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# GEODESICS ON A CENTRAL SYMMETRIC WARPED PRODUCT MANIFOLD

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Abstract. We consider a warped product Riemannian metric on the manifold $\mathbb{R}^n_0 \times \mathbb{R}^1$ with the central symmetric warping function $\phi(\mathbf{x}) = \|\mathbf{x}\|^{-2}$ mapping $\mathbb{R}^n_0 \to \mathbb{R}^1$. The orthogonal projections onto $\mathbb{R}^n_0$ of geodesics of this warped product manifold $\mathbb{R}^n_0 \times_\phi \mathbb{R}^1$ are exactly the trajectories of the mechanical systems on $\mathbb{R}^n_0$ with potential function $c\phi(\mathbf{x})^{-1}$ with arbitrary constant $c$. In this case all bounded trajectories of the mechanical system are closed. We show that the projection of geodesics are conic sections and determine the parameters of these conic sections as functions of the initial values of geodesics.

We consider the manifold $\mathbb{R}^n_0 \times \mathbb{R}^1$, (where $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \{\mathbf{0}\}$) equipped with a Riemannian scalar product $\langle .,. \rangle$ which is defined by the following properties:

(i) the projection onto $\mathbb{R}^n_0$ along $\mathbb{R}^1$ of this Riemannian scalar product $\langle .,. \rangle$ is the canonical Euclidean one,
(ii) $\mathbb{R}^1$ is orthogonal to $\mathbb{R}^n_0$ with respect to $\langle .,. \rangle$,
(iii) the projection onto $\mathbb{R}^1$ along $\mathbb{R}^n_0$ of $\langle .,. \rangle$ at $(\mathbf{a}, \alpha) \in \mathbb{R}^n_0 \times \mathbb{R}^1$ is the canonical one multiplied by the scalar coefficient $\|\mathbf{a}\|^{-1}$.

These properties determine uniquely the scalar product of vectors $(\mathbf{x}, \xi)$, $(\mathbf{y}, \eta) \in T_{(\mathbf{a},\alpha)}(\mathbb{R}^n_n \times \mathbb{R}^1)$ and it can be written in the form

$$\langle (\mathbf{x}, \xi)(\mathbf{y}, \eta) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \|\mathbf{a}\|^{-2} \xi \eta. \tag{1}$$

The above described Riemannian manifold is a special case of the so called warped product Riemannian space (cf. [4], [5]) with warping function

$$\mathbf{x} \mapsto \|\mathbf{x}\|^{-2} : \mathbb{R}^n_0 \to \mathbb{R}^1.$$

Considering an arbitrary warping function $\phi \colon \mathbb{R}^n_0 \to \mathbb{R}^1$, it is easy to see that the orthogonal projections onto $\mathbb{R}^n_0$ of geodesics of the warped product manifold

$$\mathbb{R}^n_0 \times_\phi \mathbb{R}^1$$

are exactly the trajectories of the mechanical systems on $\mathbb{R}^n_0$ with potential function $c\phi(\mathbf{x})^{-1}$ with arbitrary constant $c$ (cf. [2], [3], [5, Fact 6.2]). The geometry of the trajectories of mechanical systems with central symmetric potential function is presented in [1], Chapter 2. In particular there has been proved that all bounded trajectories of a mechanical system with central symmetric potential function are closed if and only if the potential function has the form $-k\|\mathbf{x}\|^{-1}, (k \geq 0)$ or $a\|\mathbf{x}\|^2, (a \geq 0)$. The first case gives the geometry of Kepler motions.

Now, we want to study the geometry of the central symmetric warped product manifold which corresponds to the potential function $a\|\mathbf{x}\|^2, (a \geq 0)$.

We get the following description of geodesics:

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**Theorem.** *Let $(\mathbf{x}(s), \xi(s))$ be a geodesic in $\mathbb{R}^n_0 \times \mathbb{R}^1$ with respect to the Riemannian metric* (1). *We denote its initial values at $s = 0$ by $\mathbf{x}(0) = \mathbf{x}_0$, $\xi(0) = \xi_0$, $\dot{\mathbf{x}}(0) = \mathbf{t}_0$, $\dot{\xi}(0) = \tau_0$. Then one has the following possibilities:*

a) *If $\tau_0 = 0$ then the geodesic $(\mathbf{x}(s), \xi(s))$ is contained in the line $\mathbf{x}(s) = \mathbf{t}_0 s + \mathbf{x}_0$, $\xi(s) = \xi_0$; this geodesic is complete except for the case if $\xi_0 = 0$ and the vectors $\mathbf{t}_0$ and $\mathbf{x}_0$ are collinear.*

b) *If $\tau_0 > 0$ then the projection of the geodesic onto $\mathbb{R}^n_0$ is an ellipse with centre $\mathbf{0}$. Its equation has the shape*

$$\mathbf{x}(s) = \cos(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{x}_0 + \sin(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\sqrt{\tau_0}\|\mathbf{x}_0\|\mathbf{t}_0.$$

*The corresponding geodesic is complete except for the case if the vectors $\mathbf{t}_0$ and $\mathbf{x}_0$ are collinear and the projected ellipse is degenerated to a segment with the midpoint $\mathbf{0}$.*

c) *If $\tau_0 < 0$ then the projection of the geodesic onto $\mathbb{R}^n_0$ is a hyperbola with centre $\mathbf{0}$. Its equation has the shape*

$$\mathbf{x}(s) = \cosh(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{x}_0 + \sinh(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\sqrt{\tau_0}\|\mathbf{x}_0\|\mathbf{t}_0.$$

*If the vectors $\mathbf{t}_0$ and $\mathbf{x}_0$ are collinear and the projected hyperbola is degenerated to a half line then the corresponding geodesic is complete.*

*Proof.* Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be an orthonormal basis in the vector space $\mathbb{R}^n$ satisfying $\mathbf{x}_0 = r\mathbf{e}_1$, $\mathbf{t}_0 = \cos\gamma\mathbf{e}_1 + \sin\gamma\mathbf{e}_2$ and let $\mathbf{e}_0$ be a unit vector of $\mathbb{R}^1$. In the corresponding coordinate system $\{x_0, x_1, \ldots, x_n\}$, defined by $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$ and $x_0 = \xi$, the Riemannian metric tensor $g_{ij}$ has the following components:

$$g_{\lambda\mu} = \delta_{\lambda\mu}, \quad g_{\lambda 0} = g_{0\lambda} = 0, \quad g_{00} = \|\mathbf{x}\|^{-2}, \quad (\lambda, \mu = 1, \ldots, n)$$

at the point $(\mathbf{x}, \xi)$. An easy calculation gives that the non-vanishing coefficients ${\Gamma^i}_{j\,k}$ in the equation $\ddot{x}^i + \sum \Gamma^i_{j\,k}\dot{x}^j\dot{x}^k = 0$, $(i, j, k = 0, \ldots, n)$ of geodesics can be expressed by

$${\Gamma^\lambda}_{00} = \|\mathbf{x}\|^{-3}\frac{\partial\|\mathbf{x}\|}{\partial x_\lambda}, \quad {\Gamma^0}_{\mu 0} = {\Gamma^0}_{0\mu} = -\|\mathbf{x}\|^{-1}\frac{\partial\|\mathbf{x}\|}{\partial x_\mu}.$$

It follows that the equation of a geodesic $(\mathbf{x}(s), \xi(s))$ is of the form

$$\ddot{x}^\lambda(s) + \|\mathbf{x}(s)\|^{-4}x^\lambda(s)\dot{\xi}^2(s) = 0, \quad (\lambda = 1, \ldots, n),$$

$$\ddot{\xi}(s) - 2\|\mathbf{x}(s)\|^{-2}\sum_{\mu=1}^{n} x^\mu(s)\dot{x}^\mu(s)\dot{\xi}(s) = 0,$$

where the dot denotes the derivation $\frac{d}{ds}$. These equations can be written in the form

$$\ddot{\mathbf{x}}(s) = \|\mathbf{x}(s)\|^{-4}\dot{\mathbf{x}}(s)\,\dot{\xi}^2(s), \qquad \ddot{\xi}(s) = 2\|\mathbf{x}(s)\|^{-1}\frac{d\|\mathbf{x}(s)\|}{ds}\,\dot{\xi}(s).$$

The last equation is equivalent to the expression

$$\dot{\xi}(s) = c\|\mathbf{x}(s)\|^2$$

with an arbitrary $c =$ constant. Substituting this into the preceding equations one has $\ddot{\mathbf{x}}(s) + c\mathbf{x}(s) = \mathbf{0}$.

If $\dot{\xi}(0) = \tau_0 = 0$ then $c = 0$, the function $\xi(s) = \xi_0$ is constant and the vector valued function $\mathbf{x}(s) = \mathbf{t}_0 s + \mathbf{x}_0$ is linear. It means that the corresponding geodesic is a line. If the initial values satisfy $\xi_0 = 0$ and the vectors $\mathbf{t}_0$ and $\mathbf{x}_0$ are collinear then the geodesic should contain the origin $(\mathbf{0}, 0)$ which does not belong to the manifold. Hence in this case the corresponding geodesic is non-complete.

Now, we assume $\tau_0 \neq 0$. In this case we have $\tau_0 = c\|\mathbf{x}(0)\|^2 = c\|\mathbf{x}_0\|^2$ and

$$\ddot{\mathbf{x}}(s) + \frac{\tau_0}{\|\mathbf{x}_0\|^2}\mathbf{x}(s) = \mathbf{0}.$$

If $\tau_0 > 0$ then the general solution of this equation has the following form

$$\mathbf{x}(s) = \cos(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{a} + \sin(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors satisfying $\mathbf{x}_0 = \mathbf{a}$ and $\mathbf{t}_0 = \sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}\mathbf{b}$. Clearly, if the initial values $\mathbf{x}_0$ and $\mathbf{t}_0$ are linearly independent then the solution curve is an ellipse with centre $\mathbf{0}$ which is contained in the 2-dimensional subspace $W$ of $\mathbb{R}^n$ spanned by the initial values $\mathbf{x}_0$ and $\mathbf{t}_0$. Hence the corresponding geodesic is complete.

If the initial values $\mathbf{x}_0$ and $\mathbf{t}_0$ are linearly dependent then the solution ellipse with centre $\mathbf{0}$ is degenerated to a segment containing $\mathbf{0}$. But the origin $\mathbf{0}$ does not belong to our manifold and hence the corresponding geodesic is non-complete.

If $\tau_0 < 0$ then the general solution of this equation has the following form

$$\mathbf{x}(s) = \cosh(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{a} + \sinh(\sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}s)\mathbf{b},$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors satisfying $\mathbf{x}_0 = \mathbf{a}$ and $\mathbf{t}_0 = \sqrt{\tau_0}\|\mathbf{x}_0\|^{-1}\mathbf{b}$. If $\mathbf{x}_0$ and $\mathbf{t}_0$ are linearly independent then the solution curve is a connected component of a hyperbola with centre $\mathbf{0}$ which is contained in the subspace $W$ spanned by the initial values $\mathbf{x}_0$ and $\mathbf{t}_0$. Hence the corresponding geodesic is complete.

If the vectors $\mathbf{x}_0$ and $\mathbf{t}_0$ are linearly dependent then the solution hyperbola with centre $\mathbf{0}$ is degenerated to a repeated half line fully contained in the manifold. Hence the corresponding geodesic is complete. □

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