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LARGE TIME BEHAVIOR IN A QUASILINEAR PARABOLIC-PARABOLIC-ELLIPTIC ATTRACTION-REPULSION CHEMOTAXIS SYSTEM

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Abstract. This paper deals with a quasilinear parabolic-parabolic-elliptic attraction-repulsion chemotaxis system. Boundedness, stabilization and blow-up in this system of the fully parabolic and parabolic-elliptic-elliptic versions have already been proved. The purpose of this paper is to derive boundedness and stabilization in the parabolic-parabolic-elliptic version.

1. Introduction and main result

In this paper we consider the quasilinear attraction-repulsion chemotaxis system

$$\begin{cases} u_t = \nabla\cdot\big((u+1)^{m-1}\nabla u - \chi u(u+1)^{p-2}\nabla v + \xi u(u+1)^{q-2}\nabla w\big)\,,\\ v_t = \Delta v + \alpha u - \beta v\,,\\ 0 = \Delta w + \gamma u - \delta w\,,\\ (\nabla u\cdot\nu)|_{\partial\Omega} = (\nabla v\cdot\nu)|_{\partial\Omega} = (\nabla w\cdot\nu)|_{\partial\Omega} = 0\,,\\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) \end{cases} \tag{1.1}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$ $(n\in\mathbb{N})$ with smooth boundary $\partial\Omega$. Here m, p, $q\ge 1$, χ, ξ, α, β, γ, $\delta>0$ are constants, ν is the outward normal vector to $\partial\Omega$,

$$u_0\in C^0(\overline{\Omega})\,,\quad u_0\ge 0 \ \text{ in } \overline{\Omega} \quad\text{and}\quad u_0\ne 0, \tag{1.2}$$

$$v_0\in W^{1,\theta}(\Omega) \text{ for some } \theta>n\,,\quad v_0\ge 0 \ \text{ in } \overline{\Omega} \quad\text{and}\quad v_0\ne 0. \tag{1.3}$$

The model (1.1) was proposed by [12] to describe the aggregation of microglial cells in Alzheimer's disease. Also, u, v and w represent the cell density, concentrations of attractive and repulsive chemical substances; α and γ idealize the rates at which the cell produces substances; β and δ represent the rates at which substances are transformed into another ones which do not involve in the movement of the cell.

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Let us overview previous results on the attraction-repulsion chemotaxis system

$$
\begin{cases}
u_t = \nabla \cdot \left(\nabla u - \chi u \nabla v + \xi u \nabla w\right), \\
\tau v_t = \Delta v + \alpha u - \beta v, \\
\tau w_t = \Delta w + \gamma u - \delta w,
\end{cases} \tag{1.4}
$$

where χ, ξ, α, β, γ, $\delta > 0$ are constants and $\tau \in \{0,1\}$. This system has been investigated in several studies. For instance, in the case that $\tau = 1$ boundedness (including global existence) was studied in [5], finite-time blow-up (blow-up for short) was analyzed in [9] and stabilization was studied in [11]. Also, in the simplified case that $\tau = 0$ there are more precise studies. Indeed, blow-up with logistic source was discussed in [2] and stabilization was investigated in [10, 13]. On the other hand, as to the quasilinear version, such as (1.1), of the above system (1.4) with $\tau = 0$, there are several studies. Indeed, boundedness and blow-up were classified by the size of p, q in [4] and stabilization was obtained in [1, 3].

In summary, boundedness, stabilization and blow-up in the attraction-repulsion system (1.4) have been well studied in the fully parabolic case ($\tau = 1$) and in the parabolic-elliptic-elliptic case ($\tau = 0$). However, the *quasilinear* parabolic-*parabolic*--elliptic attraction-repulsion chemotaxis system has not been analyzed. The purpose of this paper is to derive boundedness and stabilization in (1.1).

The main result of this paper reads as follows.

Theorem 1.1. *Let $n \in \mathbb{N}$. Let m, $p \geq 1$ fulfill $p - m \in [0,1]$ when $n = 1$, $p - m \in [0, \frac{2}{n}]$ when $n \geq 2$ and let $q \geq 1$. Assume that u_0, v_0 satisfy* (1.2), (1.3). *Then there exists a unique triplet (u, v, w) which solves* (1.1) *in the classical sense and is bounded, that is,*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C$$

for all $t > 0$ with some $C > 0$ in the cases that $p - m \in [0,1)$ for $n = 1$ and that $p - m \in [0, \frac{2}{n})$ for $n \geq 2$. Also, there exists $\lambda_0 > 0$ such that if

$$\|u_0\|_{L^1(\Omega)} < \lambda_0 \tag{1.5}$$

only in the cases that $p - m = 1$ for $n = 1$ and that $p - m = \frac{2}{n}$ for $n \geq 2$, then the same conclusion on boundedness holds. Moreover, assume further that u_0 satisfies

$$\chi \|u_0\|_{L^1(\Omega)}^{p-m} < \frac{1}{C_{\langle p-m \rangle}}, \tag{1.6}$$

where $C_{\langle p-m \rangle} > 0$ is a constant appearing in the Poincaré-Sobolev inequality (*see* (2.14)). *Then the bounded solution (u, v, w) has the property that*

$$(u(\cdot,t), v(\cdot,t), w(\cdot,t)) \to \left(\overline{u_0}, \frac{\alpha}{\beta}\overline{u_0}, \frac{\gamma}{\delta}\overline{u_0}\right) \text{ in } [L^\infty(\Omega)]^3 \quad \text{as } t \to \infty, \tag{1.7}$$

where $\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0$.

Remark 1.2. We need the condition (1.5) only to assert boundedness.

2. Proof of Theorem 1.1

We first give a result on local existence in (1.1).

Lemma 2.1. *Let $m, p, q \geq 1$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$. Then for all u_0, v_0 satisfying the conditions (1.2), (1.3) there exists $T_{\max} \in (0, \infty]$ such that (1.1) admits a unique classical solution (u, v, w) such that $u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$, $v, w \in C^0([0, T_{\max}); W^{1,\theta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$. Moreover, if $T_{\max} < \infty$, then $\lim_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$.*

Proof. Let $T \in (0, 1]$, $M := \|u_0\|_{L^\infty(\Omega)} + 1$ and $N := \|v_0\|_{W^{1,\theta}(\Omega)}$. We introduce the set $\mathcal{S} := \{\varphi \in X \mid 0 \leq \varphi \leq M \text{ in } \overline{\Omega} \times [0, T]\}$, where $X := C^0(\overline{\Omega} \times [0, T])$. Also, we define $\Phi(\widehat{u}) := u$ for $\widehat{u} \in \mathcal{S}$, where u is the solution of

$$u_t = \nabla \cdot \big((\widehat{u} + 1)^{m-1} \nabla u - \chi \widehat{u}(\widehat{u} + 1)^{p-2} \nabla v + \xi \widehat{u}(\widehat{u} + 1)^{q-2} \nabla w\big) \text{ in } \Omega \times (0, T)$$

with $(\nabla u \cdot \nu)|_{\partial\Omega} = 0$, $u(x, 0) = u_0(x)$, where v and w are the solutions of

$$v_t = \Delta v + \alpha \widehat{u} - \beta v \text{ in } \Omega \times (0, T)$$

with $(\nabla v \cdot \nu)|_{\partial\Omega} = 0$, $v(x, 0) = v_0(x)$ and

$$0 = \Delta w + \gamma \widehat{u} - \delta w \text{ in } \Omega \times (0, T)$$

with $(\nabla w \cdot \nu)|_{\partial\Omega} = 0$, respectively. Then, by an argument similar to that in [8,15], we can verify that Φ is a continuous and compact map of $\mathcal{S}$ into $\mathcal{S}$. Therefore, from the Schauder fixed point theorem and standard regularity theory for parabolic and elliptic equations, we obtain local existence in (1.1). □

The first purpose of this section is to derive global existence and boundedness. To achieve this, we obtain an L^r-estimate for u with sufficiently large r.

Lemma 2.2. *Let $s \in (0, T_{\max})$. Let $m, p \geq 1$ fulfill $p - m \in [0, 1]$ when $n = 1$, $p - m \in [0, \frac{2}{n}]$ when $n \geq 2$ and let $q \geq 1$. Let u_0, v_0 satisfy (1.2), (1.3). Then there exist $r_0 > 1$ and $\lambda_0 > 0$ such that if u_0 satisfies $\|u_0\|_{L^1(\Omega)} < \lambda_0$ only in the cases that $p - m = 1$ for $n = 1$ and that $p - m = \frac{2}{n}$ for $n \geq 2$, then for all $r > r_0$,*

$$\sup_{t \in (s, T_{\max})} \|u(\cdot, t)\|_{L^r(\Omega)} \leq K_r \tag{2.1}$$

with some $K_r > 0$.

Proof. Let $s \in (0, T_{\max})$ and $r > 1$. By the first equation of (1.1) and integration by parts, we have

$$\begin{aligned} \frac{1}{r} \frac{d}{dt} \|u(\cdot, t)\|_{L^r(\Omega)}^r = &- \int_\Omega (u + 1)^{m-1} \nabla u \cdot \nabla u^{r-1} \\ &+ \chi \int_\Omega u(u + 1)^{p-2} \nabla v \cdot \nabla u^{r-1} \\ &- \xi \int_\Omega u(u + 1)^{q-2} \nabla w \cdot \nabla u^{r-1} \\ =:&\, I_1(\cdot, t) + I_2(\cdot, t) + I_3(\cdot, t) \end{aligned} \tag{2.2}$$

for all $t \in (s, T_{\max})$. This corresponds to [6, (28) with $D(s) = s^{m-1}$, $S(s) = s^{p-1}$, $\varepsilon = 1$] with additional term I_3, but we use [4, (3.13) and (3.16)] to derive

$$
\begin{aligned}
(2.3) \qquad I_3(\cdot, t) &\le \frac{\xi(r-1)}{r+q-2}\Big(2\delta \int_\Omega u^{r+q-2} w + \delta c_1 \int_\Omega w - \gamma \int_\Omega u^{r+q-1}\Big) \\
&\le \frac{\xi(r-1)}{r+q-2}\Big[2\delta\Big(\frac{\gamma}{2\delta}\int_\Omega u^{r+q-1} + c_2\Big) + c_3 - \gamma \int_\Omega u^{r+q-1}\Big] \\
&= \frac{\xi(r-1)}{r+q-2}(2\delta c_2 + c_3) =: c_4
\end{aligned}
$$

for all $t \in (s, T_{\max})$ with some $c_1, c_2, c_3 > 0$. Thus, combining (2.3) with (2.2), we can observe from [6, p. 223, lines 12 and 13] that there exist $r_1, r_2 > 1$ such that

$$
\begin{aligned}
(2.4) \qquad & \frac{d}{dt}\|u(\cdot,t)\|^r_{L^r(\Omega)} \\
&\le -\|u(\cdot,t)\|^r_{L^r(\Omega)} + (c_5 r)^{c_6 r} - \frac{1}{2}A(r,m,p,u_0)\|\nabla u^{\frac{r+m-1}{2}}(\cdot,t)\|^2_{L^2(\Omega)} \\
&\quad + c_7\|\Delta v(\cdot,t)\|^{r+p-1}_{L^{r+p-1}(\Omega)} + c_8\|\Delta v(\cdot,t)\|^{r+1}_{L^{r+1}(\Omega)} + c_9
\end{aligned}
$$

for all $t \in (s, T_{\max})$ and all $r > \max\{\frac{n}{2}(p-m) - p + 1, \frac{n}{2}(2-m) - 1, r_1, r_2\}$ with some $c_5, c_6, c_7, c_8, c_9 > 0$, where $A(r,m,p,u_0) > 0$ is a constant defined as

$$
A(r,m,p,u_0) := \begin{cases} \frac{2r(r-1)}{(r+m-1)^2} & \text{if } p-m \in [0,1)\ (n=1), \\ & \quad p-m \in [0,\frac{2}{n})\ (n \ge 2), \\ \frac{4r(r-1)}{(r+m-1)^2} - c_{10} r \|u_0\|^{c_{11}(r+p-1)}_{L^1(\Omega)} & \text{if } p-m = 1\ (n=1), \\ & \quad p-m = \frac{2}{n}\ (n \ge 2) \end{cases}
$$

with $c_{10}, c_{11} > 0$, where the value $-c_{10} r \|u_0\|^{c_{11}(r+p-1)}_{L^1(\Omega)}$ in the critical case is derived from [6, p. 222, line 4]. Then, from an argument parallel to that in the derivation of [6, (38)], the differential inequality (2.4) implies that

$$
\begin{aligned}
\|u(\cdot,t)\|^r_{L^r(\Omega)} &\le \|u(\cdot,s)\|^r_{L^r(\Omega)} \\
&\quad + \Big[(c_5 r)^{c_6 r} + c_9 + (c_{12} r C_{\mathrm{MR}}^{r+p-1})^{c_{13} r} + (c_{14} r C_{\mathrm{MR}}^{r+1})^{c_{15} r}\Big] \\
&\quad + c_7 r C_{\mathrm{MR}}^{r+p-1}\|\Delta v(\cdot,s)\|^{r+p-1}_{L^{r+p-1}(\Omega)} + c_8 r C_{\mathrm{MR}}^{r+1}\|\Delta v(\cdot,s)\|^{r+1}_{L^{r+1}(\Omega)}
\end{aligned}
$$

for all $t \in (s, T_{\max})$ and all $r > \max\{\frac{n}{2}(p-m) - p + 1, \frac{n}{2}(2-m) - 1, r_1, r_2\}$ with some $C_{\mathrm{MR}}, c_{12}, c_{13}, c_{14}, c_{15} > 0$ via estimates for $\int_s^t \|\Delta e^{\frac{\sigma-t}{r+p-1}} v(\cdot,\sigma)\|^{r+p-1}_{L^{r+p-1}(\Omega)}\, d\sigma$ and $\int_s^t \|\Delta e^{\frac{\sigma-t}{r+1}} v(\cdot,\sigma)\|^{r+1}_{L^{r+1}(\Omega)}\, d\sigma$ by the maximal Sobolev regularity ([6, Lemma 2.1]) and the Young inequality. More precisely, we estimate these two terms as

$$
\begin{aligned}
\int_s^t \|\Delta e^{\frac{\sigma-t}{\theta_1}} v(\cdot,\sigma)\|^{\theta_1}_{L^{\theta_1}(\Omega)}\, d\sigma &\le c_{16} r C_{\mathrm{MR}}^{\theta_1}\|\Delta v(\cdot,s)\|^{\theta_1}_{L^{\theta_1}(\Omega)} \\
&\quad + \int_s^t e^{\sigma-t}\Big[\frac{1}{4}A(r,m,p,u_0)\|\nabla u^{\frac{r+m-1}{2}}(\cdot,t)\|^2_{L^2(\Omega)} + (c_{17} r C_{\mathrm{MR}}^{\theta_1})^{c_{18} r}\Big]\, d\sigma
\end{aligned}
$$

with $\theta_1 \in \{r+p-1, r+1\}$ and $c_{16}, c_{17}, c_{18} > 0$. Therefore, by following an argument similar to that in [6] and taking $\|u_0\|_{L^1(\Omega)}$ sufficiently small such that $A(r,m,p,u_0) > 0$ only in the cases that $p-m=1$ for $n=1$ and that $p-m=\frac{2}{n}$ for $n \geq 2$, we arrive at (2.1). □

Proof of Theorem 1.1 (*Boundedness*)**.** Taking $r = r^* > 1$ in Lemma 2.2 sufficiently large such that r^* fulfills the assumption of [14, Lemma A.1], we have $\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$, which means that $T_{\max} = \infty$ by the extensibility criterion, and boundedness holds. □

The second purpose of this section is to prove stabilization. To this end, we introduce the function

$$\Phi(s) := \int_1^s \int_1^\sigma \frac{1}{\eta(\eta+1)^{p-2}}\, d\eta d\sigma\,, \quad s \geq 0\,,$$

where $p \geq 1$ is a constant appearing in the attraction term in (1.1). In order to obtain an energy inequality we first calculate and estimate $\frac{d}{dt}\int_\Omega \Phi(u)$.

Lemma 2.3. *The first component u satisfies that*

$$\frac{d}{dt}\int_\Omega \Phi(u) + \int_\Omega \frac{(u+1)^{m-p+1}}{u}|\nabla u|^2 \leq \chi \int_\Omega \nabla u \cdot \nabla v \tag{2.5}$$

for all $t > 0$.

Proof. We see from the first equation in (1.1) and the identity $\Phi''(u) = \frac{1}{u(u+1)^{p-2}}$ as well as straightforward calculations that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \Phi(u) = &-\int_\Omega \frac{(u+1)^{m-p+1}}{u}|\nabla u|^2 + \chi \int_\Omega \nabla u \cdot \nabla v \\ &- \xi \int_\Omega (u+1)^{q-p} \nabla u \cdot \nabla w\end{aligned} \tag{2.6}$$

for all $t > 0$. Here we can estimate the third term on the right-hand side by zero. Indeed, we rewrite the third equation in (1.1) as

$$0 = \Delta\Big(w + \frac{\gamma}{\delta}\Big) + \gamma(u+1) - \delta\Big(w + \frac{\gamma}{\delta}\Big)\,, \tag{2.7}$$

and thereby we invoke integration by parts to obtain

$$\begin{aligned} I &:= -\xi \int_\Omega (u+1)^{q-p}\nabla u \cdot \nabla w \\ &= \frac{\xi}{q-p+1}\int_\Omega (u+1)^{q-p+1}\Delta\Big(w+\frac{\gamma}{\delta}\Big) \\ &= \frac{\xi\delta}{q-p+1}\int_\Omega (u+1)^{q-p+1}\Big(w+\frac{\gamma}{\delta}\Big) - \frac{\xi\gamma}{q-p+1}\int_\Omega (u+1)^{q-p+2}. \end{aligned} \tag{2.8}$$

Moreover, applying the Hölder inequality to (2.8) and noticing that (2.7) yields

$$\Big\| w(\cdot,t) + \frac{\gamma}{\delta}\Big\|_{L^{q-p+2}(\Omega)} \leq \frac{\gamma}{\delta}\|u(\cdot,t)+1\|_{L^{q-p+2}(\Omega)}$$

for all $t > 0$, we obtain

$$I \le \frac{\xi\delta}{q-p+1}\|u(\cdot,t)+1\|_{L^{q-p+2}(\Omega)}^{q-p+1} \cdot \Big(\Big\|w(\cdot,t)+\frac{\gamma}{\delta}\Big\|_{L^{q-p+2}(\Omega)} - \frac{\gamma}{\delta}\|u(\cdot,t)+1\|_{L^{q-p+2}(\Omega)}\Big) \le 0\,,$$

which along with (2.6) implies that (2.5) holds. □

In order to state the next lemma we define the function

$$V(x,t) := v(x,t) - \frac{\alpha}{\beta}\overline{u_0} \quad \text{for } x \in \Omega\,,\ t > 0\,.$$

Lemma 2.4. *The first component u satisfies that for all $t > 0$,*

$$\frac{d}{dt}\Big[\int_\Omega \Phi(u) + \frac{\chi}{2\alpha}\int_\Omega |\nabla V|^2 + \frac{\chi\beta}{\alpha}\int_\Omega V^2\Big] + \int_\Omega \frac{(u+1)^{m-p+1}}{u}|\nabla u|^2 + \frac{\chi\beta}{\alpha}\int_\Omega |\nabla V|^2 + \frac{\chi\beta^2}{\alpha}\int_\Omega V^2 + \frac{\chi}{\alpha}\int_\Omega V_t^2 \le \chi\int_\Omega (u-\overline{u_0})^2\,. \tag{2.9}$$

Proof. Noting from the second equation in (1.1) that $V_t = \Delta V + \alpha(u - \overline{u_0}) - \beta V$ and testing this equation by V_t and V, we can see that

$$\frac{d}{dt}\Big[\frac{1}{2}\int_\Omega |\nabla V|^2 + \frac{\beta}{2}\int_\Omega V^2\Big] + \int_\Omega V_t^2 = -\alpha\int_\Omega \nabla u\cdot\nabla v + \alpha\int_\Omega (u-\overline{u_0})^2 - \alpha\beta\int_\Omega (u-\overline{u_0})V, \tag{2.10}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega V^2 + \int_\Omega |\nabla V|^2 + \beta\int_\Omega V^2 = \alpha\int_\Omega (u-\overline{u_0})V \tag{2.11}$$

for all $t > 0$, respectively. Thus, multiplying (2.10) and (2.11) by $\frac{\chi}{\alpha}$ and $\frac{\chi\beta}{\alpha}$, respectively, and adding them to (2.5), we obtain (2.9). □

We finally derive an energy inequality.

Lemma 2.5. *Let m, p fulfill $p - m \in [0,1]$ when $n = 1$, $p - m \in [0, \frac{2}{n}]$ when $n \ge 2$ and let $C_{\langle p-m\rangle} > 0$ be a constant appearing in the Poincaré–Sobolev inequality (see (2.14)). Then the first component u satisfies that*

$$\frac{d}{dt}\int_\Omega F(u,v) + \Big[\frac{1}{C_{\langle p-m\rangle}\|u_0\|_{L^1(\Omega)}^{p-m}} - \chi\Big]\int_\Omega (u-\overline{u_0})^2 \le 0 \tag{2.12}$$

for all $t > 0$, where

$$F(u,v) := \int_\Omega \Phi(u) + \frac{\chi}{2\alpha}\int_\Omega |\nabla v|^2 + \frac{\chi\beta}{\alpha}\int_\Omega \Big(v - \frac{\alpha}{\beta}\Big)^2\,.$$

In particular, if u_0 meets (1.6), then

$$\int_0^\infty \int_\Omega (u-\overline{u_0})^2 < \infty\,. \tag{2.13}$$

Proof. We first see from the fact $(u+1)^{m-p+1} \geq u^{m-p+1}$, the mass conservation $\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)}$ $(t>0)$ and the Poincaré–Sobolev inequality that

$$\begin{aligned}\int_\Omega \frac{(u+1)^{m-p+1}}{u}|\nabla u|^2 &\geq \frac{1}{\|u_0\|^{p-m}_{L^1(\Omega)}}\Big(\int_\Omega |\nabla u|^{\frac{2}{p-m+1}}\Big)^{p-m+1} \\ &\geq \frac{1}{C_{\langle p-m\rangle}\|u_0\|^{p-m}_{L^1(\Omega)}}\int_\Omega (u-\overline{u_0})^2 \end{aligned} \tag{2.14}$$

for all $t>0$, which along with (2.9) implies that

$$\frac{d}{dt}\int_\Omega F(u,v) + \frac{1}{C_{\langle p-m\rangle}\|u_0\|^{p-m}_{L^1(\Omega)}}\int_\Omega (u-\overline{u_0})^2 \leq \chi\int_\Omega (u-\overline{u_0})^2$$

for all $t>0$, which entails (2.12). Also, integrating (2.12) over $(0,t)$, using the positivity of F and (1.6), and taking the limit $t\to\infty$, we derive (2.13). □

We are now in a position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1 (*Stabilization*)**.** We first derive L^∞-convergence of u. Since the first component u is bounded in time, we see from parabolic regularity theory ([7]) that there exist $\sigma\in(0,1)$ and $c_1>0$ such that

$$\|u\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\overline{\Omega}\times[1,\infty))} \leq c_1\,, \tag{2.15}$$

which implies that the function $t\mapsto \|u(\cdot,t)-\overline{u_0}\|^2_{L^2(\Omega)}$ is uniformly continuous in $[0,\infty)$. Hence, in light of time integrability of $\|u(\cdot,t)-\overline{u_0}\|^2_{L^2(\Omega)}$ (see (2.13)), we infer that $\|u(\cdot,t)-\overline{u_0}\|_{L^2(\Omega)}\to 0$ as $t\to\infty$. Also, employing the Gagliardo–Nirenberg inequality, we can find $c_2>0$ such that

$$\|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)} \leq c_2\|u(\cdot,t)-\overline{u_0}\|^{\frac{n}{n+2}}_{W^{1,\infty}(\Omega)}\|u(\cdot,t)-\overline{u_0}\|^{\frac{2}{n+2}}_{L^2(\Omega)}\,. \tag{2.16}$$

Noting from the estimate (2.15) that $\|u(\cdot,t)-\overline{u_0}\|_{W^{1,\infty}(\Omega)} \leq c_3 := c_1+\overline{u_0}$, we derive from L^2-convergence of u and the estimate (2.16) that

$$\|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t\to\infty\,.$$

We next show L^∞-convergences of v and w. We put $U(x,t) := u(x,t)-\overline{u_0}$, $V(x,t) := v(x,t)-\frac{\alpha}{\beta}\overline{u_0}$ and $W(x,t) := w(x,t)-\frac{\gamma}{\delta}\overline{u_0}$ for $x\in\Omega$, $t>0$. Then the second equation and boundary condition in (1.1) yield

$$V_t = \Delta V + \alpha U - \beta V\,, \quad (\nabla V\cdot\nu)|_{\partial\Omega} = 0\,.$$

Recalling that $(e^{t\Delta})_{t>0}$ acts as a contraction on $L^\infty(\Omega)$, we have that for all $t>0$,

$$\begin{aligned}\|V(\cdot,t)\|_{L^\infty(\Omega)} &\leq e^{-t\beta}\|e^{t\Delta}V(\cdot,0)\|_{L^\infty(\Omega)} + \alpha\int_0^t e^{-(t-s)\beta}\|e^{(t-s)\Delta}U(\cdot,s)\|_{L^\infty(\Omega)}\,ds \\ &\leq e^{-t\beta}\|V(\cdot,0)\|_{L^\infty(\Omega)} + \alpha\Big(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\Big)e^{-(t-s)\beta}\|U(\cdot,s)\|_{L^\infty(\Omega)}\,ds\,.\end{aligned}$$

Also, using boundedness of U i.e. $\|U(\cdot,s)\|_{L^\infty(\Omega)} \le c_3\,(= c_1 + \overline{u_0})$ and the estimate $e^{-(t-s)\beta} \le e^{-\frac{t}{2}\beta}$ for $s \in [0, \frac{t}{2}]$, and for all $\varepsilon > 0$, $\|U(\cdot,s)\|_{L^\infty(\Omega)} < \varepsilon$ for $s \in [\frac{t}{2}, t]$ with sufficiently large t by L^∞-convergence of U, we see that

$$\|V(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty\,.$$

On the other hand, since $0 = \Delta W + \gamma U - \delta W$ and $(\nabla W \cdot \nu)|_{\partial\Omega} = 0$, in view of the maximum principle we see from L^∞-convergence of U that

$$\|W(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{\gamma}{\delta}\|U(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty\,.$$

Therefore we arrive at (1.7). □

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References

[1] Chiyo, Y., *Stabilization for small mass in a quasilinear parabolic-elliptic-elliptic attraction-repulsion chemotaxis system with density-dependent sensitivity: repulsion-dominant case*, Adv. Math. Sci. Appl. **31** (2) (2022), 327–341.

[2] Chiyo, Y., Marras, M., Tanaka, Y., Yokota, T., *Blow-up phenomena in a parabolic-elliptic-elliptic attraction-repulsion chemotaxis system with superlinear logistic degradation*, Nonlinear Anal. **212** (2021), 14 pp., Paper No. 112550.

[3] Chiyo, Y., Yokota, T., *Stabilization for small mass in a quasilinear parabolic-elliptic-elliptic attraction-repulsion chemotaxis system with density-dependent sensitivity: balanced case*, Matematiche (Catania), to appear.

[4] Chiyo, Y., Yokota, T., *Boundedness and finite-time blow-up in a quasilinear parabolic-elliptic elliptic attraction-repulsion chemotaxis system*, Z. Angew. Math. Phys. **73** (2) (2022), 27 pp., Paper No. 61.

[5] Fujie, K., Suzuki, T., *Global existence and boundedness in a fully parabolic 2D attraction-repulsion system: chemotaxis-dominant case*, Adv. Math. Sci. Appl. **28** (2019), 1–9.

[6] Ishida, S., Yokota, T., *Boundedness in a quasilinear fully parabolic Keller-Segel system via maximal Sobolev regularity*, Discrete Contin. Dyn. Syst. Ser. S **13** (2020), 2112–232.

[7] Ladyženskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N., *Linear and Quasilinear Equations of Parabolic Type*, AMS, Providence, 1968.

[8] Lankeit, J., *Infinite time blow-up of many solutions to a general quasilinear parabolic-elliptic Keller-Segel system*, Discrete Contin. Dyn. Syst. Ser. S **13** (2) (2020), 233–255.

[9] Lankeit, J., *Finite-time blow-up in the three-dimensional fully parabolic attraction-dominated attraction-repulsion chemotaxis system*, J. Math. Anal. Appl. **504** (2) (2021), 16 pp., Paper No. 125409.

[10] Li, Y., Lin, K., Mu, C., *Asymptotic behavior for small mass in an attraction-repulsion chemotaxis system*, Electron. J. Differential Equations **2015** (146) (2015), 13 pp.

[11] Lin, K., Mu, C., Wang, L., *Large-time behavior of an attraction-repulsion chemotaxis system*, J. Math. Anal. Appl. **426** (1) (2015), 105–124.

[12] Luca, M., Chavez-Ross, A., Edelstein-Keshet, L., Mogliner, A., *Chemotactic signalling, microglia, and Alzheimer's disease senile plague: Is there a connection?*, Bull. Math. Biol. **65** (2003), 673–730.

[13] Tao, Y., Wang, Z.-A., *Competing effects of attraction vs. repulsion in chemotaxis*, Math. Models Methods Appl. Sci. **23** (2013), 1–36.

[14] Tao, Y., Winkler, M., *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity*, J. Differential Equations **252** (1) (2012), 692–715.

[15] Winkler, M., *Global classical solvability and generic infinite-time blow-up in quasilinear Keller-Segel systems with bounded sensitivities*, J. Differential Equations **266** (12) (2019), 8034–8066.

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