they denote pairwise distinct integers satisfying  $1 \leq i, j, l, h \leq 4$ , unless stated otherwise. Finally, for any positive integer  $n$ ,  $\zeta_n$  will denote a primitive  $n$ -th root of unity (without loss of generality we can take  $\zeta_n = e^{2\pi i/n}$ ).

Let  $K$  be the genus field in the narrow sense of  $k$  and let  $G := \text{Gal}(K/\mathbb{Q})$ . Then we can identify  $G$  with the direct product  $T_1 \times T_2 \times T_3 \times T_4$ , where  $T_i$  is the inertia group corresponding the ramified prime  $p_i$ . Next, we will define:

- $H := \text{Gal}(K/k)$ ,
- $m := |H|$ ,
- the canonical projections  $\pi_i: G \rightarrow T_i$ ,
- $a_i := [T_i : \pi_i(H)]$ ,
- $r_i := |H \cap \ker \pi_i|$ ,
- $s_{ij} := |H \cap \ker(\pi_i \pi_j)|$ ,
- $n_i := \frac{m}{r_i}$ ,
- $\eta := \eta_{\{p_1, p_2, p_3, p_4\}}$ ,
- $K_i$  as the maximal subfield of  $K$  ramified only at  $p_i$ .

Note that we have  $T_i = \text{Gal}(K/K_j K_l K_h) \cong \text{Gal}(K_i/\mathbb{Q})$  and

$$K = kK_i K_j K_l = K_1 K_2 K_3 K_4$$

by ramification theory.

**Assumption 3.1.** In the remainder of the paper, we will assume the following:

- $H$  is cyclic, generated by  $\tau$ ,
- each  $T_i$  is cyclic, generated by  $\sigma_i$ .

Note that the second assumption isn't very restrictive, as it is automatically true for example if all the ramified primes of  $k$  are odd (because  $T_i \cong \text{Gal}(K_i/\mathbb{Q})$  is a quotient of the Galois group  $\text{Gal}(\mathbb{Q}(\zeta_{\text{cond}(K_i)})/\mathbb{Q}) \cong (\mathbb{Z}/p_i^f)^\times$  for some positive integer  $f$ ).

**Lemma 3.2.** *Without loss of generality, we can assume  $\tau = \sigma_1^{a_1} \sigma_2^{a_2} \sigma_3^{a_3} \sigma_4^{a_4}$ .*

**Proof.** We know that  $a_i = [T_i : \pi_i(H)]$ , hence  $\pi_i(\tau)$  generates a subgroup of  $T_i$  of index  $a_i$ . The cyclicity of  $T_i$  then implies that  $\pi_i(\tau)$  must be the  $a_i$ -th power of some generator of  $T_i$ , without loss of generality  $\sigma_i$ . The statement now follows, because  $\tau$  is determined by its four projections. □

**Proposition 3.3.** *We have*

$$\begin{aligned}
 [k \cap K_i : \mathbb{Q}] &= a_i, \\
 [K : kK_i] &= r_i \\
 |T_i| &= a_i n_i, \\
 [K : kK_i K_j] &= s_{ij}, \\
 [K_i : k \cap K_i] &= |\pi_i(H)| = n_i, \\
 [K_i K_j : k \cap K_i K_j] &= |\pi_i \pi_j(H)| = \frac{m}{s_{ij}}
 \end{aligned}$$

and

$$[K_i K_j K_l : k \cap K_i K_j K_l] = |\pi_i \pi_j \pi_l(H)| = m.$$

**Proof.** Since

$$\begin{aligned}
 \text{Gal}(K/K_i) &= \text{Gal}(K/K_i K_j K_l \cap K_i K_j K_h \cap K_i K_l K_h) \\
 &= \text{Gal}(K/K_i K_j K_l) \cdot \text{Gal}(K/K_i K_j K_h) \cdot \text{Gal}(K/K_i K_l K_h) = T_j T_l T_h
 \end{aligned}$$

and  $\text{Gal}(K/k) = H$ , it follows that  $\text{Gal}(K/k \cap K_i) = T_j T_l T_h \cdot H$ . Now consider the short exact sequence

$$0 \rightarrow H \cap \ker \pi_i \rightarrow H \xrightarrow{\pi_i|_H} \pi_i(H) \rightarrow 0.$$

It follows that  $|\pi_i(H)| = \frac{m}{r_i} = n_i$  and

$$\begin{aligned}
 \pi_i(H) &\cong \frac{H}{H \cap \ker \pi_i} = \frac{H}{H \cap T_j T_l T_h} \cong \frac{T_j T_l T_h \cdot H}{T_j T_l T_h} \\
 &= \frac{\text{Gal}(K/k \cap K_i)}{\text{Gal}(K/K_i)} \cong \text{Gal}(K_i/k \cap K_i).
 \end{aligned}$$

Therefore

$$[k \cap K_i : \mathbb{Q}] = \frac{|\text{Gal}(K_i/\mathbb{Q})|}{|\text{Gal}(K_i/k \cap K_i)|} = \frac{|T_i|}{|\pi_i(H)|} = a_i$$

and

$$[K : kK_i] = \frac{|\text{Gal}(K/k)|}{|\text{Gal}(kK_i/k)|} = \frac{|H|}{|\text{Gal}(K_i/k \cap K_i)|} = \frac{m}{|\pi_i(H)|} = r_i.$$

Putting everything together, we obtain

$$|T_i| = [K_i : k \cap K_i] \cdot [k \cap K_i : \mathbb{Q}] = a_i |\pi_i(H)| = a_i n_i.$$

Next, we also have

$$\begin{aligned}
 \text{Gal}(K/K_i K_j) &= \text{Gal}(K/K_i K_j K_l \cap K_i K_j K_h) \\
 &= \text{Gal}(K/K_i K_j K_l) \cdot \text{Gal}(K/K_i K_j K_h) = T_l T_h
 \end{aligned}$$

so that  $\text{Gal}(K/k \cap K_i K_j) = T_l T_h \cdot H$ . Thus we can consider the short exact sequence

$$0 \rightarrow H \cap \ker \pi_i \pi_j \rightarrow H \xrightarrow{\pi_i \pi_j|_H} \pi_i \pi_j(H) \rightarrow 0$$

to conclude that  $|\pi_i \pi_j(H)| = \frac{m}{s_{ij}}$  and

$$\begin{aligned} \pi_i \pi_j(H) &\cong \frac{H}{H \cap \ker \pi_i \pi_j} = \frac{H}{H \cap T_l T_h} \cong \frac{T_l T_h \cdot H}{T_l T_h} \\ &\cong \frac{\text{Gal}(K/k \cap K_i K_j)}{\text{Gal}(K/K_i K_j)} \cong \text{Gal}(K_i K_j/k \cap K_i K_j). \end{aligned}$$

Then it follows that

$$[K : kK_i K_j] = \frac{|\text{Gal}(K/k)|}{|\text{Gal}(kK_i K_j/k)|} = \frac{|H|}{|\text{Gal}(K_i K_j/k \cap K_i K_j)|} = \frac{m}{|\pi_i \pi_j(H)|} = s_{ij}.$$

Finally, we have

$$\text{Gal}(K_i K_j K_l/k \cap K_i K_j K_l) \cong \text{Gal}(kK_i K_j K_l/k) = \text{Gal}(K/k) = H,$$

and we can consider the short exact sequence

$$0 \rightarrow H \cap \ker \pi_i \pi_j \pi_l \rightarrow H \xrightarrow{\pi_i \pi_j \pi_l|_H} \pi_i \pi_j \pi_l(H) \rightarrow 0$$

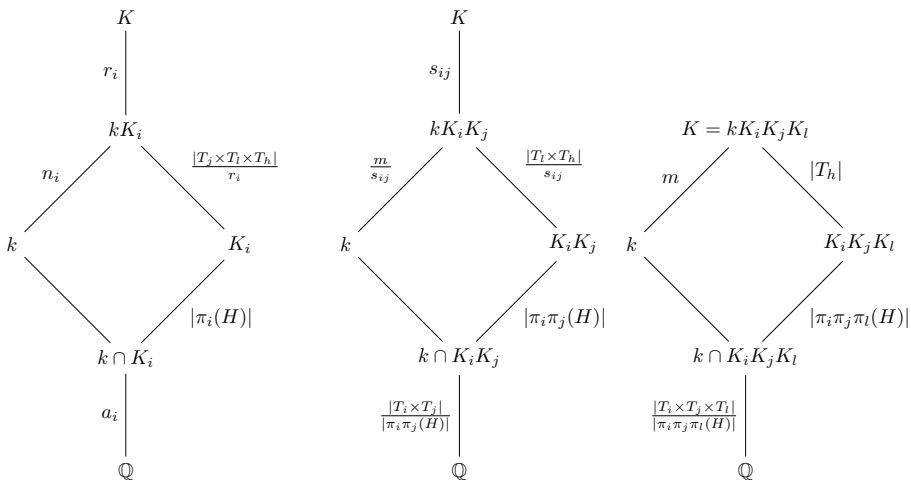
where

$$H \cap \ker \pi_i \pi_j \pi_l = H \cap T_h = \text{Gal}(K/kK_i K_j K_l) = 0.$$

Thus  $|\pi_i \pi_j \pi_l(H)| = m$  and

$$\begin{aligned} \pi_i \pi_j \pi_l(H) &\cong H \cong \frac{T_h \cdot H}{T_h} \\ &\cong \frac{\text{Gal}(K/k \cap K_i K_j K_l)}{\text{Gal}(K/K_i K_j K_l)} \cong \text{Gal}(K_i K_j K_l/k \cap K_i K_j K_l). \end{aligned}$$

□



**Remark 3.4.** Note that Proposition 3.3 implies that  $a_i n_i \neq 1$ , otherwise  $T_i$  would be trivial and  $p_i$  wouldn't ramify in  $k$ .

**Corollary 3.5.** *We have*

$$[k \cap K_i K_j : \mathbb{Q}] = a_i a_j \frac{m}{r_i r_j} s_{ij},$$

$$[k \cap K_i K_j K_l : \mathbb{Q}] = a_i a_j a_l \frac{m^2}{r_i r_j r_l}$$

and

$$[k : \mathbb{Q}] = a_1 a_2 a_3 a_4 \frac{m^3}{r_1 r_2 r_3 r_4}.$$

**Proof.** This follows from the computations

$$[k \cap K_i K_j : \mathbb{Q}] = \frac{[K_i K_j : \mathbb{Q}]}{[K_i K_j : k \cap K_i K_j]} = \frac{|T_i| \cdot |T_j|}{m/s_{ij}} = a_i a_j \frac{m}{r_i r_j} s_{ij},$$

$$[k \cap K_i K_j K_l : \mathbb{Q}] = \frac{[K_i K_j K_l : \mathbb{Q}]}{[K_i K_j K_l : k \cap K_i K_j K_l]} = \frac{|T_i| \cdot |T_j| \cdot |T_l|}{m} = a_i a_j a_l \frac{m^2}{r_i r_j r_l}$$

and

$$[k : \mathbb{Q}] = \frac{[K : \mathbb{Q}]}{[K : k]} = \frac{|T_1| \cdot |T_2| \cdot |T_3| \cdot |T_4|}{m} = a_1 a_2 a_3 a_4 \frac{m^3}{r_1 r_2 r_3 r_4}.$$

□

**Lemma 3.6.** *We have*

$$\gcd(r_i, r_j) = s_{ij},$$

$$\gcd(r_i, r_j, r_l) = 1,$$

$$\text{lcm}(n_i, n_j, n_l) = m$$

and

$$\gcd(n_i, n_j) = s_{ij} \frac{m}{r_i r_j}.$$

**Proof.** It follows from Proposition 3.3 that  $s_{ij} \mid r_i$ ,  $s_{ij} \mid r_j$  and

$$|\pi_i(H)| = n_i, \quad |\pi_i \pi_j(H)| = \frac{m}{s_{ij}} \quad \text{and} \quad |\pi_i \pi_j \pi_l(H)| = m.$$

The cyclicity of  $H$  then implies

$$\frac{m}{s_{ij}} = |\pi_i \pi_j(H)| = |\langle \pi_i \pi_j(\tau) \rangle| = |\langle \pi_i(\tau) \pi_j(\tau) \rangle| = \text{lcm}(n_i, n_j),$$

because  $\langle \pi_i(\tau) \rangle = \pi_i(H)$  and any power of the product  $\pi_i(\tau) \pi_j(\tau)$  is trivial if and only if the same power of both its factors is (since  $G$  is the direct product of the  $T_i$ 's). Now for any common divisor  $t$  of  $r_i, r_j$ , we have

$$\frac{m}{s_{ij}} = \text{lcm}(n_i, n_j) = \text{lcm}\left(\frac{m}{r_i}, \frac{m}{r_j}\right) \Big| \frac{m}{t},$$

which implies  $t \mid s_{ij}$ . Hence  $s_{ij} = \gcd(r_i, r_j)$ .

Similarly, we can compute

$$m = |\pi_i \pi_j \pi_l(H)| = |\langle \pi_i \pi_j \pi_l(\tau) \rangle| = |\langle \pi_i(\tau) \pi_j(\tau) \pi_l(\tau) \rangle| = \text{lcm}(n_i, n_j, n_l).$$

In addition, if  $t$  is any positive common divisor of  $r_i, r_j, r_l$ , we have

$$m = \text{lcm}(n_i, n_j, n_l) = \text{lcm}\left(\frac{m}{r_i}, \frac{m}{r_j}, \frac{m}{r_l}\right) \mid \frac{m}{t},$$

which implies  $t = 1$ , hence  $\gcd(r_i, r_j, r_l) = 1$ .

Finally, using the first result, we have

$$s_{ij} \frac{m}{r_i r_j} = \frac{m}{r_i r_j / s_{ij}} = \frac{m}{\text{lcm}(r_i, r_j)},$$

which clearly divides both  $\frac{m}{r_i} = n_i$  and  $\frac{m}{r_j} = n_j$ . Moreover, if  $t$  is any common divisor of  $n_i = \frac{m}{r_i}$  and  $n_j = \frac{m}{r_j}$ , then both  $r_i t$  and  $r_j t$  divide  $m$ , hence

$$t \cdot \text{lcm}(r_i, r_j) = \text{lcm}(r_i t, r_j t) \mid m.$$

Thus  $t \mid \frac{m}{\text{lcm}(r_i, r_j)}$  and we are done. □

**Remark 3.7.** If  $k$  is fixed, we have shown in Lemmas 3.3 and 3.6 and Remark 3.4 that

$$r_i \mid m, \gcd(r_i, r_j, r_l) = 1, a_i n_i \neq 1.$$

Conversely, using the theory of Dirichlet characters, it can be shown that for any choice of positive integers  $m, a_1, a_2, a_3, a_4, r_1, r_2, r_3, r_4$  satisfying

$$r_i \mid m, \gcd(r_i, r_j, r_l) = 1, a_i n_i \neq 1,$$

there exist infinitely many real abelian fields  $k$  ramified at exactly four primes satisfying the assumptions on page 223 (in particular, the family of fields we are studying is nonempty). The proof of this is analogous to the proof of a similar statement in Chapter 6 of [5] and we omit it.

**Proposition 3.8.** *We have*

$$\text{Gal}(k/\mathbb{Q}) \cong \left\{ \sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{x_3} \sigma_4^{x_4} \mid_k; 0 \leq x_1 < a_1 \frac{m}{r_1}, 0 \leq x_2 < a_2 \frac{m}{r_2 s_{34}}, \right. \\ \left. 0 \leq x_3 < a_3 \frac{m}{r_3 r_4} s_{34}, 0 \leq x_4 < a_4 \right\},$$

where each automorphism of  $k$  determines the quadruple  $(x_1, x_2, x_3, x_4)$  uniquely.

**Proof.** First note that by Lemma 3.6, we have

$$a_3 \frac{m}{r_3 r_4} s_{34} = a_3 \gcd(n_3, n_4) \in \mathbb{Z}$$

and

$$a_2 \frac{m}{r_2 s_{34}} \in \mathbb{Z}$$

(this follows from  $r_2 \mid m$ ,  $s_{34} \mid m$  and  $\gcd(r_2, s_{34}) = \gcd(r_2, r_3, r_4) = 1$ ). By Corollary 3.5, the set on the right hand side has at most  $|\text{Gal}(k/\mathbb{Q})|$  elements. Now let  $\rho$  be any automorphism of  $k$ . If we can show that  $\rho$  can be written as

$$\rho = \sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{x_3} \sigma_4^{x_4} \Big|_k$$

for a quadruple  $(x_1, x_2, x_3, x_4)$  satisfying

$$0 \leq x_1 < a_1 \frac{m}{r_1}, \quad 0 \leq x_2 < a_2 \frac{m}{r_2 s_{34}}, \quad 0 \leq x_3 < a_3 \frac{m}{r_3 r_4} s_{34}, \quad 0 \leq x_4 < a_4,$$

it will follow that the cardinalities agree and we will be done.

Since  $\text{Gal}(k \cap K_4/\mathbb{Q})$  is a cyclic group of order  $a_4$  (by Lemma 3.3) generated by  $\sigma_4|_{k \cap K_4}$  (as a quotient of  $\text{Gal}(K_4/\mathbb{Q}) = \langle \sigma_4|_{K_4} \rangle$ ), there must exist a unique  $x_4 \in \mathbb{Z}$ ,  $0 \leq x_4 < a_4$  such that  $\rho$  and  $\sigma_4^{x_4}$  have the same restrictions to  $k \cap K_4$ . Therefore  $\rho \sigma_4^{-x_4} \Big|_k \in \text{Gal}(k/k \cap K_4)$ .

Next,  $\text{Gal}(k \cap K_3 K_4/k \cap K_4)$  is a cyclic group of order  $\frac{[k \cap K_3 K_4 : \mathbb{Q}]}{[k \cap K_4 : \mathbb{Q}]} = a_3 \frac{m}{r_3 r_4} s_{34}$  (by Corollary 3.5) generated by  $\sigma_3|_{k \cap K_3 K_4}$  (as it is isomorphic by restriction to

$$\text{Gal}((k \cap K_3 K_4)K_4/K_4),$$

which is a quotient of  $\text{Gal}(K_3 K_4/K_4) = \langle \sigma_3|_{K_3 K_4} \rangle$ ), so there must exist a unique  $x_3 \in \mathbb{Z}$  with  $0 \leq x_3 < a_3 \frac{m}{r_3 r_4} s_{34}$  such that  $\rho \sigma_4^{-x_4} \Big|_k$  and  $\sigma_3^{x_3}$  have the same restriction to  $k \cap K_3 K_4$ . Therefore  $\rho \sigma_3^{-x_3} \sigma_4^{-x_4} \Big|_k \in \text{Gal}(k/k \cap K_3 K_4)$ .

Following the pattern,  $\text{Gal}(k \cap K_2 K_3 K_4/k \cap K_3 K_4)$  is a cyclic group of order

$$\frac{[k \cap K_2 K_3 K_4 : \mathbb{Q}]}{[k \cap K_3 K_4 : \mathbb{Q}]} = a_2 \frac{m}{r_2 s_{34}}$$

(by Corollary 3.5) generated by  $\sigma_2|_{k \cap K_2 K_3 K_4}$  (as it is isomorphic by restriction to

$$\text{Gal}((k \cap K_2 K_3 K_4)K_3 K_4/K_3 K_4),$$

which is a quotient of

$$\text{Gal}(K_2 K_3 K_4/K_3 K_4) = \langle \sigma_2|_{K_2 K_3 K_4} \rangle,$$

so there must exist a unique  $x_2 \in \mathbb{Z}$ ,  $0 \leq x_2 < a_2 \frac{m}{r_2 s_{34}}$  such that  $\rho \sigma_3^{-x_3} \sigma_4^{-x_4} \Big|_k$  and  $\sigma_2^{x_2}$  have the same restriction to  $k \cap K_2 K_3 K_4$ . Therefore

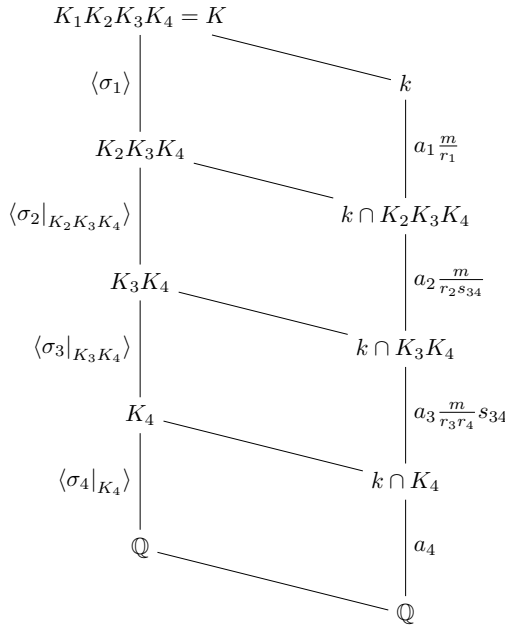
$$\rho \sigma_2^{-x_2} \sigma_3^{-x_3} \sigma_4^{-x_4} \Big|_k \in \text{Gal}(k/k \cap K_2 K_3 K_4).$$

Finally, we have

$$\text{Gal}(k/k \cap K_2 K_3 K_4) \cong \text{Gal}(k K_2 K_3 K_4/K_2 K_3 K_4) = \text{Gal}(K/K_2 K_3 K_4) = \langle \sigma_1 \rangle,$$

where the isomorphism is given by restriction. Since the order of  $\sigma_1$  is  $a_1 \frac{m}{r_1}$ , it follows that there must exist a unique  $x_1 \in \mathbb{Z}$ ,  $0 \leq x_1 < a_1 \frac{m}{r_1}$  such that  $\rho \sigma_2^{-x_2} \sigma_3^{-x_3} \sigma_4^{-x_4} \Big|_k$  and  $\sigma_1^{x_1}$  have the same restriction to  $k$ . Thus  $\rho = \sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{x_3} \sigma_4^{x_4} \Big|_k$  and the proof is finished.





□

4. GENERAL STRATEGY FOR THE CONSTRUCTION OF BASES OF CIRCULAR NUMBERS AND CIRCULAR UNITS

Our goal will be to find explicit  $\mathbb{Z}$ -bases of  $D^+$  and  $C^+$ . To achieve this, we will build upon the results in [2]. The generators of  $D^+$  are subject to norm relations that correspond to the sum of all elements of the respective inertia groups  $T_i$ . Namely, let

$$R_i = \sum_{u=0}^{a_i-1} \sigma_i^u, N_i = \sum_{u=0}^{n_i-1} \sigma_i^{ua_i}.$$

Then the norm operator from  $K$  to  $K_jK_lK_h$  can be given as  $R_iN_i$ , because both are equal to the sum of all elements from  $T_i$ . Moreover, we have

$$\text{Gal}(k/k \cap K_jK_lK_h) \cong \text{Gal}(K/K_jK_lK_h) = T_i,$$

where the first isomorphism is given by restriction, hence  $R_iN_i$  also acts as the norm operator from  $k$  to  $k \cap K_jK_lK_h$ . If we denote the congruence corresponding to the canonical projection  $\mathbb{Z}[G] \rightarrow \mathbb{Z}[G/H]$  by  $\equiv$ , then we have (using Lemma 3.2)

$$N_4 \equiv \sum_{u=0}^{n_4-1} \sigma_1^{ua_1} \sigma_2^{ua_2} \sigma_3^{ua_3}.$$

Note that any subgroup of  $k^\times$  is naturally a  $\mathbb{Z}[G/H]$ -module, since the action of  $H$  on  $k$  is trivial.

Moreover, we will denote the congruence corresponding to the composition of canonical projections

$$\mathbb{Z}[G] \rightarrow \mathbb{Z}[G/H] \rightarrow \mathbb{Z}[G/H]/(R_1N_1, R_2N_2, R_3N_3, R_4N_4)$$

by  $\sim$ , where  $(R_1N_1, R_2N_2, R_3N_3, R_4N_4)$  is the ideal generated in  $\mathbb{Z}[G/H]$  by the images of the elements  $R_iN_i$ . Lemma 2.3 shows that  $\eta \in C^+$ , therefore by Lemma 2.6, we have  $\eta^{\rho R_i N_i} \in C^+(k \cap K_j K_l K_h)$  for any  $\rho \in G$ . We will make use of this extensively, because explicit  $\mathbb{Z}$ -bases of  $D^+(k \cap K_i K_j K_l)$  and  $C^+(k \cap K_i K_j K_l)$  have already been constructed in [1] if exactly two of the primes  $p_i, p_j, p_l$  ramify in  $k \cap K_i K_j K_l$ , or in [2] if all three primes ramify in this field, as the following lemma shows. (If at most one prime ramifies in  $k \cap K_i K_j K_l$ , it is quite trivial to describe explicit  $\mathbb{Z}$ -bases of  $D^+(k \cap K_i K_j K_l)$  and  $C^+(k \cap K_i K_j K_l)$  as well.)

**Lemma 4.1.** *If the field  $k \cap K_i K_j K_l$  is ramified at all three primes  $p_i, p_j, p_l$  then this field satisfies the assumptions of [2]. In other words, if  $K'$  is the genus field in the narrow sense of  $k \cap K_i K_j K_l$ , then  $\text{Gal}(K'/k \cap K_i K_j K_l)$  is cyclic and the inertia subgroups of  $\text{Gal}(K'/\mathbb{Q})$  are all cyclic.*

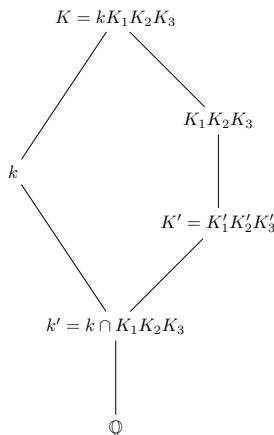
**Proof.** It's clear that  $k \cap K_i K_j K_l$  is real, abelian (its absolute Galois group is a quotient of  $G$ ) and at most three primes ramify in it. By the symmetry between the ramified primes, we can take  $\{i, j, l\} = \{1, 2, 3\}$  in the rest of the proof and we will denote  $k' := k \cap K_1 K_2 K_3$  to improve readability.

Now let  $K'$  be the genus field in the narrow sense of  $k'$ , and for any  $u \in \{1, 2, 3\}$ , let  $K'_u$  be the maximal subfield of  $K'$  ramified only at  $p_u$  and  $T'_u$  be the inertia subgroup of  $\text{Gal}(K'/\mathbb{Q})$  corresponding to  $p_u$ . Then by ramification theory, we have  $K'_u \subseteq K_u$ , hence  $T'_u \cong \text{Gal}(K'_u/\mathbb{Q})$  is isomorphic to a quotient of the cyclic group  $\text{Gal}(K_u/\mathbb{Q}) \cong T_u$ , so it must also be cyclic.

Finally note that we have  $K' = K'_1 K'_2 K'_3 \subseteq K_1 K_2 K_3$  and  $k K_1 K_2 K_3 = K$ , hence  $\text{Gal}(K'/k') = \text{Gal}(K'_1 K'_2 K'_3 / k \cap K_1 K_2 K_3)$  is a quotient of

$$\text{Gal}(K_1 K_2 K_3 / k \cap K_1 K_2 K_3) \cong \text{Gal}(K/k),$$

which is cyclic. This concludes the proof.



□

Using the results in [1] and [2], we can thus take the  $\mathbb{Z}$ -bases of

$$D^+(k \cap K_1K_2K_3), D^+(k \cap K_1K_2K_4), D^+(k \cap K_1K_3K_4), D^+(k \cap K_2K_3K_4)$$

and we will denote their union by  $B_D$ . Analogously, we can take the  $\mathbb{Z}$ -bases of

$$C^+(k \cap K_1K_2K_3), C^+(k \cap K_1K_2K_4), C^+(k \cap K_1K_3K_4), C^+(k \cap K_2K_3K_4)$$

and denote their union by  $B_C$ . Note that  $B_D$  and  $B_C$  contain the same conjugates of  $\eta_I$  for each  $I \subsetneq \{p_1, p_2, p_3, p_4\}$ ,  $|I| \geq 2$ .

To construct a  $\mathbb{Z}$ -basis of  $D^+$  (or  $C^+$ ), we will take the union of  $B_D$  (or  $B_C$ , respectively) with a set  $B$  of suitably chosen conjugates of the highest generator  $\eta$ . In order to have a chance to obtain a  $\mathbb{Z}$ -basis of  $D^+$ , this set should have cardinality

$$\begin{aligned} N &:= [k : \mathbb{Q}] + 4 - 1 - |B_D| \\ &= [k : \mathbb{Q}] + 3 - \sum_{i,j,l} ([k \cap K_iK_jK_l : \mathbb{Q}] + 2) \\ &\quad + \sum_{i,j} ([k \cap K_iK_j : \mathbb{Q}] + 1) - \sum_i [k \cap K_i : \mathbb{Q}] \\ (4.1) \quad &= a_1a_2a_3a_4 \frac{m^3}{r_1r_2r_3r_4} - \sum_{i,j,l} a_i a_j a_l \frac{m^2}{r_i r_j r_l} + \sum_{i,j} a_i a_j s_{ij} \frac{m}{r_i r_j} - \sum_i a_i + 1 \end{aligned}$$

by Proposition 2.5 and Corollary 3.5, using the principle of inclusion and exclusion (due to the fact that these bases were constructed “inductively”). Note that all conjugates of  $\eta$  are units by Lemma 2.3, so this number  $N$  will remain the same in the case of constructing a  $\mathbb{Z}$ -basis of  $C^+$ . Thus we do not have to distinguish between the cases of  $D^+$  and  $C^+$  anymore and we can take the set  $B$  to be the same for both of them.

We cannot guarantee at the moment that the set  $B_D \cup B$  (or  $B_C \cup B$ , respectively) is not linearly dependent, but if we will show how to obtain all the missing conjugates of  $\eta$  using the relations

$$R_1N_1 \sim 0, R_2N_2 \sim 0, R_3N_3 \sim 0, R_4 \sum_{u=0}^{n_4-1} \sigma_1^{ua_1} \sigma_2^{ua_2} \sigma_3^{ua_3} \sim 0$$

and their  $\mathbb{Z}[G]$ -linear combinations, it will follow that we really have a  $\mathbb{Z}$ -basis thanks to the discussion just above Lemma 4.1. A typical way to do that will be the following: if  $R \sim 0$  for some  $R \in \mathbb{Z}[G]$  and  $\eta^R$  is a product of conjugates of  $\eta$  such that we can already generate all of them except for precisely one, then we can generate the last one as well, because  $\eta^R$  can also be expressed as a  $\mathbb{Z}$ -linear combination of elements in  $B_C$ .

We will always refer to the conjugates of  $\eta$  by their coordinates  $x_1, x_2, x_3, x_4$  according to Proposition 3.8. This allows us to visualise  $\text{Gal}(k/\mathbb{Q})$  geometrically as a discrete (at most) four-dimensional cuboid.

5. THE SPECIAL CASE  $a_1 = a_2 = a_3 = r_4 = 1, r_1 \neq 1, r_2 \neq 1, r_3 \neq 1,$   
 $s_{12} = s_{13} = s_{23} = 1, \gcd(n_1, n_2, n_3) = 1$

In this case, we have

$$\text{Gal}(k/\mathbb{Q}) \cong \{\sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{x_3} \sigma_4^{x_4} |_k; 0 \leq x_1 < n_1, 0 \leq x_2 < n_2, 0 \leq x_3 < n_3, 0 \leq x_4 < a_4\},$$

$$s_{12} = s_{13} = s_{14} = s_{23} = s_{24} = s_{34} = 1$$

and

$$N_1 \sim 0, N_2 \sim 0, N_3 \sim 0, R_4 \sum_{u=0}^{m-1} \sigma_1^u \sigma_2^u \sigma_3^u \sim 0.$$

The condition  $r_1 \neq 1, r_2 \neq 1, r_3 \neq 1$  is actually not restrictive, since we will discuss the cases where it is not satisfied in Section 6.

**Lemma 5.1.** *If  $s_{12} = s_{13} = s_{23} = 1$ , the following are equivalent:*

- (1)  $\gcd(n_1, n_2, n_3) = 1,$
- (2)  $\text{lcm}(r_1, r_2, r_3) = m,$
- (3)  $r_1 r_2 r_3 = m,$
- (4)  $n_1 = r_2 r_3, n_2 = r_1 r_3, n_3 = r_1 r_2,$
- (5)  $\frac{n_1 n_2 n_3}{m} = m,$
- (6)  $\gcd(n_1, n_2) = r_3, \gcd(n_1, n_3) = r_2, \gcd(n_2, n_3) = r_1.$

**Proof.** This is just elementary number theory (recall that by Lemma 3.6, we have  $s_{ij} = \gcd(r_i, r_j)$ ). □

Thus  $\frac{n_1 n_2 n_3}{m} = m = r_2 n_2 = \gcd(n_1, n_3) n_2$  by Lemma 5.1 and using Lemma 3.6, we get

$$\begin{aligned} N &= a_4 n_1 n_2 n_3 - \frac{n_1 n_2 n_3}{m} - a_4(n_1 n_2 + n_1 n_3 + n_2 n_3) - a_4 - 2 + a_4(n_1 + n_2 + n_3) \\ &\quad + \gcd(n_1, n_2) + \gcd(n_1, n_3) + \gcd(n_2, n_3) \\ &= (a_4 - 1)(n_1 - 1)(n_2 - 1)(n_3 - 1) + (n_1 - 1)(n_2 - 1)(n_3 - 2) \\ &\quad + n_1 n_2 - (\gcd(n_1, n_3) + 1)n_2 - (n_1 - \gcd(n_1, n_3) - 1) \\ &\quad + \gcd(n_2, n_3) + \gcd(n_1, n_2) - 2 \\ &= (a_4 - 1)(n_1 - 1)(n_2 - 1)(n_3 - 1) + (n_1 - 1)(n_2 - 1)(n_3 - 2) \\ &\quad + (n_2 - 1)(n_1 - r_2 - 1) + r_1 + r_3 - 2. \end{aligned}$$

We will define  $B_5$  as the set of the following  $N$  conjugates  $\eta^{\sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{x_3} \sigma_4^{x_4}}$ :

- $0 \leq x_1 < n_1 - 1, 0 \leq x_2 < n_2 - 1, 0 \leq x_3 < n_3 - 1, 0 < x_4 \leq a_4 - 1,$
- $0 \leq x_1 < n_1 - 1, 0 \leq x_2 < n_2 - 1, 1 < x_3 \leq n_3 - 1, x_4 = 0,$
- $0 \leq x_1 < n_1 - r_2 - 1, 0 \leq x_2 < n_2 - 1, x_3 = 0, x_4 = 0,$
- $x_1 = n_1 - r_2 - 1, 0 \leq x_2 < r_1 + r_3 - 2, x_3 = 0, x_4 = 0.$

(Note that  $n_3 = r_1 r_2 \geq 4, n_1 - r_2 - 1 = r_2(r_3 - 1) - 1 > 0$  and  $n_2 - 1 > r_1 + r_3 - 2 > 0$ , since  $r_1, r_2, r_3 > 1$  and  $n_2 = r_1 r_3$ .)

First we will recover the cases  $0 < x_4 < a_4, x_1 = n_1 - 1$  or  $x_2 = n_2 - 1$  or  $x_3 = n_3 - 1$  using the relations  $N_1 \sim 0, N_2 \sim 0, N_3 \sim 0$ . From now on, we only need

to deal with the cases where  $x_4 = 0$ . Next, we will recover the cases  $1 < x_3 \leq n_3 - 1$ ,  $x_1 = n_1 - 1$  or  $x_2 = n_2 - 1$  (and always  $x_4 = 0$ ) using the relations  $N_1 \sim 0, N_2 \sim 0$  and the cases  $x_3 = x_4 = 0, 0 \leq x_1 < n_1 - r_2 - 1, x_2 = n_2 - 1$  using the relation  $N_2 \sim 0$ .

At this moment, we are only missing all the cases with  $x_3 = 1, x_4 = 0$  and some of those with  $x_3 = x_4 = 0$ . From now on, we will only focus on recovering those with  $x_3 = x_4 = 0$ , because once we have those, we can recover those with  $x_3 = 1, x_4 = 0$  just by using the relation  $N_3 \sim 0$ .

From now on, we will write  $\bar{z} := z \pmod{r_3}$  for any  $z \in \mathbb{Z}$ , hence we will always have  $\bar{z} \in \{0, 1, \dots, r_3 - 1\}$ . We will also define  $h$  to be the unique integer satisfying

$$r_1 \cdot h \equiv r_2 \pmod{r_3} \quad \text{and} \quad h \in \{0, 1, \dots, r_3 - 1\}$$

and similarly  $h'$  to be the unique integer satisfying

$$r_2 \cdot h' \equiv r_1 \pmod{r_3} \quad \text{and} \quad h' \in \{0, 1, \dots, r_3 - 1\}$$

(both are well defined, since  $\gcd(r_1, r_3) = \gcd(r_2, r_3) = 1$ ). Clearly  $h \cdot h' \equiv 1 \pmod{r_3}$ .

Let  $Q'$  be the quotient  $\mathbb{Z}[G]$ -module

$$D^+ / \langle \{\eta_I \mid \emptyset \subsetneq I \subsetneq P\} \rangle_{\mathbb{Z}[G]}$$

and let  $Q$  be the quotient  $\mathbb{Z}$ -module of  $Q'$  by the classes of conjugates we have already recovered, i.e.,

$$Q := Q' / \langle \{\eta^{\sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{x_3} \sigma_4^{x_4}}; \begin{aligned} &0 \leq x_1 < n_1, 0 \leq x_2 < n_2, 0 \leq x_3 < n_3, 0 < x_4 < a_4, \\ &\text{or } 0 \leq x_1 < n_1, 0 \leq x_2 < n_2, 1 < x_3 < n_3, x_4 = 0, \\ &\text{or } 0 \leq x_1 < n_1 - r_2 - 1, 0 \leq x_2 < n_2, x_3 = x_4 = 0, \\ &\text{or } x_1 = n_1 - r_2 - 1, 0 \leq x_2 < r_1 + r_3 - 2, x_3 = x_4 = 0 \} \rangle_{\mathbb{Z}} \end{aligned}$$

(where we denote  $\eta^\rho \in D^+$  and its class in  $Q'$  in the same way for any  $\rho \in G$ ). We will write  $Q$  additively, denoting the class of  $\eta$  in  $Q$  by  $\mu$ , hence denoting the class of  $\eta^\rho$  in  $Q$  by  $\rho \cdot \mu$  for any  $\rho \in \text{Gal}(k/\mathbb{Q})$  or  $\rho \in G$ . Showing that we have indeed chosen a basis now amounts to showing that  $Q$  is trivial. Since

$$0 = \sigma_1^{x_1} \sigma_2^{x_2} N_3 \cdot \mu = \sigma_1^{x_1} \sigma_2^{x_2} \cdot \mu + \sigma_1^{x_1} \sigma_2^{x_2} \sigma_3 \cdot \mu$$

for any  $x_1, x_2 \in \mathbb{Z}$ , this is equivalent with showing that  $\sigma_1^{x_1} \sigma_2^{x_2} \cdot \mu = 0$  for each  $0 \leq x_1 < n_1, 0 \leq x_2 < n_2$ .

The conjugates with  $x_3 = 0$  and  $x_4 = 0$  (i.e., those of the form  $\eta^{\sigma_1^{x_1} \sigma_2^{x_2}}$ ) can be visualized as a discrete rectangle with  $n_1$  rows and  $n_2$  columns. Since for each  $x_4$ , there are  $n_3$  layers of such rectangles in total, the sum  $\eta^{R_4} \sum_{u=0}^{m-1} \sigma_1^u \sigma_2^u \sigma_3^u$  must contain  $\frac{m}{n_3} = r_3$  conjugates in each of these rectangles. We will now describe the sum of these.

Let

$$T := \sum_{u=0}^{r_3-1} \sigma_1^{un_3} \sigma_2^{un_3} .$$

**Lemma 5.2.** *In  $Q$ , we have*

$$\sigma_1^{x_1} \sigma_2^{x_2} (1 - \sigma_1 \sigma_2) T \cdot \mu = 0$$

for any  $x_1, x_2 \in \mathbb{Z}$ .

**Proof.** Using the fact that every  $0 \leq w < m$  can be uniquely written as  $un_3 + v$ , where  $0 \leq u < r_3, 0 \leq v < n_3$ , together with the fact that the order of  $\sigma_3$  is  $n_3$ , we get

$$R_4 T \sum_{v=0}^{n_3-1} \sigma_1^v \sigma_2^v \sigma_3^v = R_4 \sum_{u=0}^{r_3-1} \sigma_1^{un_3} \sigma_2^{un_3} \sigma_3^{un_3} \cdot \sum_{v=0}^{n_3-1} \sigma_1^v \sigma_2^v \sigma_3^v = R_4 \sum_{w=0}^{m-1} \sigma_1^w \sigma_2^w \sigma_3^w \sim 0.$$

Together with  $N_3 \sim 0$ , this means that

$$\begin{aligned} 0 &\sim \sigma_1^{x_1} \sigma_2^{x_2} \left( R_4 T \sum_{v=0}^{n_3-1} \sigma_1^v \sigma_2^v \sigma_3^v - \sigma_1 \sigma_2 N_3 R_4 T \right) \\ &= \sigma_1^{x_1} \sigma_2^{x_2} R_4 T \sum_{v=0}^{n_3-1} (\sigma_1^v \sigma_2^v - \sigma_1 \sigma_2) \sigma_3^v \\ &= \sigma_1^{x_1} \sigma_2^{x_2} (1 - \sigma_1 \sigma_2) R_4 T + \sigma_1^{x_1} \sigma_2^{x_2} R_4 T \sum_{v=2}^{n_3-1} (\sigma_1^v \sigma_2^v - \sigma_1 \sigma_2) \sigma_3^v \\ &= \sigma_1^{x_1} \sigma_2^{x_2} (1 - \sigma_1 \sigma_2) T + \sigma_1^{x_1} \sigma_2^{x_2} (1 - \sigma_1 \sigma_2) T \sum_{u=1}^{a_4-1} \sigma_4^u \\ &\quad + \sigma_1^{x_1} \sigma_2^{x_2} R_4 T \sum_{v=2}^{n_3-1} (\sigma_1^v \sigma_2^v - \sigma_1 \sigma_2) \sigma_3^v. \end{aligned}$$

Since all the summands in the expression

$$\sigma_1^{x_1} \sigma_2^{x_2} (1 - \sigma_1 \sigma_2) T \sum_{u=1}^{a_4-1} \sigma_4^u + \sigma_1^{x_1} \sigma_2^{x_2} R_4 T \sum_{v=2}^{n_3-1} (\sigma_1^v \sigma_2^v - \sigma_1 \sigma_2) \sigma_3^v$$

have either  $x_4 > 0$  or  $x_3 > 1$  (where  $x_3$  and  $x_4$  denote the respective exponents of  $\sigma_3$  and  $\sigma_4$  in each term), the result of their action on  $\mu$  becomes trivial in  $Q$ , which yields the result.  $\square$

The rest of this section will again be stated purely algebraically, but perhaps it is helpful (although not strictly required) to see some of its parts geometrically.

We will decompose our rectangle (of conjugates of  $\eta$  having  $x_3 = x_4 = 0$ ) into  $r_3 \times r_3$  rectangular blocks of height  $r_2$  and width  $r_1$  in the natural way. In the following, by a big row (resp. a big column) we will understand a row of blocks (resp. columns), that is  $r_3$  consecutive blocks next to (resp. above) each other. Since  $r_2 \mid n_3, r_1 \mid n_3$  and the conjugates contained in  $\eta^T$  are given by  $\eta^{\sigma_1^{qn_3} \sigma_2^{qn_3}}$  for  $0 \leq q \leq r_3 - 1$ , the Chinese remainder theorem implies that  $\eta^{\sigma_1^{x_1} \sigma_2^{x_2} T}$  contains exactly one conjugate in every big row (resp. every big column) for any  $0 \leq x_1 < n_1, 0 \leq x_2 < n_2$ , and these have the same relative position in each of the respective blocks (determined only by  $\bar{r}_1, \bar{r}_2, x_1, x_2$ ). We can be even more

precise: the horizontal distance between  $\eta^{\sigma_1^{qn_3+x_1}\sigma_2^{qn_3+x_2}}$  and  $\eta^{\sigma_1^{(q+1)n_3+x_1}\sigma_2^{(q+1)n_3+x_2}}$  for  $0 \leq q \leq r_3 - 1$  and  $0 \leq x_1 < n_1, 0 \leq x_2 < n_2$  is exactly  $\overline{r_2} \cdot r_1$ , i.e.,  $\overline{r_2}$  blocks, and the vertical distance between them is exactly  $\overline{r_1} \cdot r_2$ , i.e.,  $\overline{r_1}$  blocks (again this follows easily from the Chinese remainder theorem). It follows that the horizontal distance between any two conjugates in  $\eta^T$  with a vertical distance of one block is  $h$  blocks.

For all  $0 \leq u \leq n_2$ , we will denote  $X_u := \sigma_1^{n_1-2}\sigma_2^u \cdot \mu$  and  $Y_u := \sigma_1^{r_2(r_3-1)-1}\sigma_2^u \cdot \mu$ . By definition,  $X_u$  and  $Y_u$  are elements of  $Q$ . It will be convenient to allow any integers in the indices of the  $X$ 's and  $Y$ 's and regard them only modulo  $n_2$  (to be more precise, as in the set  $\{0, 1, \dots, n_2 - 1\}$ ). Moreover note that by definition,  $Y_u = 0$  for  $0 \leq u < r_1 + r_3 - 2$ .

**Lemma 5.3.** *We have  $X_q = X_{q'}$  for any  $q \equiv q' \pmod{r_3}$ . Moreover, for any  $0 \leq x_1 < n_1, 0 \leq x_2 < n_2$ , we have*

$$\sigma_1^{x_1}\sigma_2^{x_2} \cdot \mu = \begin{cases} 0 & \text{if } 0 \leq x_1 < r_2(r_3 - 1) - 1, \\ Y_{x_2} & \text{if } x_1 = r_2(r_3 - 1) - 1, \\ X_{x_2-x_1-2} & \text{if } r_2(r_3 - 1) \leq x_1 < n_1 - 1, \\ X_{x_2-x_1-2} - Y_{x_2-h \cdot r_1} & \text{if } x_1 = n_1 - 1. \end{cases}$$

**Proof.** The first part will be proven in a moment, we will now focus on the second.

The first case ( $x_1 < r_2(r_3 - 1) - 1$ ) follows directly from the definition of  $Q$  and the second case ( $x_1 = r_2(r_3 - 1) - 1$ ) directly from the definition of  $Y_{x_2}$ .

Now for every  $0 \leq u < n_2$ , we will prove that

$$(5.1) \quad \sigma_1^{n_1-2-v}\sigma_2^{u-v} \cdot \mu = X_u$$

by induction with respect to  $v = 0, 1, \dots, r_2 - 2$ . The base step  $v = 0$  is just the definition of  $X_u$ . Now suppose that  $0 < v \leq r_2 - 2$  and the statement holds for  $v - 1$ . Then in the equality

$$(5.2) \quad \left( \sigma_1^{n_1-2-v}\sigma_2^{u-v}(1 - \sigma_1\sigma_2) \sum_{w=0}^{r_3-1} \sigma_1^{wn_3}\sigma_2^{wn_3} \right) \cdot \mu = 0,$$

which follows from Lemma 5.2, we claim that all the terms with  $w > 0$  do not contribute anything to the sum. Indeed, all the exponents of  $\sigma_1$  are pairwise congruent modulo  $r_2$  (since  $r_2 \mid n_3$ ), and since  $n_1 - r_2 \leq n_1 - 2 - v < n_1 - 2$  and  $n_1 - r_2 + 1 \leq n_1 - 1 - v < n_1 - 1$ , we have

$$\left( \sigma_1^{n_1-2-v}\sigma_2^{u-v}(1 - \sigma_1\sigma_2)\sigma_1^{wn_3}\sigma_2^{wn_3} \right) \cdot \mu = 0$$

for any  $w > 0$ , because  $r_3$  does not divide  $wn_3$  in this case. Hence (5.2) implies that

$$0 = \left( \sigma_1^{n_1-2-v}\sigma_2^{u-v}(1 - \sigma_1\sigma_2) \right) \cdot \mu = \sigma_1^{n_1-2-v}\sigma_2^{u-v} \cdot \mu - \underbrace{\sigma_1^{n_1-2-(v-1)}\sigma_2^{u-(v-1)} \cdot \mu}_{=X_u},$$

therefore  $\sigma_1^{n_1-2-v}\sigma_2^{u-v} \cdot \mu = X_u$  by the induction hypothesis. This completes the induction, so (5.1) holds.

Now for any  $0 \leq u < n_2$ , we will take  $v = r_2 - 1$  in (5.2). Again, since all the exponents of  $\sigma_1$  are pairwise congruent modulo  $r_2$  (since  $r_2 \mid n_3$ ) in this sum, the only terms which could be nonzero are those arising from  $w = 0$  and from  $w$  satisfying

$$wn_3 + n_1 - 2 - (r_2 - 1) \equiv n_1 - 1 \pmod{n_1},$$

which is equivalent to  $wn_3 \equiv r_2 \pmod{n_1}$ , which implies  $wn_3 \equiv r_2 \pmod{r_3}$ . Together with  $wn_3 \equiv 0 \pmod{r_1}$  and the fact that  $\gcd(r_1, r_3) = 1$ , this means that the only solution to the above congruence is  $wn_3 \equiv h \cdot r_1 \pmod{n_2}$ .

Thus we have

$$\begin{aligned} 0 &= \left( \sigma_1^{n_1-r_2-1} \sigma_2^{u-r_2+1} (1 - \sigma_1 \sigma_2) + \sigma_1^{n_1-1} \sigma_2^{u-r_2+1+h \cdot r_1} (1 - \sigma_1 \sigma_2) \right) \cdot \mu \\ &= \underbrace{\sigma_1^{n_1-r_2-1} \sigma_2^{u-r_2+1} \cdot \mu}_{=Y_{u-r_2+1}} - \underbrace{\sigma_1^{n_1-r_2} \sigma_2^{u-r_2+2} \cdot \mu}_{=X_u \text{ due to (5.1)}} + \underbrace{\sigma_1^{n_1-1} \sigma_2^{u-r_2+1+h \cdot r_1} \cdot \mu}_{=0} \\ &\quad - \underbrace{\sigma_1^{n_1} \sigma_2^{u-r_2+1+h \cdot r_1+1} \cdot \mu}_{=0}. \end{aligned}$$

Therefore

$$(5.3) \quad \sigma_1^{n_1-1} \sigma_2^{u-r_2+1+h \cdot r_1} \cdot \mu = X_u - Y_{u-r_2+1}.$$

Finally, for any  $0 \leq u < n_2$ , we will take  $v = r_2$  in (5.2). Again, since all the exponents of  $\sigma_1$  are pairwise congruent modulo  $r_2$  in this sum, we only get nonzero terms for  $w = 0$  and for  $w$  satisfying

$$wn_3 + n_1 - 2 - r_2 \equiv n_1 - 2 \pmod{n_1},$$

which implies (because we have got the same congruence as above)  $wn_3 \equiv h \cdot r_1 \pmod{n_2}$ .

Thus we have

$$\begin{aligned} 0 &= \underbrace{\sigma_1^{n_1-r_2-2} \sigma_2^{u-r_2} \cdot \mu}_{=0} - \underbrace{\sigma_1^{n_1-r_2-1} \sigma_2^{u-r_2+1} \cdot \mu}_{=Y_{u-r_2+1}} \\ &\quad + \underbrace{\sigma_1^{n_1-2} \sigma_2^{u-r_2+h \cdot r_1} \cdot \mu}_{=X_{u-r_2+h \cdot r_1}} - \underbrace{\sigma_1^{n_1-1} \sigma_2^{u-r_2+1+h \cdot r_1} \cdot \mu}_{=X_u - Y_{u-r_2+1} \text{ due to (5.3)}}. \end{aligned}$$

Therefore  $X_{u-r_2+h \cdot r_1} = X_u$ . Note that

$$h \cdot r_1 - r_2 \equiv 0 \pmod{r_3}$$

and

$$h \cdot r_1 - r_2 \equiv -r_2 \pmod{r_1}.$$

Since  $\gcd(-r_2, r_1) = 1$  and  $n_2 = r_1 r_3$ , this means that for all  $q, q' \in \mathbb{Z}$  satisfying

$$q \equiv q' \pmod{r_3},$$

there is some  $w \in \mathbb{Z}$  such that

$$q' \equiv w(h \cdot r_1 - r_2) + q \pmod{n_2}.$$



Without loss of generality, we can assume that  $w \geq 0$  (otherwise we can just swap  $q$  and  $q'$ ). But then

$$X_q = X_{q+(h \cdot r_1 - r_2)} = X_{q+2(h \cdot r_1 - r_2)} = \cdots = X_{q+w(h \cdot r_1 - r_2)} = X_{q'}.$$

Now for any  $x_1, x_2$  satisfying  $r_2(r_3 - 1) \leq x_1 < n_1 - 1$  and  $0 \leq x_2 < n_2$ , denoting

$$v = n_1 - 2 - x_1, \quad u = v + x_2,$$

we get  $0 \leq v \leq r_2 - 2$  and the equality (5.1) implies

$$\sigma_1^{x_1} \sigma_2^{x_2} \mu = X_{n_1 - 2 - x_1 + x_2} = X_{x_2 - x_1 - 2},$$

because  $r_3 \mid n_1$ .

Similarly, for  $x_1 = n_1 - 1$  and any  $0 \leq x_2 < n_2$ , denoting  $u = x_2 + r_2 - 1 - h \cdot r_1$ , the equality (5.3) implies that

$$\sigma_1^{x_1} \sigma_2^{x_2} \cdot \mu = X_u - Y_{u - r_2 + 1} = X_{x_2 - x_1 - 2} - Y_{x_2 - h \cdot r_1},$$

since

$$u = x_2 - 1 + r_2 - h \cdot r_1 \equiv x_2 - 1 \equiv x_2 - 2 + 1 - n_1 = x_2 - x_1 - 2 \pmod{r_3}$$

by definition of  $h$  and the fact that  $r_3 \mid n_1$ .

This concludes the proof. □

Thanks to Lemma 5.3, from now on we will regard the indices of the  $X$ 's only modulo  $r_3$ . The lemma also implies the equality

$$(5.4) \quad \sigma_1^{n_1 - 1} \sigma_2^{x_2} \cdot \mu + \sigma_1^{n_1 - r_2 - 1} \sigma_2^{x_2 - h \cdot r_1} \cdot \mu = X_{x_2 - 1} - Y_{x_2 - h \cdot r_1} + Y_{x_2 - h \cdot r_1} = X_{x_2 - 1}$$

for any  $x_2 \in \mathbb{Z}$ , which we will use several times. Another simple observation that will come in handy in the proofs of the following lemmas is that the unary operation of adding a fixed integer induces an automorphism of  $\mathbb{Z}/r_3$ , which we will not mention explicitly anymore.

To show that  $Q$  is trivial, it now suffices to show that  $X_u = 0$  for all  $0 \leq u < r_3$  and  $Y_v = 0$  for all  $r_1 + r_3 - 2 \leq v < n_2$  (knowing already that  $Y_v = 0$  for all  $0 \leq v < r_1 + r_3 - 2$ ). To achieve this, we will use linear algebra.

Let

$$\alpha := Y_{r_1 + r_3 - 2} + Y_{r_1 + r_3 - 1} + \cdots + Y_{n_2 - 1} \in Q$$

and

$$(5.5) \quad \beta := X_0 + X_1 + \cdots + X_{r_3 - 1} \in Q.$$

**Lemma 5.4.** *We have  $\alpha = \beta = 0$ .*

**Proof.** Using the relation  $N_2 \sim 0$ , we have

$$0 = \sigma_1^{r_2(r_3 - 1) - 1} N_2 \cdot \mu = \sum_{x_2=0}^{n_2 - 1} \sigma_1^{r_2(r_3 - 1) - 1} \sigma_2^{x_2} \cdot \mu = \sum_{x_2=0}^{n_2 - 1} Y_{x_2} = \alpha$$

and

$$\begin{aligned}
 0 &= \sigma_1^{r_2(r_3-1)} N_2 \cdot \mu = \sum_{x_2=0}^{n_2-1} \sigma_1^{r_2(r_3-1)} \sigma_2^{x_2} \cdot \mu = \sum_{x_2=0}^{n_2-1} X_{x_2-r_2(r_3-1)-2} \\
 &= \sum_{x_2=0}^{r_1 r_3-1} X_{x_2+r_2-2} = \sum_{u=0}^{r_1-1} \sum_{v=0}^{r_3-1} X_{ur_3+v+r_2-2} = r_1 \cdot \sum_{v=0}^{r_3-1} X_{v+r_2-2} = r_1 \cdot \beta,
 \end{aligned}$$

since each  $x_2 \in \{0, 1, \dots, r_1 r_3 - 1\}$  can be uniquely written as  $ur_3 + v$ , where  $0 \leq u < r_1, 0 \leq v < r_3$ .

Similarly, using Lemma 5.3 together with the relation  $N_1 \sim 0$  and the equality (5.4), we get

$$\begin{aligned}
 0 &= \sum_{q=0}^{r_3-1} \sigma_2^{qr_1} N_1 \cdot \mu = \sum_{q=0}^{r_3-1} \left( \sigma_1^{n_1-1} + \sigma_1^{r_2(r_3-1)-1} \right) \sigma_2^{qr_1} \cdot \mu + \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{q=0}^{r_3-1} \sigma_1^{x_1} \sigma_2^{qr_1} \cdot \mu \\
 &= \sum_{q=0}^{r_3-1} \left( \sigma_1^{n_1-1} \sigma_2^{qr_1} + \sigma_1^{r_2(r_3-1)-1} \sigma_2^{(q-h) \cdot r_1} \right) \cdot \mu + \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{q=0}^{r_3-1} \sigma_1^{x_1} \sigma_2^{qr_1} \cdot \mu \\
 &= \sum_{q=0}^{r_3-1} X_{qr_1-1} + \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{q=0}^{r_3-1} X_{qr_1-x_1-2} = \sum_{x_1=r_2(r_3-1)}^{n_1-1} \sum_{q=0}^{r_3-1} X_{qr_1-x_1-2} = r_2 \cdot \beta,
 \end{aligned}$$

since for any  $x_1$ , all possible remainders modulo  $r_3$  occur exactly once as the indices in the sum  $\sum_{q=0}^{r_3-1} X_{qr_1-x_1-2}$  (due to the fact that the order of the class of  $r_1$  is  $r_3$  in  $\mathbb{Z}/r_3$ , due to their coprimality). Since  $\gcd(r_1, r_2) = 1$ , this implies  $\beta = 0$  by Bézout's identity.  $\square$

Next, for  $0 \leq q \leq r_3 - 3$ , we will define

$$(5.6) \quad \Gamma_q := \sum_{u=0}^{r_3-h'-1} \sum_{v=0}^{\overline{r_2}-1} X_{q+v-ur_2-1} \in \mathbb{Q}.$$

**Lemma 5.5.** *For any  $0 \leq q \leq r_3 - 3$ , we have  $\Gamma_q = 0$ .*

**Proof.** Using Lemma 5.3, the relation  $N_1 \sim 0$  and the equality (5.4), we get

$$\begin{aligned}
 0 &= \sum_{u=0}^{r_3-h'-1} \sigma_2^{q-uhr_1} N_1 \cdot \mu \\
 &= \sum_{u=0}^{r_3-h'-2} \underbrace{\left( \sigma_1^{n_1-1} \sigma_2^{q-uhr_1} + \sigma_1^{r_2(r_3-1)-1} \sigma_2^{q-(u+1)hr_1} \right)}_{=X_{q-uhrr_1-1} \text{ due to (5.4)}} \cdot \mu
 \end{aligned}$$

$$\begin{aligned}
 &+ \underbrace{\sigma_1^{r_2(r_3-1)-1} \sigma_2^q \cdot \mu}_{=Y_q} + \underbrace{\sigma_1^{n_1-1} \sigma_2^{q-(r_3-h'-1)hr_1} \cdot \mu}_{=X_{q-(r_3-h'-1)hr_1-1}-Y_{q+r_1}} \\
 &+ \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{u=0}^{r_3-h'-1} \sigma_1^{x_1} \sigma_2^{q-uhr_1} \cdot \mu.
 \end{aligned}$$

Now we will use the fact that  $q \leq r_3 - 3 \leq r_1 + r_3 - 3$  (implying  $Y_q = 0$ ) and

$$q - (r_3 - h' - 1)hr_1 - hr_1 = q - r_1r_3h + r_1hh' \equiv q + r_1 \pmod{n_2},$$

since the congruence holds modulo both  $r_1$  and  $r_3$  (and  $\gcd(r_1, r_3) = 1$ ). Also note that  $Y_{q+r_1} = 0$ , since

$$r_1 \leq q + r_1 \leq r_1 + r_3 - 3,$$

which precisely justifies the bounds on  $q$  that we used in the definition of  $\Gamma_q$  and also explains why the upper bound in the first sum was chosen to be  $r_3 - h' - 1$ .

Continuing with the previous equality and using Lemma 5.3 together with the congruence  $hr_1 \equiv r_2 \pmod{r_3}$ , we thus have

$$\begin{aligned}
 0 &= \left( \sum_{u=0}^{r_3-h'-2} X_{q-uhr_1-1} \right) + X_{q-(r_3-h'-1)hr_1-1} + \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{u=0}^{r_3-h'-1} X_{q-uhr_1-x_1-2} \\
 &= \sum_{u=0}^{r_3-h'-1} X_{q-ur_2-1} + \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{u=0}^{r_3-h'-1} X_{q-ur_2-x_1-2} \\
 &= \sum_{x_1=r_2(r_3-1)}^{n_1-1} \sum_{u=0}^{r_3-h'-1} X_{q-ur_2-x_1-2}.
 \end{aligned}$$

After using the substitution  $v = n_1 - 1 - x_1$ , this becomes

$$\begin{aligned}
 0 &= \sum_{u=0}^{r_3-h'-1} \sum_{v=0}^{r_2-1} X_{q+v-ur_2-1} \\
 &= \sum_{u=0}^{r_3-h'-1} \left( \sum_{v=0}^{\bar{r}_2-1} X_{q+v-ur_2-1} + \sum_{v=\bar{r}_2}^{r_2-1} X_{q+v-ur_2-1} \right) \\
 &= \sum_{u=0}^{r_3-h'-1} \sum_{v=0}^{\bar{r}_2-1} X_{q+v-ur_2-1} + \sum_{u=0}^{r_3-h'-1} \frac{r_2 - \bar{r}_2}{r_3} \sum_{v=\bar{r}_2}^{\bar{r}_2+r_3-1} X_{q+v-ur_2-1} \\
 &= \Gamma_q + \sum_{u=0}^{r_3-h'-1} \frac{r_2 - \bar{r}_2}{r_3} \cdot \beta,
 \end{aligned}$$

which equals  $\Gamma_q$  since  $\beta = 0$  by Lemma 5.4. □

Finally, let

$$(5.7) \quad \Delta := \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{\bar{r}_2-1} \sum_{w=0}^{\bar{r}_1-1} X_{v+w-ur_2-1} \in Q.$$

**Lemma 5.6.** *We have  $\Delta = 0$ .*

**Proof.** Using Lemma 5.3, the relation  $N_1 \sim 0$  and the equality (5.4), we get

$$\begin{aligned} 0 &= \sum_{u=0}^{r_3-1} u \cdot \sum_{x_2=0}^{r_1-1} \sigma_2^{x_2-uhr_1} N_1 \cdot \mu \\ &= \sum_{u=0}^{r_3-1} u \cdot \sum_{x_2=0}^{r_1-1} \left( \sigma_1^{n_1-1} \sigma_2^{x_2-uhr_1} + \sigma_1^{r_2(r_3-1)-1} \sigma_2^{x_2-uhr_1} \right) \cdot \mu \\ &\quad + \sum_{u=0}^{r_3-1} u \cdot \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{x_2=0}^{r_1-1} \sigma_1^{x_1} \sigma_2^{x_2-uhr_1} \cdot \mu \\ &= \sum_{u=0}^{r_3-2} \sum_{x_2=0}^{r_1-1} \left( u \cdot \underbrace{\sigma_1^{n_1-1} \sigma_2^{x_2-uhr_1} \cdot \mu}_{=X_{x_2-uhr_1-1}-Y_{x_2-(u+1)hr_1}} + (u+1) \cdot \underbrace{\sigma_1^{r_2(r_3-1)-1} \sigma_2^{x_2-(u+1)hr_1} \cdot \mu}_{=Y_{x_2-(u+1)hr_1}} \right) \\ &\quad + \sum_{x_2=0}^{r_1-1} (r_3-1) \cdot \underbrace{\sigma_1^{n_1-1} \sigma_2^{x_2-(r_3-1)hr_1} \cdot \mu}_{=X_{x_2-(r_3-1)hr_1-1}-Y_{x_2-hr_1r_3}} \\ &\quad + \sum_{u=0}^{r_3-1} u \cdot \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{x_2=0}^{r_1-1} \sigma_1^{x_1} \sigma_2^{x_2-uhr_1} \cdot \mu. \end{aligned}$$

Since

$$x_2 - hr_1r_3 \equiv x_2 \pmod{n_2}$$

and  $0 \leq x_2 < r_1$ , we have  $Y_{x_2-hr_1r_3} = 0$ . Also note that for any  $r_1 \leq q < n_2$ , there exist unique

$$u \in \{0, 1, \dots, r_3 - 2\}, x_2 \in \{0, 1, \dots, r_1 - 1\}$$

such that

$$q \equiv x_2 - (u+1)hr_1 \pmod{n_2}$$

by the Chinese remainder theorem, since  $\gcd(h, r_3) = 1$  and for  $u = r_3 - 1$ , we would get  $q \equiv x_2 \pmod{n_2}$  and  $0 \leq x_2 < r_1$ . Thus we get a bijection

$$\{0, 1, \dots, r_3 - 2\} \times \{0, 1, \dots, r_1 - 1\} \rightarrow \{r_1, r_1 + 1, \dots, n_2 - 1\},$$

which we will use in a moment to transform a double sum into a simple one.

Continuing with the above equality and using the congruence  $hr_1 \equiv r_2 \pmod{r_3}$ , we thus have

$$\begin{aligned}
 0 &= \sum_{u=0}^{r_3-2} \sum_{x_2=0}^{r_1-1} u \cdot X_{x_2-ur_2-1} + \sum_{u=0}^{r_3-2} \sum_{x_2=0}^{r_1-1} Y_{x_2-(u+1)hr_1} + \underbrace{\sum_{q=0}^{r_1-1} Y_q}_{=0} \\
 &+ \sum_{x_2=0}^{r_1-1} (r_3-1) \cdot X_{x_2-(r_3-1)r_2-1} + \sum_{u=0}^{r_3-1} u \cdot \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{x_2=0}^{r_1-1} X_{x_2-ur_2-x_1-2} \\
 &= \sum_{u=0}^{r_3-1} \sum_{x_2=0}^{r_1-1} u \cdot X_{x_2-ur_2-1} + \underbrace{\sum_{q=r_1}^{n_2-1} Y_q + \sum_{q=0}^{r_1-1} Y_q}_{=\alpha} \\
 &+ \sum_{u=0}^{r_3-1} u \cdot \sum_{x_1=r_2(r_3-1)}^{n_1-2} \sum_{x_2=0}^{r_1-1} X_{x_2-ur_2-x_1-2} \\
 &= \alpha + \sum_{u=0}^{r_3-1} u \cdot \sum_{x_1=r_2(r_3-1)}^{n_1-1} \sum_{x_2=0}^{r_1-1} X_{x_2-ur_2-x_1-2}.
 \end{aligned}$$

After using the equality  $\alpha = 0$  by Lemma 5.4 and the substitutions  $v = n_1 - 1 - x_1$ ,  $w = x_2$ , this becomes

$$\begin{aligned}
 0 &= \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{r_2-1} \sum_{w=0}^{r_1-1} X_{v+w-ur_2-1} \\
 &= \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{r_2-1} \left( \sum_{w=0}^{\bar{r}_1-1} X_{v+w-ur_2-1} + \sum_{w=\bar{r}_1}^{r_1-1} X_{v+w-ur_2-1} \right) \\
 &= \sum_{u=0}^{r_3-1} u \cdot \sum_{w=0}^{\bar{r}_1-1} \sum_{v=0}^{r_2-1} X_{v+w-ur_2-1} + \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{r_2-1} \frac{r_1 - \bar{r}_1}{r_3} \cdot \beta.
 \end{aligned}$$

Using the fact that  $\beta = 0$  by Lemma 5.4, this equals

$$\begin{aligned}
 &\sum_{u=0}^{r_3-1} u \cdot \sum_{w=0}^{\bar{r}_1-1} \sum_{v=0}^{r_2-1} X_{v+w-ur_2-1} \\
 &= \sum_{u=0}^{r_3-1} u \cdot \sum_{w=0}^{\bar{r}_1-1} \left( \sum_{v=0}^{\bar{r}_2-1} X_{v+w-ur_2-1} + \sum_{v=\bar{r}_2}^{r_2-1} X_{v+w-ur_2-1} \right) \\
 &= \Delta + \sum_{u=0}^{r_3-1} u \cdot \sum_{w=0}^{\bar{r}_1-1} \frac{r_2 - \bar{r}_2}{r_3} \cdot \beta,
 \end{aligned}$$

which equals  $\Delta$  again by  $\beta = 0$ . □

Now let  $\mathcal{X}$  be the free  $\mathbb{Z}$ -module with generators  $\widehat{X}_0, \widehat{X}_1, \dots, \widehat{X}_{r_3-1}$ . Analogously to the definitions (5.5), (5.6), (5.7), we will define

$$\begin{aligned} \widehat{\beta} &:= \widehat{X}_0 + \widehat{X}_1 + \dots + \widehat{X}_{r_3-1} \in \mathcal{X}, \\ \widehat{\Gamma}_q &:= \sum_{u=0}^{r_3-h'-1} \sum_{v=0}^{\overline{r}_2-1} \widehat{X}_{q+v-ur_2-1} \in \mathcal{X}, \\ \widehat{\Delta} &:= \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{\overline{r}_2-1} \sum_{w=0}^{\overline{r}_1-1} \widehat{X}_{v+w-ur_2-1} \in \mathcal{X} \end{aligned}$$

for all  $0 \leq q \leq r_3 - 3$ . Also let  $\psi: \mathcal{X} \rightarrow Q$  be the  $\mathbb{Z}$ -module homomorphism satisfying  $\psi(\widehat{X}_u) = X_u$  for all  $0 \leq u < r_3$  (since  $\mathcal{X}$  is free, this is well defined and determines  $\psi$  uniquely). Then for all  $0 \leq q \leq r_3 - 3$ , it's clear by Lemmas 5.4, 5.5 and 5.6 that

$$\psi(\widehat{\beta}) = \beta = 0, \quad \psi(\widehat{\Gamma}_q) = \Gamma_q = 0, \quad \psi(\widehat{\Delta}) = \Delta = 0,$$

hence

$$(5.8) \quad \widehat{\beta}, \widehat{\Gamma}_q, \widehat{\Delta} \in \ker \psi.$$

Since  $\mathcal{X}$  is free, each of its elements can be expressed as  $\sum_{c=0}^{r_3-1} c_u \widehat{X}_u$  for a unique  $r_3$ -tuple of integer coefficients  $(c_0, c_1, \dots, c_{r_3-1})$ . Using this correspondence, we will now construct a matrix  $M$  with integer entries of size  $r_3 \times r_3$  (indexing its dimensions from 0 to  $r_3 - 1$ ) as follows:

- The 0-th row will correspond to the coefficients of  $\widehat{\beta}$  (i.e., it will consist of all 1's).
- The  $q$ -th row for  $1 \leq q \leq r_3 - 2$  will correspond to the coefficients of  $\widehat{\Gamma}_{q-1}$ .
- The  $(r_3 - 1)$ -th row will correspond to the coefficients of  $\widehat{\Delta}$ .

By the definition of  $M$ , we have

$$(5.9) \quad M \cdot \begin{pmatrix} \widehat{X}_0 \\ \widehat{X}_1 \\ \widehat{X}_2 \\ \widehat{X}_3 \\ \vdots \\ \widehat{X}_{r_3-2} \\ \widehat{X}_{r_3-1} \end{pmatrix} = \begin{pmatrix} \widehat{\beta} \\ \widehat{\Gamma}_0 \\ \widehat{\Gamma}_1 \\ \widehat{\Gamma}_2 \\ \vdots \\ \widehat{\Gamma}_{r_3-3} \\ \widehat{\Delta} \end{pmatrix}$$

We need to show that  $M$  is unimodular, i.e., invertible over  $\mathbb{Z}$ , from which it will follow that  $\ker \psi = \mathcal{X}$ , and consequently  $X_u = 0$  for all  $0 \leq u < r_3$ . To achieve that, we will study the effect of multiplying  $M$  by a character matrix (i.e., basically performing the discrete Fourier transform). But first we will need two technical lemmas, which will prove useful in a while.

Let

$$\begin{aligned}
 R(x) &:= \sum_{q=0}^{r_3-1} x^q \in \mathbb{Z}[x], \\
 D(x) &:= \sum_{q=0}^{r_3-1} q \cdot x^q \in \mathbb{Z}[x], \\
 P(x) &:= -x^{r_2-1} \cdot \sum_{q=0}^{r_1-1} x^q \in \mathbb{Z}[x].
 \end{aligned}$$

**Lemma 5.7.** *Let  $\zeta \neq 1$  be any  $r_3$ -th root of unity. Then we have  $R(\zeta) = 0$  and*

$$D(\zeta) \cdot (\zeta - 1) = r_3.$$

**Proof.** The first assertion is immediate since  $R(\zeta) \cdot (\zeta - 1) = \zeta^{r_3} - 1 = 0$ , but  $\zeta \neq 1$ . The second follows from the computation

$$\begin{aligned}
 D(\zeta) \cdot (\zeta - 1) &= \sum_{q=1}^{r_3-1} q \cdot \zeta^{q+1} - \sum_{q=1}^{r_3-1} q \cdot \zeta^q = \sum_{q=2}^{r_3} (q-1) \cdot \zeta^q - \sum_{q=1}^{r_3-1} q \cdot \zeta^q \\
 &= (r_3 - 1)\zeta^{r_3} + \sum_{q=1}^{r_3-1} (q-1) \cdot \zeta^q - \sum_{q=1}^{r_3-1} q \cdot \zeta^q \\
 &= r_3 - 1 - \sum_{q=1}^{r_3-1} \zeta^q \\
 &= r_3 - R(\zeta) = r_3.
 \end{aligned}$$

□

**Lemma 5.8.** *For any positive integer  $b$  and  $y \in \mathbb{C}$ , we have the equality*

$$(y - 1) \cdot \sum_{u=1}^b u \cdot y^u = (b + 1)y^{b+1} - \sum_{u=0}^b y^{u+1}.$$

**Proof.** We have

$$\begin{aligned}
 (y - 1) \cdot \sum_{u=1}^b u \cdot y^u &= \sum_{u=1}^b u \cdot y^{u+1} - \sum_{u=1}^b u \cdot y^u \\
 &= \sum_{u=0}^b u \cdot y^{u+1} - \sum_{u=0}^{b-1} (u+1) \cdot y^{u+1} \\
 &= b \cdot y^{b+1} + \sum_{u=0}^{b-1} (u - (u+1)) \cdot y^{u+1} \\
 &= (b + 1)y^{b+1} - \sum_{u=0}^b y^{u+1}.
 \end{aligned}$$

□

Now let  $\zeta$  be any  $r_3$ -th root of unity and consider the  $\mathbb{Z}$ -module homomorphism from  $\mathcal{X}$  to the cyclotomic field  $\mathbb{Q}(\zeta)$  given by

$$\sum_{u=0}^{r_3-1} c_u \widehat{X}_u \mapsto \sum_{u=0}^{r_3-1} c_u \zeta^u$$

(since  $\mathcal{X}$  is free, this is well defined and determines the homomorphism uniquely). We can apply this homomorphism to  $\widehat{\beta}$ ,  $\widehat{\Gamma}_q$ ,  $\widehat{\Delta}$  for any  $0 \leq q \leq r_3 - 3$ , and we will denote its respective values on these elements by  $\beta(\zeta)$ ,  $\Gamma_q(\zeta)$ ,  $\Delta(\zeta) \in \mathbb{Q}(\zeta)$ . Note that since  $\zeta^{r_3} = 1$ , we have  $\zeta^u = \zeta^{\bar{u}}$  for any  $u \in \mathbb{Z}$ .

**Lemma 5.9.** *Let  $\zeta \neq 1$  be any  $r_3$ -th root of unity. Then for all  $0 \leq q < r_3 - 3$ , we have*

$$\begin{aligned} \beta(\zeta) &= 0, \\ \Gamma_q(\zeta) &= \zeta^q \cdot P(\zeta) \end{aligned}$$

and

$$\Delta(\zeta) = D(\zeta) \cdot P(\zeta).$$

**Proof.** Note that  $\zeta^{-r_2} \neq 1$ , since  $\gcd(r_3, -r_2) = 1$  and  $\zeta \neq 1$ .

From the definitions and Lemma 5.7, we directly get  $\beta(\zeta) = R(\zeta) = 0$ . For the second assertion, we have

$$\begin{aligned} \Gamma_q(\zeta) &= \sum_{u=0}^{r_3-h'-1} \sum_{v=0}^{\bar{r}_2-1} \zeta^{\overline{q+v-ur_2-1}} \\ &= \zeta^{q-1} \cdot \sum_{v=0}^{\bar{r}_2-1} \zeta^v \sum_{u=0}^{r_3-h'-1} \zeta^{-ur_2} \\ &= \zeta^{q-1} \cdot \frac{\zeta^{\bar{r}_2} - 1}{\zeta - 1} \cdot \frac{\zeta^{-(r_3-h')r_2} - 1}{\zeta^{-r_2} - 1} \\ &= \zeta^{q-1} \cdot \frac{\zeta^{r_2} - 1}{\zeta^{-r_2} - 1} \cdot \frac{\zeta^{r_1} - 1}{\zeta - 1} \\ &= \zeta^q \cdot P(\zeta). \end{aligned}$$

Similarly, using Lemma 5.8 with  $y = \zeta^{-r_2}$  and  $b = r_3 - 1$ , we can see that

$$\begin{aligned} \Delta(\zeta) &= \sum_{u=0}^{r_3-1} u \cdot \sum_{v=0}^{\bar{r}_2-1} \sum_{w=0}^{\bar{r}_1-1} \zeta^{\overline{v+w-ur_2-1}} \\ &= \zeta^{-1} \cdot \sum_{v=0}^{\bar{r}_2-1} \zeta^v \sum_{w=0}^{\bar{r}_1-1} \zeta^w \sum_{u=0}^{r_3-1} u \cdot \zeta^{-ur_2} \\ &= \zeta^{-1} \cdot \frac{\zeta^{\bar{r}_2} - 1}{\zeta - 1} \cdot \frac{\zeta^{\bar{r}_1} - 1}{\zeta - 1} \cdot \frac{r_3 \zeta^{-r_2 r_3} - \sum_{u=0}^{r_3-1} \zeta^{-r_2(u+1)}}{\zeta^{-r_2} - 1} \end{aligned}$$



$$\begin{aligned}
 &= \zeta^{-1} \cdot \frac{\zeta^{\overline{r_2}} - 1}{\zeta - 1} \cdot \frac{\zeta^{\overline{r_1}} - 1}{\zeta - 1} \cdot \frac{r_3 - \zeta^{-r_2} \cdot R(\zeta^{-r_2})}{\zeta^{-r_2} - 1} \\
 &= \zeta^{-1} \cdot \frac{\zeta^{r_2} - 1}{\zeta - 1} \cdot \frac{\zeta^{r_1} - 1}{\zeta - 1} \cdot \frac{r_3}{\zeta^{-r_2} - 1} \\
 &= \zeta^{-1} \cdot \frac{r_3}{\zeta - 1} \cdot \frac{\zeta^{r_2} - 1}{\zeta^{-r_2} - 1} \cdot \frac{\zeta^{r_1} - 1}{\zeta - 1} \\
 &= D(\zeta) \cdot P(\zeta)
 \end{aligned}$$

by Lemma 5.7. □

**Proposition 5.10.** *M is unimodular.*

**Proof.** Let  $\zeta_{r_3}$  be a primitive  $r_3$ -th root of unity and let  $C$  be the corresponding  $r_3 \times r_3$  character matrix, i.e.,  $C = (\zeta_{r_3}^{r \cdot c})_{0 \leq r, c < r_3}$ . We will use the two previous lemmas together with the fact that multiplying a column of successive powers of  $\zeta_{r_3}$  by a row of  $M$  from the left corresponds to evaluating the polynomial obtained from this row at  $\zeta_{r_3}$ . Hence we have  $M \cdot C = C'$ , where  $C'_{0,0} = R(1) = r_3$  and the  $c$ -th column of  $C'$  is

$$\begin{pmatrix} R(\zeta_{r_3}^c) \\ P(\zeta_{r_3}^c) \\ \zeta_{r_3}^c \cdot P(\zeta_{r_3}^c) \\ (\zeta_{r_3}^c)^2 \cdot P(\zeta_{r_3}^c) \\ \vdots \\ (\zeta_{r_3}^c)^{r_3-3} \cdot P(\zeta_{r_3}^c) \\ D(\zeta_{r_3}^c) \cdot P(\zeta_{r_3}^c) \end{pmatrix} = \begin{pmatrix} 0 \\ P(\zeta_{r_3}^c) \\ \zeta_{r_3}^c \cdot P(\zeta_{r_3}^c) \\ \zeta_{r_3}^{2c} \cdot P(\zeta_{r_3}^c) \\ \vdots \\ \zeta_{r_3}^{(r_3-3)c} \cdot P(\zeta_{r_3}^c) \\ D(\zeta_{r_3}^c) \cdot P(\zeta_{r_3}^c) \end{pmatrix}$$

for any  $0 < c < r_3$  (we don't need to specify the rest of the 0-th column, since it doesn't influence the determinant of  $C'$ ). Thus by taking out  $P(\zeta_{r_3}^c)$  from each of these columns, we get (using that multiplication by  $r_1$  is an automorphism of  $\mathbb{Z}/r_3$ , since  $\gcd(r_1, r_3) = 1$ )

$$\begin{aligned}
 |\det C'| &= |\det C''| \cdot \left| \prod_{0 < c < r_3} P(\zeta_{r_3}^c) \right| \\
 &= |\det C''| \cdot \left| \prod_{0 < c < r_3} -\zeta_{r_3}^{c(r_2-1)} \right| \cdot \left| \prod_{0 < c < r_3} \frac{\zeta_{r_3}^{cr_1} - 1}{\zeta_{r_3}^c - 1} \right| \\
 &= |\det C''|,
 \end{aligned}$$

where

$$C'' = \begin{pmatrix} r_3 & 0 & \dots & 0 & \dots & 0 \\ * & 1 & \dots & 1 & \dots & 1 \\ * & \zeta_{r_3} & \dots & \zeta_{r_3}^c & \dots & \zeta_{r_3}^{r_3-1} \\ * & \zeta_{r_3}^2 & \dots & \zeta_{r_3}^{2c} & \dots & \zeta_{r_3}^{2(r_3-1)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ * & \zeta_{r_3}^{r_3-3} & \dots & \zeta_{r_3}^{(r_3-3)c} & \dots & \zeta_{r_3}^{(r_3-3)(r_3-1)} \\ * & D(\zeta_{r_3}) & \dots & D(\zeta_{r_3}^c) & \dots & D(\zeta_{r_3}^{r_3-1}) \end{pmatrix}.$$

On the other hand, we can take the matrix  $C$ , add all of its rows to the  $(r_3 - 1)$ -th one (thus creating  $(r_3 \ 0 \ 0 \ \dots \ 0)$  there) and then, using the equality

$$-\zeta_{r_3}^{(r_3-2)c} + \sum_{u=0}^{r_3-3} (u - r_3 + 1) \cdot \zeta_{r_3}^{uc} = \sum_{u=0}^{r_3-1} u \cdot \zeta_{r_3}^{uc} - (r_3 - 1) \cdot \underbrace{\sum_{u=0}^{r_3-1} \zeta_{r_3}^{uc}}_{=0},$$

multiply the  $(r_3 - 2)$ -th row by  $-1$  and add the  $u$ -th row multiplied by  $(u - r_3 + 1)$  for each  $0 \leq u \leq r_3 - 3$ , so that the  $(r_3 - 2)$ -th row will become

$$(* \ D(\zeta_{r_3}) \ \dots \ D(\zeta_{r_3}^c) \ \dots \ D(\zeta_{r_3}^{r_3-1})).$$

Thus we will obtain a matrix with the same determinant as  $C''$  (up to a sign). Since the elementary row operations preserve the determinant up to a sign, it follows that

$$|\det C| = |\det C''| = |\det C'| = |\det M| \cdot |\det C|.$$

Now,  $C$  can be seen as a special type of a Vandermonde matrix, so we have

$$\det C = \prod_{0 \leq r < c < r_3} (\zeta_{r_3}^r - \zeta_{r_3}^c) \neq 0$$

(in fact it is well known that  $|\det C| = \sqrt{r_3^{r_3}}$ ), which implies that  $|\det M| = 1$ , as needed. □

**Corollary 5.11.** *We have  $X_u = 0$  for all  $0 \leq q < r_3$ .*

**Proof.** Let  $M^{-1}$  be the inverse matrix to  $M$ . By Proposition 5.10, it exists and it has integer entries. From the equation (5.9), it then follows that

$$\begin{pmatrix} \widehat{X}_0 \\ \widehat{X}_1 \\ \widehat{X}_2 \\ \widehat{X}_3 \\ \vdots \\ \widehat{X}_{r_3-2} \\ \widehat{X}_{r_3-1} \end{pmatrix} = M^{-1} \cdot \begin{pmatrix} \widehat{\beta} \\ \widehat{\Gamma}_0 \\ \widehat{\Gamma}_1 \\ \widehat{\Gamma}_2 \\ \vdots \\ \widehat{\Gamma}_{r_3-3} \\ \widehat{\Delta} \end{pmatrix},$$

which implies that  $\widehat{\beta}, \widehat{\Gamma}_0, \widehat{\Gamma}_1, \dots, \widehat{\Gamma}_{r_3-3}, \widehat{\Delta}$  generate  $\mathcal{X}$ . But all of these elements lie in  $\ker \psi$  by (5.8), hence  $\ker \psi = \mathcal{X}$  and  $\psi$  is the zero homomorphism. On the

other hand, we know that the image of  $\psi$  is generated by  $X_0, X_1, \dots, X_{r_3-1}$  by the definition of  $\psi$ , so all of these must be zero as well.  $\square$

**Corollary 5.12.** *We have  $Y_u = 0$  for all  $r_1 + r_3 - 2 \leq u < n_2$ .*

**Proof.** By the Chinese remainder theorem, it suffices to show by induction with respect to  $u = 0, 1, \dots, r_3 - 1$  that for any  $0 \leq v < r_1$ , we have  $Y_{v-uhr_1} = 0$ . The base case  $u = 0$  follows directly from the definition of  $Y_u$ . Now suppose the statement is true for a given  $0 \leq u < r_3 - 1$ . Then using  $N_1 \sim 0$  and Lemma 5.3, we get

$$\begin{aligned} 0 &= \sigma_2^{v-uhr_1} N_1 \cdot \mu = \sum_{x_1=r_2(r_3-1)-1}^{n_1-1} \sigma_1^{x_1} \sigma_2^{v-uhr_1} \cdot \mu \\ &= \underbrace{Y_{v-uhr_1}}_{=0} - Y_{v-uhr_1-hr_1} + \sum_{x_1=r_2(r_3-1)}^{n_1-1} \underbrace{X_{v-uhr_1-x_1-2}}_{=0} = -Y_{v-(u+1)hr_1} \end{aligned}$$

by the induction hypothesis and by Corollary 5.11. This completes the induction.  $\square$

By Lemma 5.3, it now follows that  $Q$  is trivial, so we have proven the following theorem for the set  $B_5$  defined on page 232:

**Theorem 5.13.** *Under Assumption 3.1, if*

$a_1 = a_2 = a_3 = r_4 = 1, r_1 \neq 1, r_2 \neq 1, r_3 \neq 1, s_{12} = s_{13} = s_{23} = 1, \gcd(n_1, n_2, n_3) = 1$ , then the set  $B_5 \cup B_D$  forms a basis of  $D^+$  and the set  $B_5 \cup B_C$  forms a basis of  $C^+$ .

### 6. FOUR MORE SPECIAL CASES

In a similar, although less technical way, a  $\mathbb{Z}$ -basis of  $D^+$  and  $C^+$  can be constructed in another four cases, as given below. The details can be found in [6].

**Theorem 6.1.** *Under Assumption 3.1, if  $r_1 = r_2 = r_3 = r_4 = 1$ , then the set  $B_1 \cup B_D$  forms a basis of  $D^+$  and the set  $B_1 \cup B_C$  forms a basis of  $C^+$ , where  $B_1$  is the set of the following conjugates  $\eta^{\sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{x_3} \sigma_4^{x_4}}$ :*

- $0 \leq x_1 < a_1 m - 1, 0 \leq x_2 < a_2 m - 1, 0 \leq x_3 < a_3 m - 1, 1 \leq x_4 < a_4,$
- $0 \leq x_1 < a_1 m - 1, 0 \leq x_2 < a_2(m - 1) - 1, 0 \leq x_3 < a_3 m - 1, x_4 = 0,$
- $0 \leq x_1 < a_1 - 1, x_2 = a_2(m - 1) - 1, 0 \leq x_3 < a_3 m - 1, x_4 = 0,$
- $x_1 = a_1 - 1, x_2 = a_2(m - 1) - 1, 0 \leq x_3 < a_3(m - 1), x_4 = 0.$

**Theorem 6.2.** *Under Assumption 3.1, if  $r_1 = r_2 = a_3 = r_4 = 1$ , then the set  $B_2 \cup B_D$  forms a basis of  $D^+$  and the set  $B_2 \cup B_C$  forms a basis of  $C^+$ , where  $B_2$  is the set of the following conjugates  $\eta^{\sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{x_3} \sigma_4^{x_4}}$ :*

- $0 \leq x_1 < a_1 m - 1, 0 \leq x_2 < a_2 m - 1, 0 \leq x_3 < n_3 - 1, 1 \leq x_4 < a_4,$
- $0 \leq x_1 < a_1 m - 1, 0 \leq x_2 < a_2(m - 1) - 1, 1 \leq x_3 < n_3, x_4 = 0,$
- $0 \leq x_1 < a_1, x_2 = a_2(m - 1) - 1, 1 \leq x_3 < n_3, x_4 = 0.$

**Theorem 6.3.** *Under Assumption 3.1, if  $a_1 = a_2 = r_3 = r_4 = 1$ , then the set  $B_3 \cup B_D$  forms a basis of  $D^+$  and the set  $B_3 \cup B_C$  forms a basis of  $C^+$ , where  $B_3$  is the set of the following conjugates  $\eta^{\sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{x_3} \sigma_4^{x_4}}$ :*

- $0 \leq x_1 < n_1 - 1, 0 \leq x_2 < n_2 - 1, 0 \leq x_3 < a_3m - 1, 0 < x_4 \leq a_4 - 1,$
- $0 \leq x_1 < n_1 - 1, 0 \leq x_2 < n_2 - 1, a_3 < x_3 < a_3m, x_4 = 0,$
- $1 \leq x_1 < \gcd(n_1, n_2), x_2 = 0, x_3 = 0, x_4 = 0.$

**Theorem 6.4.** *Under Assumption 3.1, if*

$$a_1 = a_2 = a_3 = r_4 = 1, \gcd(n_1, n_2, n_3) = \gcd(n_1, n_2),$$

*then the set  $B_4 \cup B_D$  forms a basis of  $D^+$  and the set  $B_4 \cup B_C$  forms a basis of  $C^+$ , where  $B_4$  is the set of the following  $N$  conjugates  $\eta^{\sigma_1^{x_1} \sigma_2^{x_2} \sigma_3^{x_3} \sigma_4^{x_4}}$ :*

- $0 \leq x_1 < n_1 - 1, 0 \leq x_2 < n_2 - 1, 0 \leq x_3 < n_3 - 1, 0 < x_4 \leq a_4 - 1,$
- $0 \leq x_1 < n_1 - 1, 0 \leq x_2 < n_2 - 1, 1 < x_3 \leq n_3 - 1, x_4 = 0,$
- $1 \leq x_1 < n_1, \gcd(n_2, n_3) \leq x_2 < n_2, x_3 = 0, x_4 = 0,$
- $\gcd(n_1, n_3) \leq x_1 < n_1, 1 \leq x_2 < \gcd(n_2, n_3), x_3 = 0, x_4 = 0,$
- $1 \leq x_1 < \gcd(n_1, n_2), x_2 = 0, x_3 = 0, x_4 = 0.$

### 7. THE MODULE OF RELATIONS

In this section, we will study the relations between the generators of the group of circular numbers more abstractly, following the approach in [2]. Sometimes we will only state the results and just outline the proofs or even omit them altogether.

Consider the (additively written)  $\mathbb{Z}[G]$ -module

$$\begin{aligned} \mathcal{X} &:= \bigoplus_{\emptyset \subsetneq I \subseteq \{1,2,3,4\}} \mathbb{Z}[\text{Gal}(k \cap \prod_{i \in I} K_i) / \mathbb{Q}] \\ &= \mathbb{Z}[\text{Gal}(k / \mathbb{Q})] \oplus \bigoplus_{i,j,l} \mathbb{Z}[\text{Gal}(k \cap K_i K_j K_l) / \mathbb{Q}] \\ &\quad \oplus \bigoplus_{i,j} \mathbb{Z}[\text{Gal}(k \cap K_i K_j) / \mathbb{Q}] \oplus \bigoplus_i \mathbb{Z}[\text{Gal}(k \cap K_i) / \mathbb{Q}], \end{aligned}$$

where  $G$  acts on each summand via restriction. For any  $\emptyset \subsetneq I \subseteq \{1, 2, 3, 4\}$ , we will denote  $x_I$  the element of  $\mathcal{X}$  having all coordinates zero except for 1 at the position corresponding to  $I$ . To simplify the notation, we will sometimes write simply

$$x := x_{\{1,2,3,4\}}, x_{ijl} := x_{\{i,j,l\}}, x_{ij} := x_{\{i,j\}}, x_i := x_{\{i\}}$$

and similarly

$$\eta_{ijl} := \eta_{\{p_i, p_j, p_l\}}, \eta_{ij} := \eta_{\{p_i, p_j\}}, \eta_i := \eta_{\{p_i\}}.$$

Therefore we have

$$\mathcal{X} = \langle x, x_{123}, x_{124}, x_{134}, x_{234}, x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, x_1, x_2, x_3, x_4 \rangle_{\mathbb{Z}[G]}$$

and

$$D^+ = \langle \eta, \eta_{123}, \eta_{124}, \eta_{134}, \eta_{234}, \eta_{12}, \eta_{13}, \eta_{14}, \eta_{23}, \eta_{24}, \eta_{34}, \eta_1, \eta_2, \eta_3, \eta_4 \rangle_{\mathbb{Z}[G]}.$$

Since

$$\eta \in k, \quad \eta_{ijl} \in k \cap K_i K_j K_l, \quad \eta_{ij} \in k \cap K_i K_j \quad \text{and} \quad \eta_i \in k \cap K_i,$$

this gives us a surjective homomorphism of  $\mathbb{Z}[G]$ -modules  $\varphi: \mathcal{X} \rightarrow D^+$  defined by

$$\varphi(x) = \eta, \quad \varphi(x_{ijkl}) = \eta_{ijkl}, \quad \varphi(x_{ij}) = \eta_{ij}, \quad \varphi(x_i) = \eta_i.$$

Then  $\ker \varphi$  is a  $\mathbb{Z}[G]$ -submodule of  $\mathcal{X}$ , and we will call it *the module of relations* of  $k$ , because we can regard its elements as the relations between the generators of the group of circular numbers of  $k$ .

Lemmas 2.3 and 2.6 imply that for any  $I \subseteq \{1, 2, 3, 4\}$ ,  $|I| \geq 2$  and  $i \in I$ , we have

$$\mathbb{N}_{k \cap \prod_{u \in I} K_u / k \cap \prod_{u \in I \setminus \{i\}} K_u} \varphi(x_I) \in C^+ \left( k \cap \prod_{u \in I \setminus \{i\}} K_u \right),$$

hence there exists some

$$\rho_{i,I} \in \langle \{x_J \mid \emptyset \subsetneq J \subseteq I \setminus \{i\}\} \rangle_{\mathbb{Z}[G]}$$

such that

$$N_{i,I} := \mathbb{N}_{k \cap \prod_{u \in I} K_u / k \cap \prod_{u \in I \setminus \{i\}} K_u} x_I - \rho_{i,I} \in \ker \varphi.$$

We will call  $N_{i,I}$  a *norm relation*. Note that for  $I = \{1, 2, 3, 4\}$ , we have

$$\mathbb{N}_{k \cap \prod_{u \in I} K_u / k \cap \prod_{u \in I \setminus \{i\}} K_u} x = R_i N_{i,I}.$$

**Remark 7.1.** In fact, the relation  $N_{i,I}$  can be described much more explicitly using the Frobenius automorphisms, but we won't go into the details here.

Now let  $M$  be the  $\mathbb{Z}[G]$ -submodule of  $\ker \varphi$  generated by the norm relations  $N_{i,I}$  for all possible  $I \subseteq \{1, 2, 3, 4\}$ ,  $|I| \geq 2$  and  $i \in I$ . Our goal will be to describe the quotient  $\mathbb{Z}[G]$ -module  $\ker \varphi / M$ , which we will call *the module of Ennola relations* of  $k$ . (However, to follow the terminology in [2], by an *Ennola relation* we will mean an element of  $\ker \varphi \setminus M$  rather than  $\ker \varphi / M$ .)

Let  $E_{ijkl}$  be the Ennola relation described by Theorem 10 in [2] applied to the field  $k \cap K_i K_j K_l$ . By Theorem 19 there,  $E_{ijkl}$  generates all the Ennola relations (modulo the norm relations) for this field.

**Proposition 7.2.** *In all the cases described in sections 5 and 6, the  $\mathbb{Z}[G]$ -module  $\ker \varphi / M$  is generated by the classes of  $E_{123}, E_{124}, E_{134}, E_{234}$ . In addition, the action of  $G$  on  $\ker \varphi / M$  is trivial.*

**Proof.** For any case described in Sections 5 and 6, let  $B$  be a  $\mathbb{Z}$ -basis of  $D^+$ . For any element of  $B$ , we will fix its preimage with respect to  $\varphi$ ; let  $Y$  be the set of these fixed preimages. Then the elements of  $Y$  are  $\mathbb{Z}$ -linearly independent and we have  $\mathcal{X} = \ker \varphi \oplus Y$ . Recall that in order to construct  $B$ , we always used only ( $\mathbb{Z}[G]$ -linear combinations of) norm relations together with the four implicit Ennola relations  $E_{123}, E_{124}, E_{134}, E_{234}$  from [2]. This shows that  $\ker \varphi$  is generated by  $M \cup \{E_{123}, E_{124}, E_{134}, E_{234}\}$ , which proves the first part of the proposition. The second part follows from the observation that the action of  $G$  on  $E_{ijkl}$  is the same as the action of  $\text{Gal}(k \cap K_i K_j K_l / \mathbb{Q})$  on  $E_{ijkl}$ , which is trivial by Theorem 19 in [2].  $\square$

In certain cases, we can say something stronger. For the rest of this paragraph and for the next lemma, we will only use Assumption 3.1 (so we're not focusing on the five cases in Sections 5 and 6 yet). As in the proof of Lemma 4.1, let  $K'$  be the genus field in the narrow sense of  $k' = k \cap K_1K_2K_3$ , and for any  $u \in \{1, 2, 3\}$ , let  $K'_u$  be the maximal subfield of  $K'$  ramified only at  $p_u$ ,  $T'_u$  be the inertia subgroup of  $\text{Gal}(K'/\mathbb{Q})$  corresponding to  $p_u$  and  $r'_u := [K' : k'K'_u]$ . The order of  $E_{123}$  in the module  $\ker \varphi/M$  of Ennola relations of  $k$  is a divisor of the order of  $E_{123}$  in the module of Ennola relations of  $k'$ , which is equal to  $\frac{[K':k']}{r'_1r'_2r'_3}$  by Theorem 19 of [2].

**Lemma 7.3.** *Let  $s_{ij} = 1$  for all  $i, j \in \{1, 2, 3\}$ . Then we have  $T'_u = T_u$ ,  $K_u = K'_u$  and  $r'_u = r_u$  for all  $u \in \{1, 2, 3\}$  and moreover  $[K' : k'] = m$ .*

**Proof.** Using Proposition 3.3, the ramification index of  $p_1$  in  $k'/\mathbb{Q}$  is

$$|T'_1| = [k' : k' \cap K_2K_3] = \frac{[K_1K_2K_3 : k \cap K_2K_3]}{[K_1K_2K_3 : k \cap K_1K_2K_3]} = \frac{|T_1| \cdot \frac{m}{s_{23}}}{m} = \frac{|T_1|}{s_{23}}.$$

Since  $s_{23} = 1$ , we have  $|T'_1| = |T_1|$  and  $K'_1 = K_1$ . Similarly,  $|T'_2| = |T_2|$ ,  $|T'_3| = |T_3|$  and  $K'_2 = K_2$ ,  $K'_3 = K_3$ . Hence

$$[K' : k'] = [K_1K_2K_3 : k \cap K_1K_2K_3] = m$$

and

$$r'_1 = [K' : K'_1k'] = \frac{[K' : k']}{[K_1k' : k']} = \frac{m}{[K_1 : K_1 \cap k']} = \frac{m}{[K_1 : k \cap K_1]} = \frac{m}{r_1} = r_1$$

by Proposition 3.3 again (and similarly  $r'_2 = r_2$ ,  $r'_3 = r_3$ ). □

Obvious analogies of Lemma 7.3 could be also stated for the fields  $k \cap K_1K_2K_4$ ,  $k \cap K_1K_3K_4$  and  $k \cap K_2K_3K_4$ , but we would get a collision in notation, because of our assymmetric definition of  $k'$ . The proofs would be exactly the same though, because the only assymetry present is purely notational.

**Corollary 7.4.** *In the following three cases, more can be said about  $\ker \varphi/M$ :*

- (i) *If  $r_1 = r_2 = r_3 = r_4 = 1$ , then  $\ker \varphi/M$  is a quotient of  $(\mathbb{Z}/m)^4$ .*
- (ii) *If  $r_1 = r_2 = a_3 = r_4 = 1$ , then  $\ker \varphi/M$  is a quotient of  $(\mathbb{Z}/n_3)^3 \times \mathbb{Z}/m$ .*
- (iii) *If*

$$a_1 = a_2 = a_3 = r_4 = 1, \quad r_1 \neq 1, \quad r_2 \neq 1, \quad r_3 \neq 1, \\ s_{12} = s_{13} = s_{23} = 1, \quad \gcd(n_1, n_2, n_3) = 1,$$

*then  $\ker \varphi/M$  is a quotient of  $\mathbb{Z}/r_1 \times \mathbb{Z}/r_2 \times \mathbb{Z}/r_3$ , so it is in particular cyclic.*

**Proof.** This follows immediately from Proposition 7.2 and the mentioned analogies of Lemma 7.3 (in the third part we also use the pairwise coprimality of  $r_1, r_2, r_3$  by Lemma 3.6). □

**Remark 7.5.** In the case

$$a_1 = a_2 = a_3 = a_4 = r_1 = r_2 = r_3 = r_4 = 1$$

(which is a special case of the first case in Section 6), it can be shown that  $\ker \varphi/M \cong (\mathbb{Z}/m)^4$ , which is a stronger result than in Corollary 7.4. The proof is too technical to be included here, but essentially it consists of constructing a  $\mathbb{Z}$ -module (not  $\mathbb{Z}[G]$ -module!) homomorphism from  $\mathcal{X}$  to  $\mathbb{Z}/m$  and showing that all the norm relations together with three of the four Ennola relations lie in its kernel, while the fourth Ennola relation maps to the class of 1 modulo  $m$ . It is possible that a similar approach could be used in other cases to improve the bounds in Corollary 7.4.

**Remark 7.6.** A crucial part of the proof of Proposition 7.2 was the fact that in all of the cases studied in Sections 5 and 6, we never encountered any new Ennola relation, i.e., an element of  $\ker \varphi \setminus M$  having a nonzero coefficient at  $x$ . This will not always be the case though, because we have already found a new Ennola relation  $E$  in the special case

$$m = a_3 = r_3 = 2, a_1 = a_2 = a_4 = r_1 = r_2 = r_4 = 1.$$

It's not very hard to show that  $E \notin M$  (and  $2E \in M$ ), but the proof that  $E \notin \ker \varphi$  is again too technical to be described here. Note that in this case, we have  $N = 0$  (recall that  $N$  was defined by the equation (4.1)), but it is still possible to recover all the conjugates of  $\eta$  using this new Ennola relation  $E$ .

In fact, it appears quite plausible that a new Ennola relation could arise whenever we have  $a_i > 1$  and  $r_i > 1$  at the same time. It is not a coincidence that this didn't happen in any of the cases studied in Sections 5 and 6, because it seems that this assumption will drastically increase the difficulty of the construction of  $\mathbb{Z}$ -bases of  $D^+$  and  $C^+$ .

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE,  
MASARYK UNIVERSITY,  
611 37 BRNO, CZECH REPUBLIC  
*E-mail:* vlada.sedlacek@mail.muni.cz