An action-free characterization of weak Hopf-Galois extensions

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Abstract

We define comodule algebras and Galois extensions for actions of bialgebroids. Using just module conditions we characterize the Frobenius extensions that are Galois as depth two and right balanced extensions. As a corollary, we obtain characterizations of certain weak and ordinary Hopf-Galois extensions without reference to action in the hypothesis.

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1 Introduction

A finite Hopf H-Galois extension $A|B$ is a Frobenius extension and has many characterizations as an H^*-module algebra A with invariants B satisfying various conditions, or dually, as an H-comodule algebra A with coinvariants B satisfying various conditions [8]; and has many interesting applications [7, 8]. There is recently in [5, 6] a characterization of certain noncommutative Hopf-Galois extensions - those with trivial centralizer (and arising in subfactor theory) - in terms of module-theoretic conditions of depth two on the tensor-square of the extension and "balanced" on the module A_B. The main thesis in [6] is that to a depth two ring extension $A|B$ one associates by construction two bialgebroids over the unrestricted centralizer R, a left bialgebroid $S := \mathrm{End}\,_B A_B$ and a right bialgebroid $T := (A \otimes_B A)^B$, the B-central elements with multiplication induced from $T \cong \mathrm{End}\,_A(A \otimes_B A)_A$. The bialgebroids S and T are simultaneously each other's left and right R-dual bialgebroids and they act on A and $\mathrm{End}\,_B A$ respectively. If B is trivial, we obtain the two main examples of Lu bialgebroids. If R is trivial, R-bialgebroids are usual bialgebras, and if R is a separable algebra, R-bialgebroids are weak bialgebras: antipodes may be added to create weak Hopf algebras if $A|B$ is additionally a Frobenius extension. If A_B is balanced, the invariants $A^S = B$ and the endomorphism ring $\mathrm{End}\, A_B$ is a smash product of A and S (tensoring over R), which signals a Galois extension. Good definitions of Galois extension in terms of applications have appeared very recently [1, 2]. In this paper we extend the main theorems in [5, 6] to weak Hopf-Galois extension and Galois extensions for bialgebroids.

2 Preliminaries

Let $B \subseteq A$ be an associative not necessarily commutative subring pair sharing 1, also referred to here as a ring extension $A|B$.

Definition 2.1 *A ring extension $A|B$ is depth two (D2) if the tensor-square $A \otimes_B A$ is isomorphic both as natural B-A-bimodules (left D2) and as A-B-bimodules (right D2) to a direct summand of a finite direct sum of A with itself. Equivalently, $A|B$ is D2 if there are (left D2 quasibase) elements $\beta_i \in S$, $t_i \in T$ such that*

$$a \otimes_B a' = \sum_i t_i \beta_i(a) a' \tag{1}$$

and (right D2 quasibase) elements $\gamma_j \in S$, $u_j \in T$ such that

$$a \otimes_B a' = \sum_j a \gamma_j(a') u_j \tag{2}$$

for all $a, a' \in A$.

Example 2.2 *A finite dimensional algebra is D2 with dual bases as a vector space corresponding to D2 quasibases. Given a subgroup of a finite group $H < G$, the complex group subalgebra pair $\mathcal{C}H \subseteq \mathcal{C}G$ is D2 iff $H \lhd G$ [4]. Another related example: a normal Hopf subalgebra pair is D2. Yet another is a finite weak Hopf-Galois extension [3, 3.1].*

Recall from [6] that a right R'-bialgebroid T' are two rings R' and T' with two maps $\tilde{s}, \tilde{t} : R' \to T'$, a ring homomorphism and anti-homomorphism resp., such that $\tilde{s}(r)\tilde{t}(r') = \tilde{t}(r')\tilde{s}(r)$ for all $r, r' \in R'$, $(T',\ \Delta : T' \to T' \otimes_{R'} T', \varepsilon : T' \to R')$ is an R'-coring w.r.t. the R'-R'-bimodule $r \cdot x \cdot r' = x\tilde{t}(r)\tilde{s}(r')$ such that $(\tilde{s}(r) \otimes 1)\Delta(x) = (1 \otimes \tilde{t}(r))\Delta(x)$, $\Delta(xy) = \Delta(x)\Delta(y)$ (which makes sense thanks to the previous axiom), $\Delta(1) = 1 \otimes 1$, $\varepsilon(1_{T'}) = 1_{R'}$ and $\varepsilon(xy) = \varepsilon(\tilde{s}(\varepsilon(x))y) = \varepsilon(\tilde{t}(\varepsilon(x))y)$ for all $x, y \in T', r, r' \in R'$. A left bialgebroid is just a right bialgebroid with three of the axioms transposed [6].

Example 2.3 *[6, section 5] Given a D2 extension $A|B$, the ring $T = (A \otimes_B A)^B$ is a right bialgebroid over the centralizer $R := C_A(B)$ with maps $\tilde{s}(r) = 1 \otimes r$, $\tilde{t}(r) = r \otimes 1$, $\Delta(x) = \sum_j (x^1 \otimes_B \gamma_j(x^2)) \otimes_R u_j$ and $\varepsilon(x) = x^1 x^2$ for all $x = x^1 \otimes x^2 \in T$. Note then that ${}_R T_R$ is given by $r \cdot x \cdot r' = r x^1 \otimes x^2 r'$ and ${}_R T$ is finite projective from eq. (2).*

The dual left R-bialgebroid is S, there being two R-valued nondegenerate pairings of S and T; e.g., $\langle \alpha, t \rangle = t^1 \alpha(t^2)$ for each $\alpha \in S$, $t \in T$. The left bialgebroid structure is given by $\overline{s}(r) = \lambda_r$, left multiplication by $r \in R$, $\overline{t}(r) = \rho_r$, right multiplication, $\Delta(\alpha) = \sum_i \alpha(?t_i^1)t_i^2 \otimes \beta_i$ and $\varepsilon_S(\alpha) = \alpha(1)$.

The R-bialgebroid S acts on A by $\alpha \triangleright a = \alpha(a)$ (where R acts as a subring of A) with invariants $A^S = \{a \in A | \alpha \triangleright a = \varepsilon_S(\alpha)a \, \forall \alpha \in S\} \supseteq B$, and if A_B is balanced, $A^S = B$. A is thereby a left S-module algebra (or algebroid [6, 2.1]). We need the dual notion:

Definition 2.4 *Let T' be a right R'-bialgebroid $(T', \tilde{s}, \tilde{t}, \Delta, \varepsilon)$. A (right) T'-comodule algebra A' is a ring A' with ring homomorphism $R' \to A'$ together with a coaction $\delta : A' \to A' \otimes_{R'} T'$, where values $\delta(a)$ are denoted by the Sweedler notation $a_{(0)} \otimes a_{(1)}$, such that A' is a right T'-comodule over the R'-coring T' [1, 18.1], $\delta(1_{A'}) = 1_{A'} \otimes 1_{T'}$, $ra_{(0)} \otimes a_{(1)} = a_{(0)} \otimes \tilde{t}(r)a_{(1)}$ for all $r \in R'$, and $\delta(aa') = \delta(a)\delta(a')$ for all $a, a' \in A'$. The subring of coinvariants is ${A'}^{\mathrm{co}\,\mathrm{T}'} := \{a \in A' | \delta(a) = a \otimes 1_{T'}\}$. Consequently R' and ${A'}^{\mathrm{co}\,\mathrm{T}'}$ commute in A'.*

For example, T' is a comodule algebra over itself. A D2 extension $A|B$ has T-comodule algebra A [3, 5.1], indeed a T-Galois extension, which we define as follows.

Definition 2.5 *Let T' be a left finite projective right R'-bialgebroid. A T'-comodule algebra A' is a (right) T'-Galois extension of its coinvariants B' if the (Galois) mapping $\beta : A' \otimes_{B'} A' \to A' \otimes_{R'} T'$ defined by $\beta(a \otimes a') = a{a'}_{(0)} \otimes {a'}_{(1)}$ is bijective.*

3 D2 characterization of Galois extensions

In this section we provide characterizations of generalized Hopf-Galois extensions in analogy with the Steinitz characterization of Galois extension of fields as being separable and normal.

Theorem 3.1 *Let $A \,|\, B$ be a Frobenius extension. The extension $A|B$ is T-Galois for some left finite projective right bialgebroid T over some ring R if and only if $A|B$ is D2 with A_B balanced.*

Proof. ($\Rightarrow$) Since ${}_RT \oplus * \cong {}_RR^t$ for some positive integer t, we apply to this the functor $A \otimes_R -$ from left R-modules into A-B-bimodules which results in ${}_AA \otimes_B A_B \oplus * \cong {}_AA^t_B$, after using the Galois A-B-isomorphism $A \otimes_B A \cong A \otimes_R T$. Hence, $A|B$ is right D2, and left D2 since $A|B$ is Frobenius [6, 6.4].

Let $\mathcal{E} := \operatorname{End} A_B$. The module A_B is balanced iff the natural bimodule ${}_{\mathcal{E}}A_B$ is faithfully balanced, which we proceed to show based on the following claim. Let R be a ring, M_R and ${}_RV$ modules with ${}_RV$ finite projective. If

$\sum_j m_j \phi(v_j) = 0$ for all ϕ in the left R-dual ${}^*V$, then $\sum_j m_j \otimes_R v_j = 0$. This claim follows immediately by using dual bases $f_i \in {}^*V$, $w_i \in V$.

Given $F \in \mathrm{End}\,_{\mathcal{E}}A$, it suffices to show that $F = \rho_b$ for some $b \in B$. Since $\lambda_a \in \mathcal{E}$, $F \circ \lambda_a = \lambda_a \circ F$ for all $a \in A$, whence $F = \rho_{F(1)}$. Designate $F(1) = a$. If we show that $a_{(0)} \otimes a_{(1)} = a \otimes 1$ after applying the right T-valued coaction on A, then $a \in A^{\mathrm{co\,T}} = B$. For each $\alpha \in {}^*({}_RT)$, define $\overline{\alpha} \in \mathrm{End}\,A_B$ by $\overline{\alpha}(x) = x_{(0)}\alpha(x_{(1)})$. Since $\rho_r \in \mathcal{E}$ for each $r \in R$ by 2.4,

$$a\alpha(1_T) = F(\overline{\alpha}(1)) = \overline{\alpha}(F(1)) = a_{(0)}\alpha(a_{(1)})$$

for all $\alpha \in {}^*T$. By the claim $a_{(0)} \otimes_R a_{(1)} = a \otimes 1_T$.

($\Leftarrow$) Let T be the left projective right bialgebroid $(A \otimes_B A)^B$ over $R = C_A(B)$. Using a right D2 quasibase, we give A the structure of a right T-comodule algebra via $a_{(0)} \otimes a_{(1)} := \sum_j \gamma_j(a) \otimes u_j \in A \otimes_R T$, the details of which are in [3, 5.1]. The D2 condition ensures that $\theta : A \otimes_R T \to A \otimes_B A$ defined by $\theta(a \otimes t) = at^1 \otimes t^2$ is an isomorphism.

Note that for each $b \in B$

$$b_{(0)} \otimes b_{(1)} = \sum_j \gamma_j(b) \otimes_R u_j = b \otimes \sum_j \gamma_j(1)u_j = b \otimes 1_T$$

so $B \subseteq A^{\mathrm{co}\,\rho}$. The converse: if $\rho(x) = x \otimes 1_T = \sum_j \gamma_j(x) \otimes u_j$ applying θ we obtain $x \otimes_B 1 = 1 \otimes_B x$. Since A_B is balanced, we know $A^S = B$ under the action $\triangleright$ of S on A [6, 4.1]. Applying $\mu(\alpha \otimes \mathrm{id})$ for each $\alpha \in S$, where μ is multiplication, we obtain $\alpha \triangleright x = \alpha(x) = \alpha(1)x$, whence $x \in B$.

The Galois mapping $\beta : A \otimes_B A \to A \otimes_R T$ given by

$$\beta(a \otimes a') := aa'_{(0)} \otimes_R a'_{(1)} \tag{3}$$

is an isomorphism since θ is an inverse by eq. (2). *Q.e.d.*

Corollary 3.2 *Let k be a field and $A|B$ be a Frobenius extension of k-algebras with centralizer R a separable k-algebra. The extension $A|B$ is weak Hopf-Galois iff $A|B$ is D2 with A_B balanced.*

The proof of $\Leftarrow$ depends first on recalling that the right R-bialgebroid T is a weak bialgebra since R has an index-one Frobenius system ($\phi : R \to k, e_i, f_i \in R$) where $\sum_i e_i f_i = 1_A$ and $\sum_i \phi(re_i)f_i = r = \sum_i e_i\phi(f_i r)$ for all $r \in R$, whence $\Delta(t) = \sum_i t_{(1)}e_i \otimes_k f_i t_{(2)}$ and $\varepsilon(t) = \phi(t^1t^2)$ satisfy the axioms of a weak bialgebra [6, (96)]. Since $A|B$ is Frobenius, the dual bases tensor is a nondegenerate right integral in T, whence T is weak Hopf algebra by the Larson-Sweedler-Vecsernyes theorem. The coaction on A has values in $A \otimes_R T \cong (A \otimes_k T)\Delta(1)$, given by $a \otimes t \mapsto \sum_i ae_i \otimes f_i t$, an isomorphism of the Galois A-corings in [3, 5.1] and in [2, 2.1]. The proof of $\Rightarrow$ follows from the fact that a weak Hopf-Galois extension is D2 and an argument that A_B is balanced like the one above.

Example 3.3 *A separable field extension $\overline{k}|k$ is a weak Hopf-Galois extension, since $\overline{k}$ is a separable k-algebra.*

The theorem provides another proof and extends the theorems [6, 8.14] and [5, 6.6] as we see below. We define an *irreducible* k-algebra extension to be an extension where the centralizer is the trivial $k1$.

Corollary 3.4 *Let $A|B$ be an irreducible extension. The extension $A|B$ is finite Hopf-Galois $\Longleftrightarrow$ $A|B$ is a D2, right balanced extension.*

The proof of $\Rightarrow$ does not require the centralizer to be trivial. The Frobenius condition may be dropped here from the proof of $\Leftarrow$ since the bialgebra T acts Galois implies it is a Hopf algebra [9]. If the characteristic of k is zero, the Larson-Radford theorem permits the condition "right balanced" to be replaced by "separable extension" [3, 4.1].

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