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ON THE LIMIT OF A SEQUENCE

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Abstract. The object of this article is to examine the sequence

$$a_n = \frac{\sum_{i=0}^{n} \frac{n^i}{i!}}{e^n}$$

well known from probability theory. We prove that the sequence is bounded, strictly monotonously decreasing, and $\lim_{n\to\infty} a_n = \frac{1}{2}$. The last two statements are proved by analytical means. Finally, a modification and a generalization of (a_n) will be mentioned, and the sketch of a second analytical proof for the original limit will be given.

1. On p. 288 of [1] (under 6.1) the following theorem is to be found: For $\lambda \longrightarrow \infty$,

$$e^{-\lambda\Theta} \cdot \sum_{k \le \lambda x} \frac{(\lambda\Theta)^k}{k!} \longrightarrow \begin{cases} 0, & \text{if} \quad \Theta > x \\ 1, & \text{if} \quad \Theta < x. \end{cases}$$

[1] has no reference to the case $\Theta = x$. The sequence (a_n) of the present article is a reformulation of this specific case.

The main problem to be discussed in this article was raised by Professor *Zoltán László* of Veszprém University several years ago.

Initially I was motivated to find an elementary solution to the problem, but the cul-de-sacs have convinced me that this is hardly viable.

Let $a_n = \dfrac{\sum_{i=0}^{n} \frac{n^i}{i!}}{e^n}$. Then the usual questions are likely to arise: Is the sequence monotonous? Is it bounded? Does a limit exist?

2.1. Boundedness is relatively easy to decide: The sequence is bounded from below, as a sum of positive terms is divided by a positive number, so $a_n > 0$ holds; on the other hand, it is known that for all given n $\sum_{i=0}^{\infty} \frac{n^i}{i!} = e^n$, and the numerator of a_n is a partial sum of this very series. So $a_n < 1$ follows.

2.2. The remaining two questions are more difficult to answer; here we have to resort to other means. On integrating by parts we obtain

$$\int_0^n \frac{e^{-x} \cdot x^n}{n!}\, dx = 1 - e^{-n} \cdot \left(1 + n + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\right) .$$

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So for a_n

$$a_n = 1 - \int_0^n \frac{e^{-x} \cdot x^n}{n!}\, dx \ .$$

Now we shall prove that the sequence (a_n) is strictly monotonously decreasing.

Statement: $a_n > a_{n+1}$.

Proof: By reason of the above formula for a_n we are to show that

$$\int_0^n \frac{e^{-x} \cdot x^n}{n!}\, dx < \int_0^{n+1} \frac{e^{-x} \cdot x^{(n+1)}}{(n+1)!}\, dx \ .$$

By decomposition of the integral

$$\int_0^{n+1} \frac{e^{-x} \cdot x^{(n+1)}}{(n+1)!}\, dx = \int_0^n \frac{e^{-x} \cdot x^{(n+1)}}{(n+1)!}\, dx + \int_n^{n+1} \frac{e^{-x} \cdot x^{(n+1)}}{(n+1)!}\, dx \ .$$

Hereafter we shall denote the first and second integrals on the right side by I_1, and I_2 respectively.
First we shall give an estimate for I_2. For the derivative $f'(x)$ of the function

$$f(x) := \frac{e^{-x} \cdot x^{(n+1)}}{(n+1)!}$$

$$f'(x) = \frac{e^{-x} \cdot x^n}{n!} \cdot \left(1 - \frac{x}{n+1}\right) \geq 0 \ , \quad \text{if} \quad x \in [0, n+1] \ ,$$

which means that in this interval $f(x)$ is monotonously increasing. So

$$I_2 \geq 1 \cdot \frac{e^{-n} \cdot n^{(n+1)}}{(n+1)!} \ .$$

Let us deal now with the first integral. By integration by parts

$$I_1 = \int_0^n \frac{e^{-x} \cdot x^{(n+1)}}{(n+1)!}\, dx = \left[-\frac{e^{-x} \cdot x^{(n+1)}}{(n+1)!}\right]_0^n + \int_0^n \frac{e^{-x} \cdot x^n}{n!}\, dx \ .$$

So we obtain that

$$\int_0^{n+1} \frac{e^{-x} \cdot x^{(n+1)}}{(n+1)!}\, dx = I_2 + \int_0^n \frac{e^{-x} \cdot x^n}{n!}\, dx - \frac{e^{-n} \cdot n^{n+1}}{(n+1)!} \geq$$

$$\geq \int_0^n \frac{e^{-x} \cdot x^n}{n!}\, dx + \frac{e^{-n} \cdot n^{n+1}}{(n+1)!} - \frac{e^{-n} \cdot n^{n+1}}{(n+1)!} \ ,$$

as $I_2 \geq \dfrac{e^{-n} \cdot n^{(n+1)}}{(n+1)!}$. $\diamondsuit$

Thus we have proved that the sequence (a_n) is bounded from below and strictly monotonously decreasing, consequently a limit exists.

2.3. Next, we shall try to find this limit. To this end, we are going to use the following lemma (without proof):

Lemma:

$$\left(\frac{e}{n}\right)^n \cdot n! = \sqrt{2\pi n} + O\left(\frac{1}{\sqrt{n}}\right) , \tag{1}$$

whence

$$\frac{n^n}{n! \cdot e^n} = \frac{1}{\sqrt{2\pi n}} \cdot \left[1 + O\left(\frac{1}{n}\right)\right] .$$

(The proof can be found in numerous places. E.g. [3].)

¿From the formula for a_n

$$\lim_{n\to\infty} a_n = 1 - \lim_{n\to\infty} \int_0^n \frac{e^{-x} \cdot x^n}{n!} \, dx ,$$

so, to find the limit of the sequence, we have to calculate

$$\lim_{n\to\infty} \int_0^n \frac{e^{-x} \cdot x^n}{n!} \, dx.$$

Let $\eta = n^{-\frac{1}{2}+\varepsilon}$, where $0 < \varepsilon < \frac{1}{6}$. Then, by the substitution $x = n \cdot (z+1)$

$$\int_0^n \frac{e^{-x} \cdot x^n}{n!} \, dx = \int_{-1}^0 \frac{e^{-n\cdot(z+1)} \cdot (z+1)^n \cdot n^{(n+1)}}{n!} \, dz = n \cdot \frac{n^n}{n! \cdot e^n} \cdot \int_{-1}^0 e^{-nz} \cdot (z+1)^n =$$

$$= \sqrt{\frac{n}{2\pi}} \cdot \left[1 + O\left(\frac{1}{n}\right)\right] \cdot \int_{-1}^0 \left[e^{-z} \cdot (1+z)\right]^n dz =$$

(Here we used Lemma (1) .) Transforming the integral further

$$= \sqrt{\frac{n}{2\pi}} \cdot \left[1 + O\left(\frac{1}{n}\right)\right] \cdot \int_{-1}^0 \left[e^{-z} \cdot (1+z)\right]^n dz =$$

$$= \sqrt{\frac{n}{2\pi}} \cdot \left[1 + O\left(\frac{1}{n}\right)\right] \cdot \left[\int_{-1}^{-\eta} \left[e^{-z} \cdot (1+z)\right]^n dz + \int_{-\eta}^0 \left[e^{-z} \cdot (1+z)\right]^n dz\right] .$$

As for the derivative $f'(z)$ of the function $f(z) := e^{-z} \cdot (1+z)$ $f'(z) = -e^{-z} \cdot z$, for $z \le 0$ $f'(z) \ge 0$ holds, which means that the above function is monotonously increasing in the interval $[-1, -\eta]$. So

$$\int_{-1}^{-\eta} \left[e^{-z} \cdot (1+z)\right]^n dz < (1-\eta) \cdot \left[e^{-\eta} \cdot (1-\eta)\right]^n < \left[e^{-\eta} \cdot (1-\eta)\right]^n ,$$

that is, for the above integral

$$\sqrt{\frac{n}{2\pi}} \cdot \left[1 + O\left(\frac{1}{n}\right)\right] \cdot \int_{-1}^0 \left[e^{-z} \cdot (1+z)\right]^n dz =$$

$$= \sqrt{\frac{n}{2\pi}} \cdot \left[1 + O\left(\frac{1}{n}\right)\right] \cdot \int_{-\eta}^0 \left[e^{-z} \cdot (1+z)\right]^n dz + O\left(\sqrt{n} \cdot e^{-\frac{1}{2}n^{2\varepsilon}}\right) .$$

On the remaining segment of the interval, using the equalities

$$f(z) = e^{ln\, f(z)} \ , \quad \text{and} \quad ln\, f(z) = -z + ln\,(z+1) \ ,$$

$(z \in [-\eta, 0])$, and MacLaurin's series for the function $ln\,(z+1)$:

$$ln\,(z+1) = \frac{z}{2} - \frac{z^2}{3} + \frac{z^3}{4 \cdot (1+\vartheta(z))^4}$$

for the expansion of the integrand we obtain

$$f(z) = e^{-\frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4 \cdot (1+\vartheta(z))^4}} \ ,$$

where $0 < \vartheta(x) < 1$. The factor $e^{-n \cdot \frac{z^4}{4(1+\vartheta(z))^4}}$ is of the form $1 + O\left(n^{-1+n\varepsilon}\right)$ so the integral assumes the following form

$$\sqrt{\frac{n}{2\pi}} \cdot \left[1 + O\left(n^{-1+4\varepsilon}\right)\right] \cdot \int_{-\eta}^{0} e^{-n\left(\frac{z^2}{2} - \frac{z^3}{3}\right)} dz + O\left(\sqrt{n} \cdot e^{-\frac{1}{2}n^{2\varepsilon}}\right) \ .$$

It is known that

$$e^{n\frac{z^3}{3}} = 1 + n \cdot \frac{z^3}{3} + O\left(n^{-1+6\varepsilon}\right) \ ,$$

and

$$\int_{-\eta}^{0} e^{n \cdot \frac{z^2}{2}} dz$$

is of the order $n^{-1/2}$, the order term in $e^{n \cdot \frac{z^3}{3}}$ is $O\left(n^{-1+6\varepsilon}\right)$, so

$$\sqrt{\frac{n}{2\pi}} \cdot \left[1 + O\left(n^{-1+4\varepsilon}\right)\right] \cdot \int_{-\eta}^{0} e^{-n\left(\frac{z^2}{2} - \frac{z^3}{3}\right)} dz =$$

$$= \sqrt{\frac{n}{2\pi}} \cdot \left[1 + O\left(n^{-1+4\varepsilon}\right)\right] \cdot \int_{-\eta}^{0} e^{-n \cdot \frac{z^2}{2}} \cdot \left(1 + n \cdot \frac{z^3}{3}\right) dz + O\left(n^{1+6\varepsilon}\right) =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-n^{\varepsilon}}^{0} e^{-\frac{\vartheta^2}{2}} \cdot \left(1 + \frac{\vartheta^3}{3\sqrt{n}}\right) d\vartheta + O\left(n^{-1+6\varepsilon}\right) =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{0} e^{-\frac{\vartheta^2}{2}} d\vartheta + \frac{1}{\sqrt{2\pi n}} \cdot \int_{-\infty}^{0} \frac{\vartheta^3 \cdot e^{-\frac{\vartheta^2}{2}}}{3} d\vartheta + O\left(n^{-1+6\varepsilon}\right) \ .$$

The first integral equals $1/2$ (Gaussian integral), while the second one will be transformed further:

$$\int_{-\infty}^{0} \frac{\vartheta^3 \cdot e^{-\frac{\vartheta^2}{2}}}{3} d\vartheta = -\left[\frac{\vartheta^2 \cdot e^{-\frac{\vartheta^2}{2}}}{3}\right]_{-\infty}^{0} + \frac{2}{3} \cdot \int_{-\infty}^{0} \vartheta \cdot e^{-\frac{\vartheta^2}{2}} d\vartheta = 0 + \frac{2}{3} \cdot \left[-e^{-\frac{\vartheta^2}{2}}\right]_{-\infty}^{0} = -\frac{2}{3} \ .$$

(using integration by parts). Thus, for the limit

$$\lim_{n\to\infty} \frac{1}{\sqrt{2\pi n}} \cdot \left(-\frac{2}{3}\right) = 0 \ , \quad \text{és} \quad \lim_{n\to\infty} O\left(n^{-1+6\varepsilon}\right) = 0 \ ,$$

i.e. the value of the original integral is $\frac{1}{2}$, so we have proved the following

Theorem: $\lim_{n\to\infty} a_n = \dfrac{1}{2}$. $\diamondsuit$

3.1. Remark: Defining the sequence b_n as

$$b_n = \frac{\sum_{i=0}^{2n} \frac{n^i}{i!}}{e^n}$$

it can be shown that this sequence is strictly monotonously increasing. Integration by substitution is used and the relationship $f \le g \Longrightarrow \int_0^n f \le \int_0^n g$ is applied. Boundedness is proved in practically the same manner as in the previous case.The calculation of the limit is performed similarly, and

$$\lim_{n\to\infty} b_n = 1 \ .$$

is obtained.

Moreover, it can be shown that by modifying the limits of the summation the limit can assume any value in the interval [0,1].

3.2. Remark: To give a further proof for the limit of (a_n) we shall use the following theorem: (p. 128 of [2])

If the functions $\varphi(x)$, $h(x)$ and $f(x) = e^{h(x)}$ defined for the finite or infinite interval $[a,b]$ satisfy the following conditions:

(i) $\varphi(x) \cdot [f(x)]^n$ is absolute integrable in $[a,b]$ for $\forall\ n \in N$,

(ii) $h(x)$ assumes its maximum only in ξ of (a,b), and the least upper bound of $h(x)$ is less than $h(\xi)$-t for all closed intervals not including ξ; furthermore, a neighbourhood of ξ exists such that $h''(x)$ exists and is continuous; finally, $h''(x) < 0$,

(iii) $\varphi(x)$ is continuous in $x = \xi$, $\varphi(x) \neq 0$,

then, for $\forall\ \alpha \in R$

$$\int_a^{\xi+\frac{\alpha}{\sqrt{n}}} \varphi(x) \cdot [f(x)]^n \, dx \quad \sim \quad \varphi(\xi) \cdot e^{n \cdot h(\xi)} \cdot \frac{1}{\sqrt{-n \cdot h''(\xi)}} \cdot \int_{-\infty}^{\alpha \cdot c} e^{-\frac{t^2}{2}} \, dt \ , \qquad (2)$$

where $c = \sqrt{-h''(\xi)}$. $\diamondsuit$

Let now $a = 0$, $b = n+1$, $\varphi(x) \equiv 1$, $\alpha = 0$. So condition (iii) is satisfied.

Let $h(x) = ln\, x - \dfrac{x}{n} - \dfrac{ln\, n!}{n}$. Then $h'(x) = \frac{1}{x} - \frac{1}{n}$, whence we get that h has got a maximum in $x = n$, and $h(x)$ is strictly monotonously increasing in $(0,n]$, is strictly monotonously decreasing in $[n, \infty)$, and $\xi = n$. $-\dfrac{1}{x^2} < 0$, so condition (ii) is also automatically satisfied.

On the other hand,

$$\frac{1}{n!}\Gamma(n-1) = \int_0^\infty \frac{e^{-x}\cdot x^n}{n!}\,dx$$

is absolute integrable, so (i) is also satisfied. Substituting this for formula (2) we get

$$\int_0^n \frac{e^{-x}\cdot x^n}{n!}\,dx \quad \sim \quad \frac{e^{-n}\cdot n^n}{n!}\cdot\frac{1}{\sqrt{\frac{1}{n}}}\cdot\int_{-\infty}^0 e^{-\frac{t^2}{2}}\,dt\ ,$$

then, using Stirling's Formula the result 1/2 is received.

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