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APPROXIMATION OF LIMIT CYCLE OF DIFFERENTIAL SYSTEMS WITH VARIABLE COEFFICIENTS

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Dedicated to Professor Tetsuo Furumochi on the occasion of his 75th birthday

Abstract. The behavior of the approximate solutions of two-dimensional nonlinear differential systems with variable coefficients is considered. Using a property of the approximate solution, so called conditional Ulam stability of a generalized logistic equation, the behavior of the approximate solution of the system is investigated. The obtained result explicitly presents the error between the limit cycle and its approximation. Some examples are presented with numerical simulations.

1. Introduction

We consider the two-dimensional nonlinear differential system

$$
\begin{aligned}
x' &= f(t)x + g(t)y - \frac{f(t)}{\kappa} x \left(x^2+y^2\right)^{\frac{\alpha}{2}}\,, \\
y' &= -g(t)x + f(t)y - \frac{f(t)}{\kappa} y \left(x^2+y^2\right)^{\frac{\alpha}{2}}\,,
\end{aligned}
\tag{1.1}
$$

and its perturbed system

$$
\begin{aligned}
x' &= f(t)x + g(t)y - \frac{f(t)}{\kappa} x \left(x^2+y^2\right)^{\frac{\alpha}{2}} + p_1(t)\,, \\
y' &= -g(t)x + f(t)y - \frac{f(t)}{\kappa} y \left(x^2+y^2\right)^{\frac{\alpha}{2}} + p_2(t)\,,
\end{aligned}
\tag{1.2}
$$

where f, g, p_1 and p_2 are real-valued continuous functions for $t \geq 0$, and α and κ are positive constants. If $f = g \equiv 1$, $\alpha = 2$ and $\kappa = 1$, then (1.1) is reduces to the

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differential system

$$x' = x + y - x\left(x^2 + y^2\right)\,, \qquad (1.3)$$
$$y' = -x + y - y\left(x^2 + y^2\right)\,.$$

This system is well-known to have exactly one stable limit cycle $x^2 + y^2 = 1$ (see, [13]); that is, on the phase plane, there is the orbit of the unique periodic solution of (1.3) that rotates infinitely on the unit circle, and any orbit of the solution except the zero solution $(x(t), y(t)) \equiv (0,0)$ and the periodic solution approaches while rotating to the unit circle.

On the other hand, if $p_1 \not\equiv 0 \not\equiv p_2$, then the differential system

$$x' = x + y - x\left(x^2 + y^2\right) + p_1(t)\,, \qquad (1.4)$$
$$y' = -x + y - y\left(x^2 + y^2\right) + p_2(t)\,.$$

does not have the zero solution and it is unknown whether it has a periodic solution. Needless to say, it will be very difficult to derive the conditions for the system to have a limit cycle because (1.4) is a nonautonomous differential system. If it is an autonomous system, many tools can be used, for example, the well-known Poincaré-Bendixon theorem, but a different approach will be needed for nonautonomous systems. See [2, 3, 7, 9, 10, 11, 14, 20] for recent results related to limit cycles. Here, instead of looking for the periodic orbit or limit cycle of (1.4), it can be regarded as a perturbed system of (1.3). If we impose some constraints on p_1 and p_2, we would expect the solution of (1.4) to be an approximation of the solution of (1.3). A well-known tool is the linear approximation method, but unfortunately (1.3) dealt with here is a nonlinear system. In this study, we will introduce a new tool for approximating nonlinear systems. It provides an approximation of the limit cycle by using a property called conditional Ulam stability for a scalar nonlinear equation. The definition of conditional Ulam stability will be given in the next section.

Define $\|(x, y)\| := \sqrt{x^2 + y^2}$. The main result of this study is as follows.

Theorem 1.1. *Suppose that there exists $\underline{f} > 0$ such that*

$$f(t) \geq \underline{f} \quad \text{for} \quad t \geq 0\,. \qquad (1.5)$$

Let $\varepsilon \in \left(0, \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\right]$, $\|(x_0, y_0)\| \in \left[\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}, \infty\right)$, *let* $(x(t), y(t))$ *and* $(\xi(t), \eta(t))$ *be the solutions of* (1.1) *and* (1.2) *with*

$$\big(x(0), y(0)\big) = \big(\xi(0), \eta(0)\big) = (x_0, y_0)\,, \qquad (1.6)$$

respectively. If

$$\|\big(p_1(t), p_2(t)\big)\| \leq \varepsilon \quad \text{for} \quad t \geq 0\,, \qquad (1.7)$$

then $(x(t), y(t))$ *and* $(\xi(t), \eta(t))$ *exist on* $[0, \infty)$. *Furthermore,*

$$\min\left\{\|\big(\xi(t), \eta(t)\big)\|, \|\big(x(t), y(t)\big)\|\right\} \geq \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}},$$

and

$$\big|\|\big(\xi(t),\eta(t)\big)\| - \|\big(x(t),y(t)\big)\|\big| \le \max\Big\{\frac{\alpha+1}{\alpha\underline{f}}, \frac{\alpha+1}{\alpha^2\underline{f}}\Big\}\varepsilon$$

for $t \in [0,\infty)$.

If $f = g \equiv 1$, $\alpha = 2$ and $\kappa = 1$, then we immediately obtain the following result.

Corollary 1.2. *Let* $\varepsilon \in \Big(0, \frac{2}{3\sqrt{3}}\Big]$, $\|(x_0,y_0)\| \in \Big[\frac{1}{\sqrt{3}}, \infty\Big)$, *let* $(x(t),y(t))$ *and* $(\xi(t),\eta(t))$ *be the solutions of* (1.3) *and* (1.4) *with* (1.6), *respectively. If* (1.7) *holds, then* $(x(t),y(t))$ *and* $(\xi(t),\eta(t))$ *exist on* $[0,\infty)$. *Furthermore,*

$$\min\big\{\|\big(\xi(t),\eta(t)\big)\|, \|\big(x(t),y(t)\big)\|\big\} \ge \frac{1}{\sqrt{3}},$$

and

$$\big|\|\big(\xi(t),\eta(t)\big)\| - \|\big(x(t),y(t)\big)\|\big| \le \frac{3}{2}\varepsilon$$

for $t \in [0,\infty)$.

We denote a circle with radius $R > 0$ centered at the origin by C_R. Let (x_0,y_0) on the circle $C_{\frac{1}{\sqrt{3}}}$ or, be outside the circle $C_{\frac{1}{\sqrt{3}}}$. Now we consider the solutions $(x(t),y(t))$ and $(\xi(t),\eta(t))$ of (1.3) and (1.4) with (1.6) and

$$p_1(t) = \frac{2}{3\sqrt{3}}\big(1 - 2\max\big\{\cos\sqrt{t}, 0\big\}\big) \quad \text{and} \quad p_2(t) = 0\,, \tag{1.8}$$

respectively. From $\|(p_1(t),p_2(t))\| \le \frac{2}{3\sqrt{3}}$ for $t \ge 0$, we can choose $\varepsilon = \frac{2}{3\sqrt{3}}$, and using Corollary 1.2, we see that $(x(t),y(t))$ and $(\xi(t),\eta(t))$ on the circle $C_{\frac{1}{\sqrt{3}}}$ or, are outside the circle $C_{\frac{1}{\sqrt{3}}}$, and

$$\big|\|\big(\xi(t),\eta(t)\big)\| - \|\big(x(t),y(t)\big)\|\big| \le \frac{3}{2}\varepsilon = \frac{1}{\sqrt{3}} \quad \text{for} \quad t \in [0,\infty)\,. \tag{1.9}$$

Figure 1 shows the orbits corresponding to $(x(t),y(t))$ and $(\xi(t),\eta(t))$ of (1.3) and (1.4) with (1.6), (1.8) and

$$(x_0,y_0) = \Big(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\Big)\,. \tag{1.10}$$

Figure 2 shows the orbits corresponding to $(x(t),y(t))$ and $(\xi(t),\eta(t))$ of (1.3) and (1.4) with (1.6), (1.8) and

$$(x_0,y_0) = (1.2, 1.2)\,. \tag{1.11}$$

The circle $C_{\frac{1}{\sqrt{3}}}$ is drawn with broken line. Figure 3 shows the orbits corresponding to $(x(t),y(t))$ and $(\xi(t),\eta(t))$ of (1.3) and (1.4) with (1.6), (1.8) and

$$(x_0,y_0) = \Big(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\Big)\,; \tag{1.12}$$

that is, (x_0,y_0) on the unit circle. This means that the orbit of $(x(t),y(t))$ represent the limit cycle. From this, we can conclude that the orbit of $(\xi(t),\eta(t))$ represent an approximation of the limit cycle. Note here that all orbits in Figures 1–3 are drawn for $0 \le t \le 150$. If we draw more time than 150, the red curve will fill the inside of the lip-like area. From (1.9), we see that the orbit corresponding to $(\xi(t), \eta(t))$ of (1.4) with (1.6), (1.8) and (1.12) is inside the circle $C_{1+\frac{1}{\sqrt{3}}}$ for $t \in [0, \infty)$. The purpose of this study is to explicitly present the error between the limit cycle of (1.1) and its approximation.

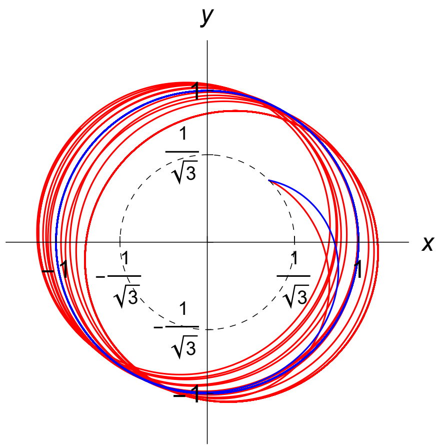

FIG. 1: Orbits of (1.3) (blue) and (1.4) (red) with (1.8) and (1.10).

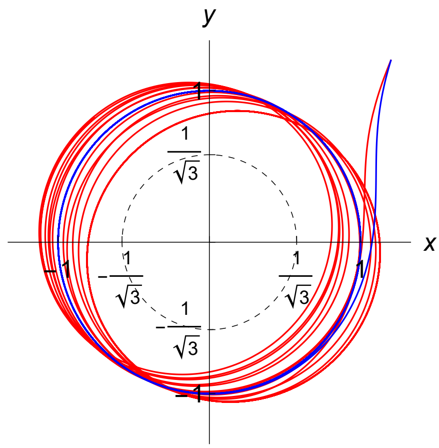

FIG. 2: Orbits of (1.3) (blue) and (1.4) (red) with (1.8) and (1.11).

In Section 2, we introduce the concept of conditional Ulam stability and present previous results that play an important role in this study. In Section 3, we prove Theorem 1.1 by using a previous result. In Section 4, we present the second main result and prove it. In Section 5, we give two examples of variable coefficients and present numerical simulations.

2. Conditional Ulam stability

In this section, we consider the nonautonomous generalized logistic equation

$$z' = h(t)z\left(1 - \frac{z^\alpha}{K}\right), \tag{2.1}$$

where h is a positive continuous function for $t \geq 0$, and α and K are positive constants. Especially when $h(t)$ is a constant, (2.1) is called the Richards model, which is one of the models that describe infectious diseases. Here, z, h, and K represent the cumulative number of cases/deaths, growth rate, and final epidemic size, respectively. In [16], the present author studied conditional Ulam stability of (2.1). Conditional Ulam stability is a property that guarantees the difference between the approximate solution and the exact solution to be finite. The exact definition is as follows.

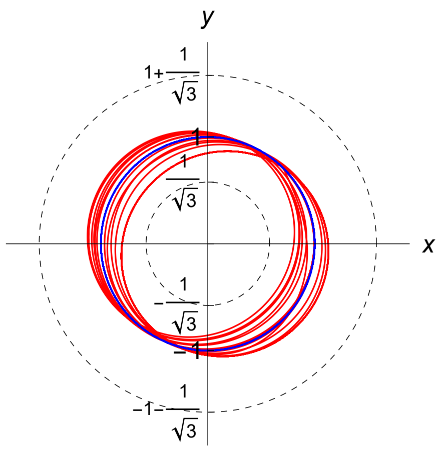

FIG. 3: Orbits of (1.3) (blue) and (1.4) (red) with (1.8) and (1.12).

Definition 2.1. Let $A \subseteq (0,\infty)$ and $B \subseteq \mathbb{R}$ be nonempty sets. Define the class $\mathcal{C}_B := \left\{x \in C^1[0,T_x) : x(0) \in B, T_x > 0 \text{ with } T_x = \infty \text{ or } |x(t)| \to \infty \text{ as } t \nearrow T_x\right\}$. Note that $[0,T_x)$ refers to the maximal existence interval of $x(t)$. The nonlinear differential equation

$$z' = F(t,z) \tag{2.2}$$

is *conditionally Ulam stable* on $[0,\min\{T_z,T_\zeta\})$ with A in the class $\mathcal{C}_B$ if there exists $L>0$ such that for any $\varepsilon \in A$ and any approximate solution $\zeta \in \mathcal{C}_B$ that satisfy

$$|\zeta' - F(t,\zeta)| \le \varepsilon \quad \text{for} \quad t \in [0,T_\zeta)\ ,$$

there exists a solution $z \in \mathcal{C}_B$ of (2.2) such that $|\zeta(t)-z(t)| \le L\varepsilon$ for $t \in [0,\min\{T_z,T_\zeta\})$. We call such an L an *Ulam constant* for (2.2) on $[0,\min\{T_z,T_\zeta\})$.

If $A=(0,\infty)$ and $B=\mathbb{R}$, then this definition is exactly the same as that for the standard Ulam stability. See [1, 4, 5, 6, 8, 12, 15, 17, 18, 19] for previous studies on standard and conditional Ulam stabilities. In [16], the present author obtained the following results.

Theorem 2.2 ([16, Theorem 5.1])**.** *Suppose that there exists* $\underline{h}>0$ *such that*

$$h(t) \ge \underline{h} \quad \textit{for} \quad t \ge 0\,.$$

Let $\varepsilon \in \left(0, \frac{\alpha \underline{h} K^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\right]$, $z_0 \in \left[\left(\frac{K}{\alpha+1}\right)^{\frac{1}{\alpha}}, \infty\right)$, *let* $z \in C^1[0,T_z)$ *be the solution of* (2.1) *with* $z(0)=z_0$, *and let* $\zeta \in C^1[0,T_\zeta)$ *be the solution of the perturbed* *nonautonomous Richards model*

$$\zeta' = h(t)\zeta\Big(1 - \frac{\zeta^\alpha}{K}\Big) + p(t)\,, \quad |p(t)| \le \varepsilon \tag{2.3}$$

with $\zeta(0) = z_0$*, where* $p(t)$ *is a real-valued continuous function. Then the global existence of the solutions of* (2.1) *and* (2.3) *with* $z(0) = \zeta(0) = z_0$ *is guaranteed. That is,* $T_z = T_\zeta = \infty$ *holds. Furthermore,*

$$|\zeta(t) - z(t)| \le \max\Big\{\frac{\alpha+1}{\alpha\underline{h}}, \frac{\alpha+1}{\alpha^2\underline{h}}\Big\}\varepsilon \quad for \quad t \in [0,\infty)\,.$$

Theorem 2.3 ([16, Theorem 5.2])**.** *Suppose that there exists* $\underline{h} > 0$ *such that*

$$h(t) \ge \underline{h} \quad for \ t \ge 0.$$

Let $A = \left(0, \frac{\alpha\underline{h}K^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\right]$ *and* $B = \left[\left(\frac{K}{\alpha+1}\right)^{\frac{1}{\alpha}}, \infty\right)$*. Then* (2.1) *is conditionally Ulam stable on* $[0,\infty)$ *with* A *in the class* $\mathcal{C}_B$*. Furthermore,* $L = \max\left\{\frac{\alpha+1}{\alpha\underline{h}}, \frac{\alpha+1}{\alpha^2\underline{h}}\right\}$ *is an Ulam constant on* $[0,\infty)$*.*

In this study, we will especially use Theorem 2.2, which is clearly given the initial values, to help analyze the approximate solutions of (1.1).

3. Proof of main result

Using the polar transformation $x = r\cos\theta$ and $y = r\sin\theta$ to (1.1) and (1.2), we obtain the systems

$$\begin{aligned} r' &= f(t)r\Big(1 - \frac{r^\alpha}{\kappa}\Big)\,, \\ r\theta' &= -g(t)r\,, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} r' &= f(t)r\Big(1 - \frac{r^\alpha}{\kappa}\Big) + p_1(t)\cos\theta + p_2(t)\sin\theta, \\ r\theta' &= -g(t)r - p_1(t)\sin\theta + p_2(t)\cos\theta \end{aligned} \tag{3.2}$$

for $t \ge 0$, respectively. In this section, first we present the proof of main result by using (3.1), (3.2) and Theorem 2.2.

Proof of Theorem 1.1. Suppose that there exists $\underline{f} > 0$ such that (1.5) holds. Given an arbitrary $\varepsilon \in \left(0, \frac{\alpha\underline{f}\kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\right]$, suppose that (1.7) holds. Let $(x(t), y(t))$ and $(\xi(t), \eta(t))$ be the solutions of (1.1) and (1.2) with (1.6) and

$$\|(x_0, y_0)\| \in \Big[\Big(\frac{\kappa}{\alpha+1}\Big)^{\frac{1}{\alpha}}, \infty\Big)\,,$$

respectively. Now we consider the solution $(\rho(t),\phi(t))$ of (3.2) corresponding to $(\xi(t),\eta(t))$. Then, by (1.7), we see that

$$\begin{aligned}\left\|\begin{pmatrix}\rho'-f(t)\rho\left(1-\frac{\rho^\alpha}{\kappa}\right)\\ \rho\big(\phi'+g(t)\big)\end{pmatrix}\right\| &= \left\|\begin{pmatrix}p_1(t)\cos\phi+p_2(t)\sin\phi\\ -p_1(t)\sin\phi+p_2(t)\cos\phi\end{pmatrix}\right\|\\ &=\sqrt{(p_1(t)\cos\phi+p_2(t)\sin\phi)^2+(-p_1(t)\sin\phi+p_2(t)\cos\phi)^2}\\ &=\sqrt{p_1^2(t)+p_2^2(t)}=\|\big(p_1(t),p_2(t)\big)\|\le\varepsilon\quad\text{for}\quad t\ge 0\,.\end{aligned}$$

Hence we obtain

$$\left|\rho'-f(t)\rho\left(1-\frac{\rho^\alpha}{\kappa}\right)\right|\le\left\|\begin{pmatrix}\rho'-f(t)\rho\left(1-\frac{\rho^\alpha}{\kappa}\right)\\ \rho\big(\phi'+g(t)\big)\end{pmatrix}\right\|\le\varepsilon\quad\text{for}\quad t\ge 0\,. \tag{3.3}$$

Moreover, by (1.6) we know that

$$\rho(0)=\|\big(\xi(0),\eta(0)\big)\|=\|(x_0,y_0)\|\in\left[\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}},\infty\right).$$

Next we consider the solution $(r(t),\theta(t))$ of (3.1) corresponding to $(x(t),y(t))$. Then, from (1.6) it follows that

$$r(0)=\|(x(0),x(0))\|=\rho(0).$$

Using Theorem 2.2 with $h=f$, $K=\kappa$, $z=r$, $\zeta=\rho$, we conclude that $r(t)$ and $\rho(t)$ exist on $[0,\infty)$ and

$$|\rho(t)-r(t)|\le\max\left\{\frac{\alpha+1}{\alpha\underline{f}},\frac{\alpha+1}{\alpha^2\underline{f}}\right\}\varepsilon\quad\text{for}\quad t\in[0,\infty)\,;$$

that is,

$$\big|\|\big(\xi(t),\eta(t)\big)\|-\|\big(x(t),y(t)\big)\|\big|\le\max\left\{\frac{\alpha+1}{\alpha\underline{f}},\frac{\alpha+1}{\alpha^2\underline{f}}\right\}\varepsilon\quad\text{for}\quad t\in[0,\infty)\,.$$

Note here that $r(t)$ and $\rho(t)$ are non-negative on $[0,\infty)$ because $r(t)=\|(x(t),y(t))\|$ and $\rho(t)=\|(\xi(t),\eta(t))\|$.

Next, we will show that

$$\|(\xi(t),\eta(t))\|=\rho(t)\ge\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}\quad\text{for}\quad t\in[0,\infty)\,.$$

To prove this fact, we assume that there exists $t_1>0$ such that

$$\rho(t_1)<\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}.$$

From the continuity of $\rho(t)$, $\rho(t)$ is negative near $t=t_1$. Using this with $\rho(0)\in\left[\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}},\infty\right)$, we see that there exists $0\le t_2<t_1$ such that

$$\rho(t_2)=\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}},$$

and

$$\rho(t) < \Big(\frac{\kappa}{\alpha+1}\Big)^{\frac{1}{\alpha}} \quad \text{for} \quad t \in (t_2, t_1]\,. \tag{3.4}$$

Now we consider the case (i) $\varepsilon \in \left(0, \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\right)$. From (1.5) and (3.3), we have

$$\rho'(t_2) \ge f(t_2)\rho(t_2)\Big(1 - \frac{\rho^\alpha(t_2)}{\kappa}\Big) - \varepsilon = f(t_2)\frac{\alpha\kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}} - \varepsilon$$
$$\ge \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}} - \varepsilon > 0\,.$$

From this with the continuity of $\rho'(t)$, $\rho'(t)$ is positive near $t = t_2$. Thus, we see that there exists $0 < \delta \le t_1 - t_2$ such that

$$\rho'(t) > 0 \quad \text{for} \quad t \in [t_2, t_2 + \delta]\,.$$

Therefore,

$$\rho(t) \ge \rho(t_2) = \Big(\frac{\kappa}{\alpha+1}\Big)^{\frac{1}{\alpha}} \quad \text{for} \quad t \in [t_2, t_2 + \delta]\,.$$

This contradicts (3.4).

Next we consider the case (ii) $\varepsilon = \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}$. From $\rho \ge 0$, (1.5) and (3.4), we have

$$f(t)\rho(t)\Big(1 - \frac{\rho^\alpha(t)}{\kappa}\Big) \ge f(t)\rho(t)\Big(1 - \frac{1}{\alpha+1}\Big) > \frac{\alpha \underline{f}}{\alpha+1}\rho(t) \ge 0 \quad \text{for} \quad t \in (t_2, t_1]\,.$$

Hence, by (3.3), we have

$$\Big(\rho(t)e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)}\Big)' = \Big(\rho'(t) - \frac{\alpha \underline{f}}{\alpha+1}\rho(t)\Big)\, e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)}$$
$$> \Big[\rho'(t) - f(t)\rho(t)\Big(1 - \frac{\rho^\alpha(t)}{\kappa}\Big)\Big]\, e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)}$$
$$\ge -\varepsilon e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)}$$

for $t \in (t_2, t_1]$. Integrating this inequality and using

$$\rho(t_2) = \Big(\frac{\kappa}{\alpha+1}\Big)^{\frac{1}{\alpha}} \quad \text{and} \quad \varepsilon = \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}},$$

we obtain

$$\rho(t)e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)} > \rho(t_2) + \frac{\alpha+1}{\alpha \underline{f}}\varepsilon\Big(e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)} - 1\Big)$$
$$= \Big(\frac{\kappa}{\alpha+1}\Big)^{\frac{1}{\alpha}} + \frac{\alpha+1}{\alpha \underline{f}}\frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\Big(e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)} - 1\Big)$$
$$= \Big(\frac{\kappa}{\alpha+1}\Big)^{\frac{1}{\alpha}} e^{-\frac{\alpha \underline{f}}{\alpha+1}(t-t_2)}$$

for $t \in (t_2, t_1]$. This contradicts (3.4). Hence we can conclude that

$$\rho(t) \geq \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}} \quad \text{for } \ t \in [0, \infty).$$

If $p_1 = p_2 \equiv 0$, then $\rho \equiv r$. Therefore,

$$r(t) \geq \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}} \quad \text{for } \ t \in [0, \infty).$$

This completes the proof of Theorem 1.1. □

4. Second main result

By assuming stronger condition to $\|(p_1(t), p_2(t))\|$, we can also obtain a relationship between θ and ϕ. The following theorem is the second main result in this paper.

Theorem 4.1. *Suppose that there exists* $\underline{f} > 0$ *such that* (1.5) *holds. Let* $\varepsilon \in \left(0, \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\right]$, $\|(x_0, y_0)\| \in \left[\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}, \infty\right)$, *let* $(x(t), y(t))$ *and* $(\xi(t), \eta(t))$ *be the solutions of* (1.1) *and* (1.2) *with* (1.6), *respectively. If there exists* $\beta > 1$ *such that*

$$\|(p_1(t), p_2(t))\| \leq \frac{\varepsilon}{(t+1)^{\beta}} \quad \textit{for} \quad t \geq 0, \tag{4.1}$$

then the global existence of $(x(t), y(t))$ *and* $(\xi(t), \eta(t))$ *is guaranteed. Furthermore, the following holds: Let* $(r(t), \theta(t))$ *and* $(\rho(t), \phi(t))$ *be the solutions of* (3.1) *and* (3.2) *corresponding to* $(x(t), y(t))$ *and* $(\xi(t), \eta(t))$, *respectively. Then* $(r(t), \theta(t))$ *and* $(\rho(t), \phi(t))$ *exist on* $[0, \infty)$, *and*

$$\min\{\rho(t), r(t)\} \geq \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}},$$

$$|\rho(t) - r(t)| \leq \max\left\{\frac{\alpha+1}{\alpha \underline{f}}, \frac{\alpha+1}{\alpha^2 \underline{f}}\right\}\varepsilon \quad \textit{and} \quad |\phi(t) - \theta(t)| \leq \left(\frac{\alpha+1}{\kappa}\right)^{\frac{1}{\alpha}} \frac{\varepsilon}{\beta - 1}$$

for $t \in [0, \infty)$.

Proof. Suppose that there exists $\underline{f} > 0$ such that (1.5) holds. Given an arbitrary $\varepsilon \in \left(0, \frac{\alpha \underline{f} \kappa^{\frac{1}{\alpha}}}{(\alpha+1)^{\frac{\alpha+1}{\alpha}}}\right]$, suppose that (1.7) holds. Let $(x(t), y(t))$ and $(\xi(t), \eta(t))$ be the solutions of (1.1) and (1.2) with (1.6) and $\|(x_0, y_0)\| \in \left[\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}, \infty\right)$, respectively. In addition, let $(r(t), \theta(t))$ and $(\rho(t), \phi(t))$ be the solutions of (3.1) and (3.2) corresponding to $(x(t), y(t))$ and $(\xi(t), \eta(t))$, respectively. Note that (4.1) implies (1.7). Then, using the same method as the proof of Theorem 1.1, we see that

$$r(0) = \rho(0) = \|(\xi(0), \eta(0))\| = \|(x_0, y_0)\| \in \left[\left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}, \infty\right);$$

$$|\rho(\phi' + g(t))| \leq \left\| \begin{pmatrix} \rho' - f(t)\rho\left(1 - \frac{\rho^\alpha}{\kappa}\right) \\ \rho(\phi' + g(t)) \end{pmatrix} \right\| \leq \frac{\varepsilon}{(t+1)^\beta} \quad \text{for} \quad t \geq 0; \tag{4.2}$$

and $r(t)$ and $\rho(t)$ exist on $[0, \infty)$; and

$$\min\{\rho(t), r(t)\} \geq \left(\frac{\kappa}{\alpha+1}\right)^{\frac{1}{\alpha}}, \tag{4.3}$$

and

$$|\rho(t) - r(t)| \leq \max\left\{\frac{\alpha+1}{\alpha \underline{f}}, \frac{\alpha+1}{\alpha^2 \underline{f}}\right\}\varepsilon \quad \text{for} \quad t \in [0, \infty).$$

Next we will prove that $\theta(t)$ and $\phi(t)$ exist on $[0, \infty)$ and

$$|\phi(t) - \theta(t)| \leq \left(\frac{\alpha+1}{\kappa}\right)^{\frac{1}{\alpha}} \frac{\varepsilon}{\beta - 1} \quad \text{for } t \in [0, \infty).$$

Define

$$q(t) := \rho(t)(\phi'(t) + g(t))$$

for $t \in [0, \infty)$. Then, by (4.2), we have

$$|q(t)| \leq \frac{\varepsilon}{(t+1)^\beta} \quad \text{for} \quad t \geq 0.$$

Since $\rho(t)$ is positive on $[0, \infty)$, we can solve the above differential equation. Then we obtain

$$\phi(t) = \phi(0) + \int_0^t \left(g(s) + \frac{q(s)}{\rho(s)}\right) ds \quad \text{for } t \in [0, \infty).$$

Because $\rho(t)$ exists on $[0, \infty)$, $\phi(t)$ exists on $[0, \infty)$. Obviously, $\theta'(t) + g(t) = 0$ is also solved and we obtain

$$\theta(t) = \theta(0) + \int_0^t g(s) ds \quad \text{for } t \in [0, \infty).$$

By (1.6), we have $\phi(0) = \theta(0)$, and so that

$$|\phi(t) - \theta(t)| \leq \int_0^t \frac{|q(s)|}{\rho(s)} ds \leq \int_0^t \frac{\varepsilon}{\rho(s)(s+1)^\beta} ds \quad \text{for } t \in [0, \infty).$$

From this with (4.3) it follows that

$$\begin{aligned} |\phi(t) - \theta(t)| &\leq \left(\frac{\alpha+1}{\kappa}\right)^{\frac{1}{\alpha}} \varepsilon \int_0^t \frac{1}{(s+1)^\beta} ds \\ &\leq \left(\frac{\alpha+1}{\kappa}\right)^{\frac{1}{\alpha}} \frac{\varepsilon}{\beta - 1} \left[1 - \frac{1}{(t+1)^{\beta-1}}\right] \\ &< \left(\frac{\alpha+1}{\kappa}\right)^{\frac{1}{\alpha}} \frac{\varepsilon}{\beta - 1} \end{aligned}$$

for $t \in [0, \infty)$. This completes the proof of Theorem 4.1. □

5. Examples

In this section, we will present two examples of variable coefficients. Let (x_0, y_0) on the circle $C_{\frac{1}{\sqrt{3}}}$ or, be outside the circle $C_{\frac{1}{\sqrt{3}}}$. Consider the the solutions $(x(t), y(t))$ and $(\xi(t), \eta(t))$ of (1.1) and (1.2) with (1.6), (1.8) and

$$\alpha = 2, \quad \kappa = 1, \quad f(t) = \frac{|\sin t|}{t+1} + 1, \quad g(t) = \cos t + 0.5, \tag{5.1}$$

respectively. From $\underline{f} = 1$ and $\|(p_1(t), p_2(t))\| \le \frac{2}{3\sqrt{3}}$ for $t \ge 0$, we can choose $\varepsilon = \frac{2}{3\sqrt{3}}$, and using Theorem 1.1, we conclude that $(x(t), y(t))$ and $(\xi(t), \eta(t))$ on the circle $C_{\frac{1}{\sqrt{3}}}$ or, are outside of $C_{\frac{1}{\sqrt{3}}}$, and (1.9) holds. Figure 4 shows the orbits corresponding to $(x(t), y(t))$ and $(\xi(t), \eta(t))$ of (1.1) and (1.2) with (1.6), (1.8), (1.12) and (5.1), for $0 \le t \le 150$. From (3.1), we see that (1.1) has a limit cycle as the unit circle. Thus, the orbit of $(x(t), y(t))$ represent the limit cycle, and the orbit of $(\xi(t), \eta(t))$ represent an approximation of the limit cycle. Note that $(x(t), y(t))$ rotates in the opposite direction each time the sign of g changes on the unit circle. However, because $\lim_{t\to\infty} \theta(t) = -\infty$ holds, we will call the unit circle the limit cycle here.

Next we consider the the solutions $(x(t), y(t))$ and $(\xi(t), \eta(t))$ of (1.1) and (1.2) with (1.6), (1.12), (5.1) and

$$p_1(t) = \frac{2}{3\sqrt{3}(t+1)^2}\left(1 - 2\max\left\{\cos\sqrt{t}, 0\right\}\right) \quad \text{and} \quad p_2(t) = 0, \tag{5.2}$$

respectively. From $\underline{f} = 1$ and

$$\|(p_1(t), p_2(t))\| \le \frac{2}{3\sqrt{3}(t+1)^2} \le \frac{2}{3\sqrt{3}} \quad \text{for} \quad t \ge 0,$$

we can choose $\varepsilon = \frac{2}{3\sqrt{3}}$ and $\beta = 2$. Let $(r(t), \theta(t))$ and $(\rho(t), \phi(t))$ be the solutions of (3.1) and (3.2) corresponding to $(x(t), y(t))$ and $(\xi(t), \eta(t))$, respectively. Using Theorem 4.1, we have

$$|\rho(t) - r(t)| \le \frac{3}{2}\varepsilon = \frac{1}{\sqrt{3}} \quad \text{and} \quad |\phi(t) - \theta(t)| \le \sqrt{3}\varepsilon = \frac{2}{3}$$

for $t \in [0, \infty)$. Figure 5 shows the orbits corresponding to $(x(t), y(t))$ and $(\xi(t), \eta(t))$ of (1.1) and (1.2) with (1.6), (1.12), (5.1) and (5.2), for $0 \le t \le 50$.

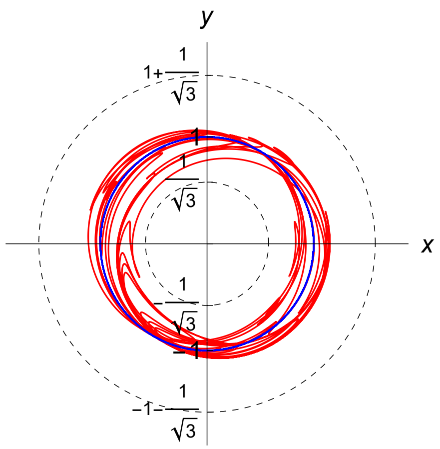

Fig. 4: Orbits of (1.1) (blue) and (1.2) (red) with (1.6), (1.8), (1.12) and (5.1).

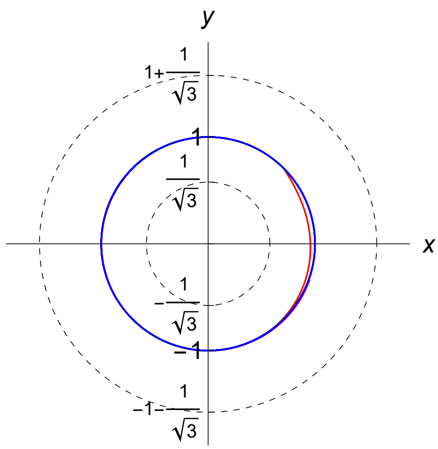

Fig. 5: Orbits of (1.1) (blue) and (1.2) (red) with (1.6), (1.12), (5.1) and (5.2).

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