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CYCLIC CONNECTIVITY CLASSES OF DIRECTED GRAPHS

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Dedicated to Á. Varecza on his sixtieth birthday

Abstract. We show that most connectivity types of simple directed graphs defined by A. Ádám in [1] are nonempty. The nonexistence of the three types that remain missing is the consequence of a fairly plausible conjecture, stated at the end of this paper.

Introduction

This paper gives an almost complete answer to a question raised by A. Ádám in [1] concerning the connectivity types of simple directed graphs. In his paper A. Ádám defined ten properties pertinent to the cyclic structure of a simple directed graph. He also explored most (possibly all) implications between these properties, resulting in a hierarchy of cyclic connectivity, represented on Figure 1. By studying the acyclic directed graph of logical dependencies he concluded that his proposed classification may yield at most twenty one disjoint types of cyclic connectivity for directed graphs. He also constructed examples for ten types and raised the question whether all other eleven types are nonempty.

The main result of this paper is that at least eight of these missing types is not missing any more. Furthermore, the techniques used to construct our examples reveal interesting connections between Ádám's cyclic connectivity classes, some of which are either invariant or go consistently into the same other class under the effect of such simple operations as contracting edges or expanding vertices. These operations may, by the way, yield a graph that is not simple any more but we indicate at least one standard method –the use of "compasses"– to get around this issue. This technique hints how to extend Ádám's theory to non-simple directed graphs as well.

We conclude our paper with an important conjecture which, if true, would explain why the remaining three missing types are still missing: our conjecture implies that no graph of these types would exist. Our conjecture, if true may reveal a deep interrelation between the cyclic connectivity properties of directed graphs.

1. Basic concepts and facts

We define a *directed graph* G as a pair (V, E) of a finite nonempty vertex set V and of a subset E of $V \times V$. We consider only graphs which are *simple* in the following sense: they contain no loops (edges of the form (v, v)) and there is at most one edge between any fixed pair of vertices. Furthermore, we denote the *indegree* resp. *outdegree* (the number of incoming resp. outgoing edges) of a vertex a by

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$\delta^-(a)$ and $\delta^+(a)$ respectively. Two edges $e = (a, b)$ and $f = (c, d)$ are *adjacent* if the set $\{a, b, c, d\}$ has three elements. If $b = c$ or $a = d$ then we say that e and f are *consecutively* adjacent, if $a = c$ or $b = d$ then e and f are *oppositely* adjacent.

The classification proposed by Ádám in [1] restricts it attention to graphs satisfying

$$\delta^-(a) \cdot \delta^+(a) \geq 2 \text{ for any vertex } a. \tag{1}$$

In particular, these graphs do not have *transient* vertices, i.e., vertices satisfying $\delta^-(a) = \delta^+(a) = 1$. As Ádám noted in [1, Remark 1], when we study the cycle structure of graphs, this restriction is not really serious. All the examples presented in this paper will satisfy Ádám's condition. In our proofs, however, when we verify the properties of our examples we will take induced subgraphs, and remove edges from our examples. The resulting subgraphs will not necessarily satisfy condition (1).

A *directed path* is a list $(a_1, a_2, \ldots, a_k)$ vertices such that for each

$$i \in \{1, 2, \ldots, k-1\}$$

the ordered pair (a_i, a_{i+1}) is an edge of the graph. A graph G is *strongly connected* if for each ordered pair (a, b) of vertices there exist a directed path from a to b. A *cycle* is a cyclically ordered list $(a_1, a_2, \ldots, a_k)$ of pairwise different vertices such that (a_i, a_{i+1}) is an edge for $i = 1, 2, \ldots, k-1$, and (a_k, a_1) is also an edge. We say that an edge or vertex x is *cyclic* if there is a cycle Z containing x. Let y be an edge or a vertex different from x. We say that x and y are *cyclically completeable* if there is a cycle containing both x and y.

2. Ádám's connectivity classes for directed graphs

In his paper [1] Ádám introduced the following properties of a directed graph $G = (V, E)$.

- (A) Any vertex of G is cyclic.
- (B) Any edge of G is cyclic.
- (C) Any pair a, b of vertices is cyclically completeable.
- (D) Any pair $a \in V$, $e \in E$ of a vertex and an edge is cyclically completeable.
- (E) Any pair e, f of nonadjacent or consecutively adjacent edges is cyclically completeable.
- (F) There exist a vertex a such that any pair $a, b(\in V \setminus \{a\})$ is cyclically completeable.
- (G) There exist a vertex a such that any pair $a, e(\in E)$ is cyclically completeable.
- (H) There exist an edge e such that any pair $a(\in V), e$ is cyclically completeable.
- (J) There exist an edge e such that any pair $e, f(\in E \setminus \{e\})$ is cyclically completeable.
- (K) There exist an edge e such that any pair $e, f(\in E \setminus \{e\})$ of nonadjacent or consecutively adjacent edges is cyclically completeable.

Ádám observed the following implications, and provided a proof for the ones that are not direct consequences of the definitions: (B) $\Rightarrow$ (A), (C) $\Rightarrow$ (F), (G) $\Rightarrow$ (F), (H) $\Rightarrow$ (F), (J) $\Rightarrow$ (K), (J) $\Rightarrow$ (G), (E) $\Rightarrow$ (K), (D) $\Rightarrow$ (C), (D) $\Rightarrow$ (G), (D) $\Rightarrow$ (H), (E) $\Rightarrow$ (D), (K) $\Rightarrow$ (H), and (F) $\Rightarrow$ (B). These implications induce a hierarchy on the classes defined. This hierarchy is represented on Figure 1. The main question is whether there are further interrelations among the defined graph classes or not. The hierarchy on Figure 1 represents a cycle free directed graph that has twenty-one nonempty independent vertex sets. Let $\{\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_\mathrm{g}\}$ be any of these twenty one sets (here $1 \leq g \leq 3$). We say that a graph G has *type* $(\mathrm{X}_1 \ldots \mathrm{X}_\mathrm{g})$ precisely if

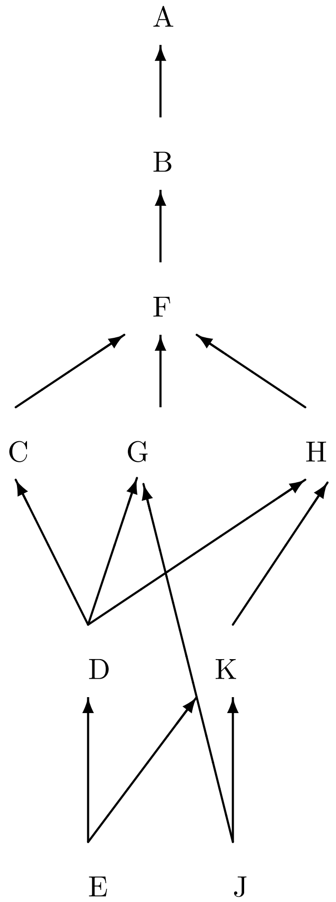

FIGURE 1. Hierarchy of Ádám's connectivity properties

(α) G has properties $(\mathrm{X}_1), \ldots, (\mathrm{X}_\mathrm{g})$.
(β) G does not have any property (out of (A),...,(K)) that is not accessible from the vertices $\mathrm{X}_1, \ldots, \mathrm{X}_\mathrm{g}$ in the graph in Figure 1.

For example a graph G has type (CJ) if it has properties (C), (J) (and also (A), (B), (F), (G), (H), (K)) but G does not have property (D) (thus it does not have property (E) either).

In his paper [1] Ádám has shown the existence of graphs of the following types: (CJ), (G), (EJ), (CGK), (GK), (J), (E), (A), (B), (F). He raised the question in [1, Problem 2] whether there is a graph for each of the remaining eleven types.

Our main result is the following:

Theorem 1. *For each of the following eight types there exists a directed graph G having exactly that type: (K), (H), (CK), (H), (DJ), (CH), (DK), (CGH), (GH).*

It is still open whether there is a graph of type (C), (D), or (CG). We have a conjecture implying that the remaining types are empty. This conjecture will be stated and explained in section 4.

3. The proof of the main theorem

Example 1. *The graph G_1 on Figure 2 has type (K).*

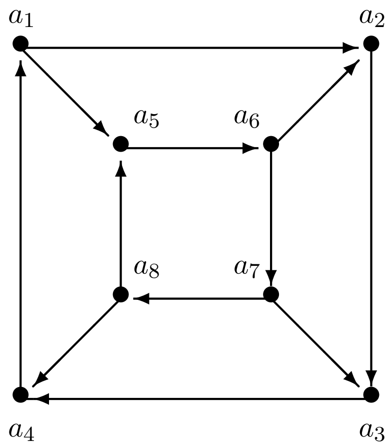

FIGURE 2. Graph G_1 of type (K)

Property (K) is satisfied since each nonadjacent or consecutive edge is contained in at least one of the following cycles: $(a_1, a_5, a_6, a_7, a_8, a_4)$, $(a_1, a_5, a_6, a_2, a_3, a_4)$, and $(a_1, a_5, a_6, a_7, a_3, a_4)$.

The only edge leading from the set $V_1 = \{a_1, a_2, a_3, a_4\}$ into the set $V_2 = \{a_5, a_6, a_7, a_8\}$ is the edge (a_1, a_5). (There are three edges leading from V_2 into V_1: (a_6, a_2), (a_7, a_3), and (a_8, a_4). Hence the graph G_1 does not have property (C): the vertices $a_2 \in V_1$ and $a_8 \in V_2$ are not cyclically completeable. In fact, the only way to go from a_2 to a_1 is by using the directed path (a_2, a_3, a_4, a_1), which excludes the use of any edge from V_2 to V_1.

The graph G_1 does not have property (G) either. Let $e_1 = (a_8, a_5)$ and $e_2 = (a_1, a_2)$. Given any vertex $a \in V_i$ $(i = 1, 2)$, the pair $\{a, e_i\}$ is not cyclically completeable, since both e_1 and e_2 are oppositely adjacent with the edge (a_1, a_5).

Remark 1. The graph G_1 does not have property (J) since property (J) implies property (G). In general, if we know about a graph G that it does not satisfy a collection of properties that form a cutset in Figure 1 then G does not have any property represented below that cutset.

Remark 2. It is interesting to note that adding the edge (a_8, a_6) to the graph G_1 results in the loss of property (K): the new graph will have type (H). Note also that the class of graphs having property (H) is closed under adding new edges.

Example 2. *The graph G_2 on Figure 3 has type (H).*

Observe first that G_2 is obtained from G_1 by adding the (a_8, a_6). The same justification as the one for G_1 shows that G_2 has none of the properties (C) or (G). G_2 does not have property (K) either: if the edge e connects two vertices in $V_1 = \{a_1, a_2, a_3, a_4\}$ then it is not cyclically completeable with $f_1 = (a_8, a_5)$, otherwise e connects two vertices in $V_2 = \{a_5, a_6, a_7, a_8\}$ or it is one of the edges (a_8, a_4), (a_7, a_3) and then it it is not cyclically completeable with with $f_2 = (a_1, a_2)$. In fact, a cycle containing f_1 or f_2 can not contain the oppositely adjacent edge (a_1, a_5) without which there is no way to get from V_1 into V_2. The edge $e = (a_1, a_5)$ that induced property (K) in G_1 is not cyclically completeable with the new edge $f = (a_8, a_6)$: after removing the edges (a_8, a_5) and (a_5, a_6) which are oppositely adjacent to f, the outdegree of a_5 becomes zero, and so (a_1, a_5) can not be used in any cycle. Finally the edge (a_6, a_2) is not cyclically completeable with $f = (a_7, a_3)$: after removing the edges (a_6, a_7) and (a_7, a_8) –which are oppositely adjacent to either e or f– the indegree of a_7 becomes zero.

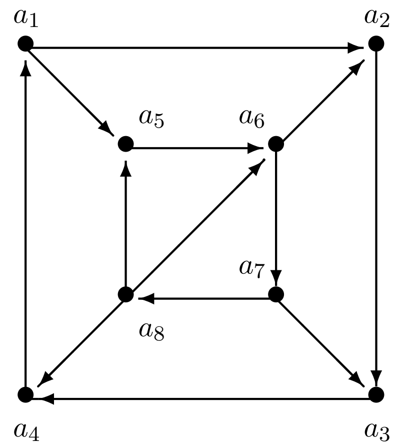

FIGURE 3. Graph G_2 of type (H)

G_2 has property (H) since for $e = (a_1, a_5)$ all vertices of G_2 are covered by the cycles listed in Example 1 which all contain e.

Example 3. *The graph G_3 obtained by replacing the edge (a_7, a_3) with (a_3, a_7) in G_1 (Figure 4) has type (CK).*

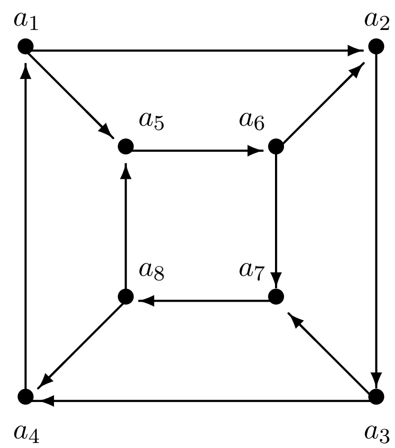

FIGURE 4. Graph G_3 of type (CK)

The graph has properties (C) and (H) since the cycle $(a_1, a_5, a_6, a_2, a_3, a_7, a_8, a_4)$ covers all vertices of the graph. The graph has property (K) since for the edge $e = (a_1, a_5)$ every nonadjacent or consecutively adjacent edge is covered by one of the following three cycles that contain

$$(a_1, a_5) : (a_1, a_5, a_6, a_7, a_8, a_4), (a_1, a_5, a_6, a_2, a_3, a_7, a_8, a_4), (a_1, a_5, a_6, a_2, a_3, a_4).$$

G_3 does not have property (G). The vertex $a = a_1$ is not cyclically completeable with the edge $e = (a_8, a_5)$: after removing the edges (a_1, a_5) and (a_8, a_4) which are oppositely adjacent to e, any cycle Z containing a and e has to contain the directed path (a_3, a_4, a_1, a_2). After removing the edges (a_6, a_7) and (a_3, a_7) which are oppositely adjacent to some edge in this path one may see that Z can not contain any of the vertices a_5, a_6, a_7, a_8. It may be shown in a similar manner that the vertex $a = a_2$ is not cyclically completeable with the the edge $e = (a_6, a_7)$.

Since the graph is invariant under 180^o rotation, property (G) may not be satisfied neither by the choice $a = a_3$ nor by the choice $a = a_4$. Finally, observe that the map $x \mapsto f(x)$ given in the following table is a graph isomorphism:

x	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
$f(x)$	a_6	a_7	a_8	a_5	a_2	a_3	a_4	a_1

Hence property (G) can not be satisfied by choosing any vertex a from

$$\{a_5, a_6, a_7, a_8\}.$$

The graph G_3 does not have property (J) since property (J) implies property (G).

Example 4. *The graph G_4 obtained from G_1 by reversing the orientation of the edge (a_6, a_2) in G_1 has also type (H).*

The justification is similar to the one in the previous example.

Example 5. *The graph G_5 (Figure 5) obtained from G_3 by reversing the orientation of the edges in the cycle (a_5, a_6, a_7, a_8) has type (DJ).*

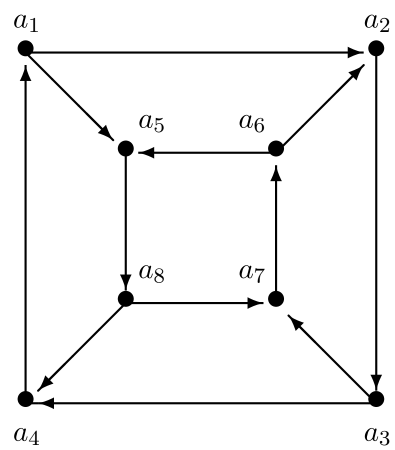

FIGURE 5. Graph G_5 of type (DJ)

The graph has property (D) since every edge is contained in at least one of the two following Hamiltonian cycles: $Z_1 = (a_1, a_5, a_8, a_7, a_6, a_2, a_3, a_4)$ and $Z_2 = (a_1, a_2, a_3, a_7, a_6, a_5, a_8, a_4)$. (A Hamiltonian cycle is a cycle containing every vertex of a graph.) Since the edge $e = (a_4, a_1)$ belongs to both cycles, G_5 also has property (J). The graph does not have property (E) since there is no cycle containing both edges $e = (a_1, a_2)$ and $f = (a_8, a_7)$. In fact, after removing the edges (a_1, a_5), (a_6, a_2), (a_8, a_4), (a_3, a_7) which are oppositely adjacent to e or f, the graph becomes disconnected containing e and f in different connected components.

Example 6. *Add a new vertex a_9 to the graph G_3 and draw edges from a_5, a_6, a_7 and a_8, as shown on Figure 6. The resulting graph G_6 has type (CH).*

The graph has property (C): the cycle $Z_1 = (a_1, a_5, a_6, a_2, a_3, a_7, a_8, a_4)$ contains all vertices except a_9. Vertex a_9 is cyclically completeable with any other vertex: any other vertex is either on the cycle $Z_2 = (a_9, a_5, a_6, a_2, a_3, a_7, a_8)$ or on the cycle $Z_3 = (a_9, a_7, a_8, a_4, a_1, a_5, a_6)$. The graph has property (H) since the edge $e = (a_1, a_5)$ belongs to both Z_1 and Z_3 these cycles cover all vertices of the graph.

Let us show next that the graph does not have property (K). For this purpose, we partition the set of its edges into pairs $\{e, f\}$ as shown in the first two rows of

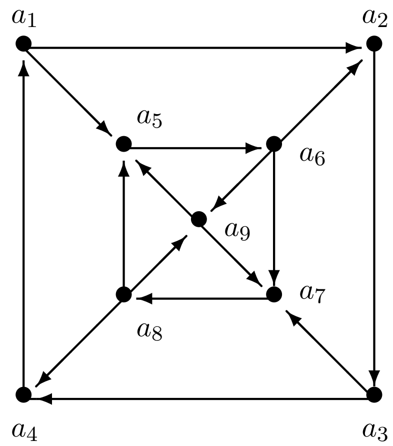

FIGURE 6. Graph G_6 of type (CH)

following table:

e	(a_1, a_5)	(a_6, a_2)	(a_3, a_7)	(a_8, a_4)	(a_1, a_2)	(a_2, a_3)	(a_3, a_4)	(a_4, a_1)
f	(a_8, a_9)	(a_9, a_7)	(a_6, a_9)	(a_9, a_5)	(a_5, a_6)	(a_6, a_7)	(a_7, a_8)	(a_8, a_5)
a	a_7	a_8	a_5	a_6				

For a fixed pair $\{e, f\}$ let us discard all edges from G_6 that are oppositely adjacent to either e or f. In the first four cases the resulting graph has a cut-vertex a, the deletion of which leads to a graph in which e and f are in distinct connected components. (This vertex a is shown in the third row of our table.) Hence e and f are not cyclically completeable in G_6. In the second four cases, after removing all edges that are oppositely adjacent to e or f, there are ostensibly at most two cycles containing e, none of which contain f:

e	f	the cycles
(a_1, a_2)	(a_5, a_6)	(a_1, a_2, a_3, a_4) and $(a_1, a_2, a_3, a_7, a_8, a_4)$
(a_2, a_3)	(a_6, a_7)	(a_1, a_2, a_3, a_4)
(a_3, a_4)	(a_7, a_8)	(a_1, a_2, a_3, a_4) and $(a_1, a_5, a_6, a_2, a_3, a_4)$
(a_4, a_1)	(a_8, a_5)	(a_1, a_2, a_3, a_4)

Finally, we show that the graph does not have property (G). Since G_6 is invariant under a 180^o rotation around a_9, it is sufficient that for every vertex $a \in \{a_1, a_2, a_5, a_6, a_9\}$ there is an edge e which is not cyclically completeable with a. Such an edge for each is given in the following table:

a	a_1	a_2	a_5	a_6	a_9
e	(a_8, a_5)	(a_6, a_7)	(a_1, a_2)	(a_1, a_2)	(a_1, a_2)

For $a = a_1$, after removing the edges (a_1, a_5) and (a_8, a_4), oppositely adjacent to $e = (a_8, a_5)$, it is clear that any cycle Z containing both a_1 and e must contain the path (a_3, a_4, a_1, a_2), and so we may remove the edges (a_3, a_7) and (a_6, a_2), oppositely adjacent to some edge in this path. After the removal of these edges G_6 falls apart into two connected components, and a and e end up being in different components. For $a = a_2$ and $e = (a_6, a_7)$, after removing the edges (a_6, a_2), (a_6, a_9), (a_3, a_7) and (a_9, a_7) that are oppositely adjacent to e, it is clear that the edge e belongs to only three cycles. These are $Z_1 = (a_5, a_6, a_7, a_8, a_9)$, $Z_2 = (a_5, a_6, a_7, a_8)$, and $Z_3 = (a_1, a_5, a_6, a_7, a_8, a_4)$, none of them containing a_2. In the last three cases, after removing the edges (a_1, a_5) and (a_6, a_2) which are oppositely adjacent to

$e = (a_1, a_2)$ we can see that e is contained only in two cycles, (a_1, a_2, a_3, a_4) and $(a_1, a_2, a_3, a_7, a_8, a_4)$. None of these contains any of a_5, a_6, or a_9. The graph G_6 does not have property (D) since property (D) implies property (G).

Example 7. *Let G_7 be the graph obtained by orienting the edges of an octahedron, as shown on Figure 7. This graph has type (DK).*

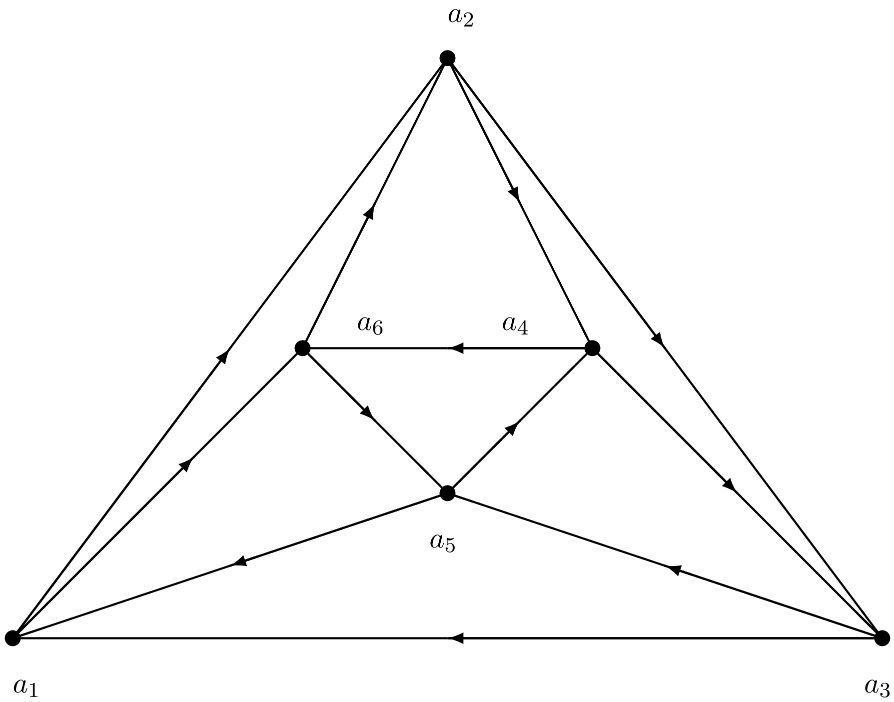

FIGURE 7. Graph G_7 of type (DK)

Since the graph has a Hamiltonian cycle $Z_1 = (a_1, a_6, a_2, a_4, a_3, a_5)$, it has properties (C) and (H). More is true: the graph also has property (D). Because of the rotational symmetry it is sufficient to show that a vertex $a \in \{a_1, a_4\}$ and any edge is cyclically completeable. Assume first $a = a_1$. The aforementioned cycle Z_1, and the cycles $Z_2 = (a_1, a_2, a_3)$, $Z_3 = (a_1, a_6, a_5, a_4, a_3)$, and $Z_4 = (a_1, a_2, a_4, a_6, a_5)$ pass through a_1, and together they cover all edges of the graph. In the case when $a = a_4$ the cycles Z_1, Z_3, Z_4, furthermore $Z_5 = (a_2, a_3, a_5, a_4, a_6)$ all pass through a_4, and together they cover all edges of the graph.

The graph does not have property (E), since the edges $e = (a_1, a_2)$ and $f = (a_5, a_4)$ are not cyclically completeable. In fact, after removing the edges (a_1, a_6), (a_6, a_2), (a_2, a_4), (a_5, a_1) which are oppositely adjacent to either e or f, every path from the set $\{a_1, a_2\}$ to the set $\{a_4, a_5, a_6\}$ passes through the cut-vertex a_3.

The graph has property (K) as it is shown by the choice $e = (a_1, a_6)$, since the cycles Z_1, Z_3, $Z_6 = (a_1, a_6, a_5)$, and $Z_7 = (a_1, a_6, a_2, a_3)$ containing it cover all edges of the graph. Finally the graph does not have property (J) since for every edge another oppositely adjacent edge may be found.

Example 8. *The graph G_8 on Figure 8 ha type (CGH).*

The graph has the properties (C) and (H) since it has a Hamiltonian cycle: $Z_1 = (a_1, a_2, a_2', a_3, a_4, a_4', a_5, a_6, a_6')$. The choice $a = a_1$ shows that the graph has property (G) since any edge not contained in Z_1 is contained in at least one of the

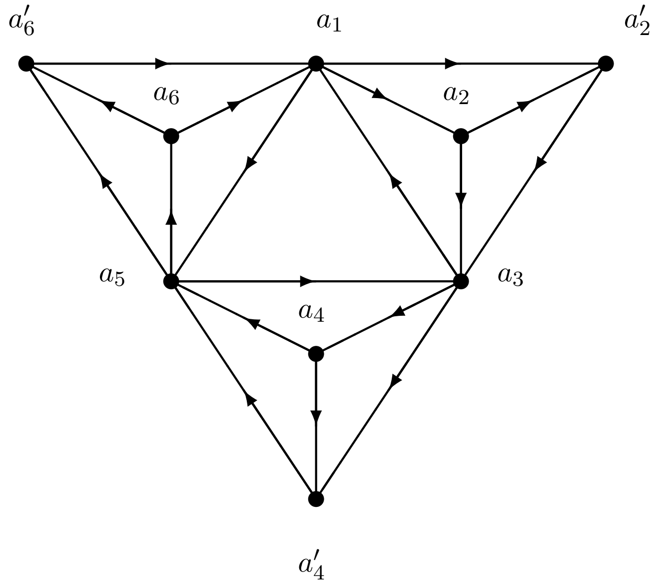

FIGURE 8. Graph G_8 of type (CGH)

cycles $Z_2 = (a_1, a_2, a_3, a_4, a_5, a_6)$, $Z_3 = (a_1, a_2', a_3, a_4', a_5, a_6')$, and $Z_4 = (a_1, a_5, a_3)$, and these cycles all pass through a_1.

The graph does not have property (K). Since the graph has a rotational symmetry, it is sufficient to show for any edge e of the subgraph induced by the vertex set $\{a_1, a_2, a_2', a_3\}$ that an edge f may be found with which e is not cyclically completeable. Let us choose f according to the following table:

e	$(a_1, a_2), (a_1, a_2')$ or (a_2, a_2')	(a_2, a_3) or (a_2', a_3)	(a_3, a_1)
f	(a_5, a_3)	(a_1, a_5)	(a_5, a_6)

After removing all edges oppositely adjacent to e or f the vertex a_1 is a cut vertex of the resulting graph G^*: any directed path from a vertex adjacent to e to a vertex adjacent to f or vice versa passes through a_1. Hence e and f are not cyclically completeable. The graph G_8 does not have property (J) since property (J) implies property (K).

To show that the graph does not have property (D) it is sufficient to find a vertex a and and edge e that are not cyclically completeable. The vertex $a = a_2$ and the edge $e = (a_1, a_5)$ are such, since (a_1, a_2), the only edge going into a_2, is oppositely adjacent to (a_1, a_5).

Example 9. *The graph G_9, obtained by adding some new edges and vertices to the graph G_8, as shown on Figure 9, has type (GH).*

The graph G_8 had properties (C) and (H) by the existence of the cycle

$$Z_1 = (a_1, a_2, a_2', a_3, a_4, a_4', a_5, a_6, a_6')$$

which, from the point of view of property (H) also means that the edge (a_1, a_5) is cyclically completeable with the vertices $a_1, a_2, a_2', a_3, a_4, a_4', a_5, a_6$ and a_6'. Because of the existence of the cycle $Z_5 = (a_1, a_2, a_2', a_7, a_7')$, the edge (a_1, a_2) is also cyclically completeable with the vertices a_7 and a_7'. Hence G_9 too, has property (H).

The choice $a = a_1$ showed that G_8 has property (G), that is, a_1 is cyclically completeable with any edge of G_8. By the existence of Z_5, the vertex a_1 is also

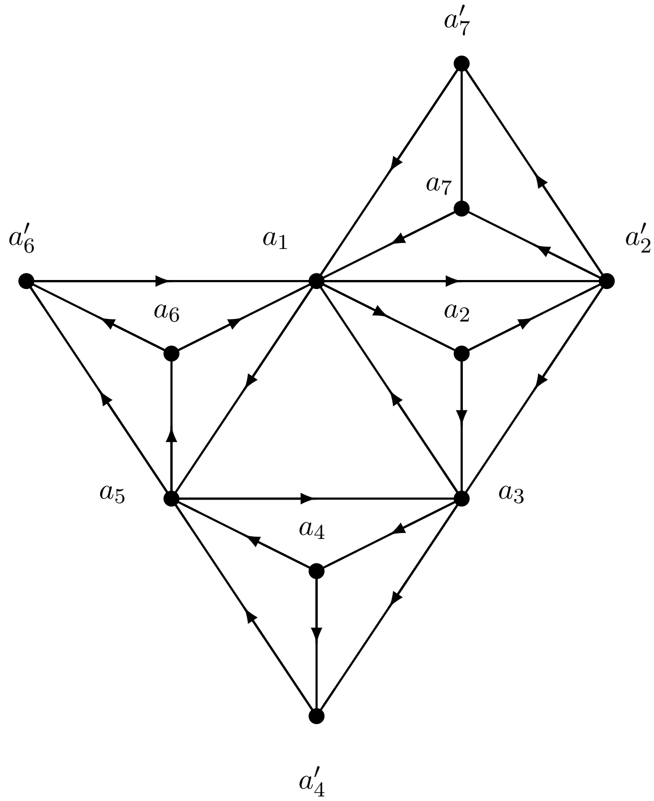

Figure 9. Graph G_9 of type (GH)

cyclically completeable with the edges (a_2', a_7), (a_7, a_7'), and (a_7', a_1). The remaining edges (a_7, a_1) and (a_2', a_7') lie on the cycles $Z_6 = (a_1, a_2', a_7)$ resp. $Z_7 = (a_1, a_2', a_7')$, both of which contain a_1. Hence G_9 too has property (G).

G_9 can not have property (C) since a_5 and a_7 are not cyclically completeable. In fact, in order to get to the vertex set $V_0 = \{a_1, a_2, a_2', a_3, a_7, a_7'\}$ from an outside vertex one needs to pass through either a_1 or a_3. Hence the edge (a_3, a_1) is not contained in any cycle Z that visits both a_5 and a_7: we may remove it without destroying Z. But after removing this edge, we can not get to any other vertex of V_0 from a_3 in the subgraph induced by V_0.

The graph G_9 does not have property (K). To prove this, we only need to show that there is no such edge among the ones complementing G_8 to G_9 which would be cyclically completeable with any nonadjacent or consecutive edge, for example with (a_1, a_5). If there was such a cycle Z, it would pass through a_5 and a directed path from a_2' to a_1 would contain the new edge. This, however, is impossible, since by what was said earlier, the vertices a_5 and a_7 (and similarly a_5 and a_7') are not cyclically completeable.

4. Concluding remarks

In our quest for examples of missing types the following observations proved to be useful. Let us call the graph on Figure 10 a *"compass"*.

Given a directed graph G, let us replace one of its edges (a, b) or one of its directed paths (a, c, b) containing the transient vertex c with a compass. Since the edges of our compass are covered by the directed paths (a, c_1, b), (a, c_2, b), and

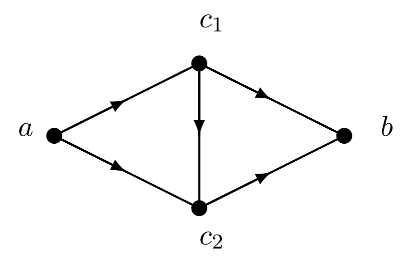

FIGURE 10. A "compass"

(a, c_1, c_2, b), an edge of the compass is cyclically completeable with some edge or vertex x outside the compass, if and only if x was on a cycle containing (a, b) or (a, c, b) in the original graph G. This observation may be used to construct a directed graph satisfying condition (1) from Example 8 in [1] which is a graph with property (A) but containing transient vertices. By replacing the transient vertices with compasses, we obtain a new type (A) graph in which no vertex violates restriction (1). Similarly, graph G_8 in our paper may be generated from the simpler graph with transient vertices shown on Figure 11. The use of compasses may also

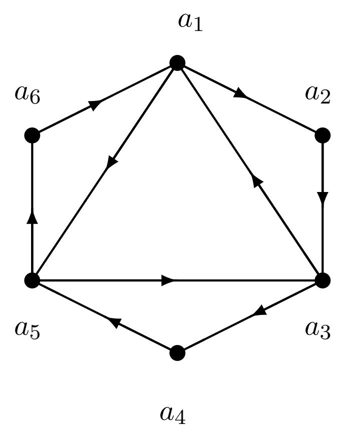

FIGURE 11. Replacing the transient vertices with compasses in this graph yields G_8

help to generate from a graph having multiple edges between a pair of vertices a graph that is simple and satisfies restriction (1). For example, the simplest way to generate a type (CGK) graph is to take the directed graph G with vertex set $V = \{a, b\}$ and edge set $E = \{(a, b), (b, a)\}$ and replace both edges with a compass. The resulting graph is shown on Figure 12.

Other useful operations include *contracting edges* and *expanding vertices*. We may contract any edge (a, b) as follows: we remove the vertices a, b and the edge (a, b) from the graph, we introduce a new vertex d and we replace each edge of the form (x, a) or (b, y) respectively with an edge (x, d) or (d, y) respectively. Multiple edges may arise (which may be handled by the use of compasses). Similarly we may expand any vertex a to an edge, by removing a, introducing to new vertices a_1 and a_2 and by introducing the edges (a_1, a_2), all edges (x, a_1) respectively $a_2, y)$ such that (x, a) respectively (a, y) were edges of the original graph. If the indegree or outdegree of a is 1 then a transient vertex arises.

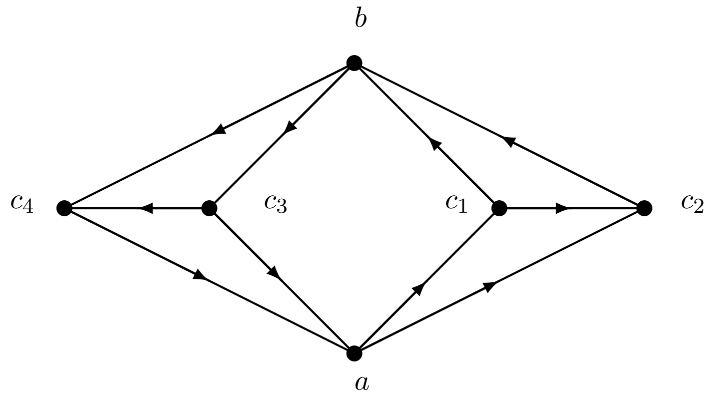

FIGURE 12. A small simple graph of type (CGK)

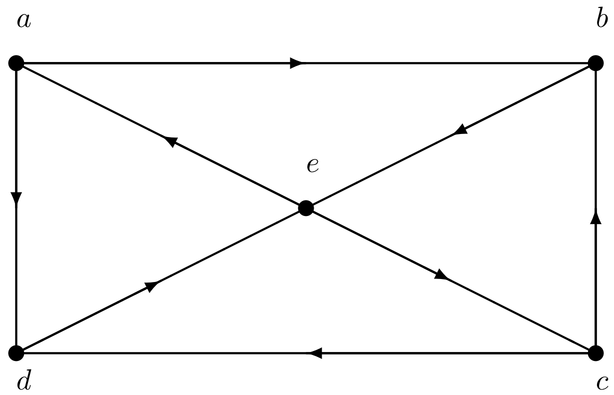

FIGURE 13. Example of a graph of type (G) from Ádám's paper

It is a straightforward consequence of the definitions that expanding a vertex a in a type (J) graph yields a type (EJ) graph with $e = (a_1, a_2)$ showing that property (J) is still satisfied. (The existence of oppositely adjacent edges is preserved under expanding vertices: if for example (x_1, a) and (x_2, a) are oppositely adjacent edges in the original graph then (x_1, a_1) and (x_2, a_1) are oppositely adjacent in the expanded graph.) Hence the study of type (EJ) graphs may be essentially reduced to the study of type (E) graphs. This may indicate that the answer to [1, Problem 5] asking for a characterization of graphs of type (EJ) may not be that easy at the end of all.

If a graph has property (J) with edge $e = (a_1, a_2)$ being cyclically completeable with all other edges then contracting e yields a graph with (G). (The resulting graph may not be simple any more, but this issue may be resolved by using compasses.) Conversely, given a graph with property (G) such that vertex a is cyclically completeable with any edge, we may expand a to an edge (a_1, a_2) and get a graph with property (J). (If the indegree of a is 1 and e is the only edge ending in a then the expansion is not necessary, the graph already has property (J) as it is shown by the existence of e. The situation is similar if the outdegree of a is 1.) For example we may use the graph from Example 2 of [1] shown on Figure 13 to generate a graph of type (J) shown on Figure 14.

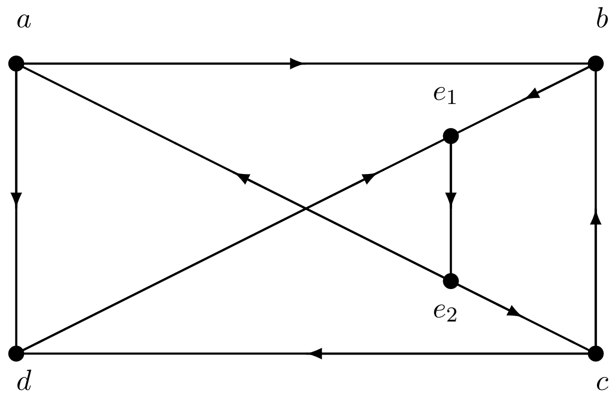

FIGURE 14. Graph of type (J) obtained by expanding vertex e on Figure 13

Let us address finally the issue of the missing types (C), (D), and (CG). Based on our research we conjecture the following.

Conjecture 1. *The implications* (C) $\Rightarrow$ (H) *and* (D) $\Rightarrow$ (K) *hold.*

If this conjecture is true then there is no graph of type (C), (D), or (CG), only graphs of type (CH), (DK), and (CGH) may exist, and these types are in fact nonempty.

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[1] A. Ádám, On Some Cyclic Connectivity Properties of Directed Graphs (Examples and Problems), *Acta Cybernetica* **14** (1999) 1–12.

[2] F. Harary, R. Z. Norman, D. Cartwright, "Structural models: An introduction to the theory of directed graphs", Wiley, New York, 1965. (In French translation: Dunod, Paris, 1968).

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