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CONVERGENCE OF CESÀRO MEANS OF FUNCTIONS WITH RESPECT TO UNBOUNDED VILENKIN SYSTEMS

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This paper is dedicated to Professor William Wade on the occasion of his sixtieth birthday

Abstract. One of the most celebrated problems in dyadic harmonic analysis is the pointwise convergence of the Fejér (or $(C, 1)$) means of functions on unbounded Vilenkin groups. The aim of this paper is to give a résumé of the recent developments concerning this matter. Above all, we prove that the maximal operator $\sup |\sigma_{M_n}|$ is of type (H, L^1) on unbounded Vilenkin groups.

First, we give a brief introduction to the theory of Vilenkin systems. These orthonormal systems were introduced by N.Ja. Vilenkin in 1947 (see e.g. [25, 1]) as follows.

Let $m := (m_k, k \in \mathbb{N})$ $(\mathbb{N} := \{0, 1, \dots\}, \mathbb{P} := \mathbb{N} \setminus \{0\})$ be a sequence of integers each of them not less than 2. Let Z_{m_k} denote the discrete cyclic group of order m_k. That is, Z_{m_k} can be represented by the set $\{0, 1, \dots, m_k - 1\}$, with the group operation $\mod m_k$ addition. Since the groups is discrete, then every subset is open. The normalized Haar measure on Z_{m_k}, μ_k is defined by $\mu_k(\{j\}) := 1/m_k$ $(j \in \{0, 1, \dots, m_k - 1\})$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

Then every $x \in G_m$ can be represented by a sequence $x = (x_i, i \in \mathbb{N})$, where $x_i \in Z_{m_i}$ $(i \in \mathbb{N})$. The group operation on G_m (denoted by $+$) is the coordinate-wise addition (the inverse operation is denoted by $-$), the measure (denoted by μ), which is the normalized Haar measure, and the topology are the product measure and topology. Consequently, G_m is a compact Abelian group. If $\sup_{n\in\mathbb{N}} m_n < \infty$, then we call G_m a bounded Vilenkin group. If the generating sequence m is not bounded, then G_m is said to be an unbounded Vilenkin group. The Vilenkin group is metrizable in the following way:

$$d(x, y) := \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{M_{i+1}} \quad (x, y \in G_m).$$

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The topology induced by this metric, the product topology, and the topology given by intervals defined below, are the same. A base for the neighborhoods of G_m can be given by the intervals:

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbb{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

for $x \in G_m, n \in \mathbb{P}$. Let $0 = (0, i \in \mathbb{N}) \in G_m$ denote the nullelement of G_m.

Furthermore, let $L^p(G_m)\,(1 \leq p \leq \infty)$ denote the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) on G_m, $\mathcal{A}_n$ the σ-algebra generated by the sets $I_n(x)\,(x \in G_m)$, and E_n the conditional expectation operator with respect to $\mathcal{A}_n\,(n \in \mathbb{N})$ $(f \in L^1)$.

The concept of the maximal Hardy space ([21]) $H^1(G_m)$ is defined by the maximal function $f^* := \sup_n |E_n f|$ $(f \in L^1(G_m))$, saying that f belongs to the Hardy space $H^1(G_m)$ if $f^* \in L^1(G_m)$. $H^1(G_m)$ is a Banach space with the norm $\|f\|_{H^1} := \|f^*\|_1$.

The so-called atomic Hardy space $H(G_m)$ is defined for bounded Vilenkin groups as follows [21, 22]. A function $a \in L^\infty(G_m)$ is called an atom, if either $a = 1$ or a has the following properties: supp $a \subseteq I_a$, $\|a\|_\infty \leq \frac{1}{\mu(I_a)}$, $\int_{I_a} a = 0$, where $I_a \in \mathfrak{I} := \{I_n(x) : x \in G_m, n \in \mathbf{N}\}$. The elements of $\mathfrak{I}$ are called intervals on G_m. We say that the function f belongs to $H(G_m)$, if f can be represented as $f = \sum_{i=0}^\infty \lambda_i a_i$, where a_i-s are atoms and for the coefficients $\lambda_i\,(i \in \mathbf{N})$ $\sum_{i=0}^\infty |\lambda_i| < \infty$ is true. It is known that $H(G_m)$ is a Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^\infty |\lambda_i|,$$

where the infinum is taken all over decompositions

$$f = \sum_{i=0}^\infty \lambda_i a_i \in H(G_m).$$

If the sequence m is not bounded, then we define the set of intervals in a different way [22], that is we have "more" intervals than in the bounded case.

A set $I \subset G_m$ is called an interval if for some $x \in G_m$ and $n \in \mathbb{N}_0, I$ is of the form $I = \bigcup_{k \in U} I_n(x,k)$ where U is one of the following sets

$$U_1 = \left\{0, \dots, \left[\frac{m_n}{2}\right] - 1\right\}, U_2 = \left\{\left[\frac{m_n}{2}\right], \dots, m_n - 1\right\}$$

$$U_3 = \left\{0, \dots, \left[\frac{[m_n/2] - 1}{2}\right] - 1\right\}, U_4 = \left\{\left[\frac{[m_n/2] - 1}{2}\right] - 1, \dots, \left[\frac{m_n}{2}\right] - 1\right\}, \dots$$

etc., and $I_n(x,k) := \{y \in G_m : y_j = x_j (j < n), y_n = k\}$, $(x \in G_m, k \in Z_{m_n}, n \in \mathbb{N}_0)$. The rest of the definition of the atomic Hardy space H is the same as in the bounded case.

It is known that if the sequence m is bounded, then $H^1 = H$, otherwise H is a proper subset of H^1 [16].

We say that the function $f \in L^1(G_m)$ belongs to the logarithm space $L\log^+ L(G_m)$ if the integral

$$\|f\|_{L\log^+ L} := \int_{G_m} |f(x)| \log^+(|f(x)|) d\mu(x)$$

is finite. The positive logarithm $\log^+$ is defined as

$$\log^+(x) := \begin{cases} x & \text{if } x > 1, \\ 0 & \text{otherwise .} \end{cases}$$

Let X and Y be either $H^1(G_m)$ or $L^p(G_m)$ for some $1 \le p \le \infty$ with norms $\|.\|_X$ and $\|.\|_Y$. We say that operator T is of type (X, Y) if there exist an absolute constant $C > 0$ for which $\|Tf\|_Y \le C\|f\|_X$ for all $f \in X$. If $X = Y = L^p(G_m)$ then we often say that T is of type (p, p) instead of type (L^p, L^p). T is of weak type (L^1, L^1) (or weak type $(1, 1)$) if there exist an absolute constant $C > 0$ for which $\mu(Tf > \lambda) \le C\|f\|_1/\lambda$ for all $\lambda > 0$ and $f \in L^1(G_m)$. It is known that the operator which maps a function f to the maximal function f^* is of weak type (L^1, L^1), and of type (L^p, L^p) for all $1 < p \le \infty$ (see e.g. [3]).

Let $M_0 := 1, M_{n+1} := m_n M_n$ $(n \in \mathbb{N})$ be the so-called generalized powers. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, \dots, m_i - 1\},\ i \in \mathbb{N}),$$

where only a finite number of n_i-s differ from zero. The generalized Rademacher functions are defined as

$$r_n(x) := \exp(2\pi\imath \frac{x_n}{m_n}) \quad (x \in G_m, n \in \mathbb{N}, \imath := \sqrt{-1}).$$

It is known that $\sum_{i=0}^{m_n-1} r_n^i(x) = \begin{cases} 0 & \text{, if } x_n \ne 0, \\ m_n & \text{, if } x_n = 0 \end{cases}$ $(x \in G_m,\ n \in \mathbb{N})$. The n^{th} Vilenkin function is

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbb{N}).$$

The system $\psi := (\psi_n : n \in \mathbb{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m, and all the characters of G_m are of this form. Define the m -adic addition as

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\bmod m_j)) M_j \quad (k, n \in \mathbb{N}).$$

Then , $\psi_{k \oplus n} = \psi_k \psi_n,\ \psi_n(x + y) = \psi_n(x)\psi_n(y),\ \psi_n(-x) = \bar{\psi}_n(x),\ |\psi_n| = 1$ $(k, n \in \mathbb{N}, x, y \in G_m)$.

Define the Fourier coefficients, the partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the Vilenkin system ψ as follows

$$\hat{f}(n) := \int_{G_m} f\bar{\psi}_n,\ S_n f := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k,\ D_n(y, x) = D_n(y - x) := \sum_{k=0}^{n-1} \psi_k(y)\bar{\psi}_k(x),$$

$$\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f,\ K_n(y, x) = K_n(y - x) := \frac{1}{n} \sum_{k=0}^{n-1} D_k(y - x),$$

$$(n \in \mathbb{P}, y, x \in G_m,\ \hat{f}(0) := \int_{G_m} f,\ S_0 f = D_0 = K_0 = 0,\ f \in L^1(G_m)).$$

It is well-known that

$$S_n f(y) = \int_{G_m} f(x) D_n(y-x) d\mu(x),$$
$$\sigma_n f(y) = \int_{G_m} f(x) K_n(y-x) d\mu(x) \quad (n \in \mathbb{P},\, y \in G_m,\, f \in L^1(G_m)).$$

It is also well-known that

$$D_{M_n}(x) = \begin{cases} M_n & \text{if } x \in I_n(0) \\ 0 & \text{if } x \notin I_n(0) \end{cases},$$
$$S_{M_n} f(x) = M_n \int_{I_n(x)} f = E_n f(x) \quad (f \in L^1(G_m), n \in \mathbb{N}).$$

Next we introduce some notation with respect to the theory of two-dimensional Vilenkin systems. Let $\tilde{m}$ be a sequence like m. The relation between the sequence $(\tilde{m}_n)$ and $(\tilde{M}_n)$ is the same as between sequence (m_n) and (M_n). The group $G_m \times G_{\tilde{m}}$ is called a two-dimensional Vilenkin group. The normalized Haar measure is denoted by μ, just as in the one-dimensional case. It will not cause any misunderstood.

The two-dimensional Fourier coefficients, the rectangular partial sums of the Fourier series, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the two-dimensional Vilenkin system are defined as follows:

$$\hat{f}(n_1, n_2) := \int_{G_m \times G_{\tilde{m}}} f(x^1, x^2) \bar{\psi}_{n_1}(x^1) \bar{\psi}_{n_2}(x^2) d\mu(x^1, x^2),$$
$$S_{n_1,n_2} f(y^1, y^2) := \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \hat{f}(k_1, k_2) \psi_{k_1}(y^1) \psi_{k_2}(y^2),$$
$$D_{n_1,n_2}(y, x) \ = D_{n_1}(y^1 - x^1) D_{n_2}(y^2 - x^2)$$
$$:= \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} \psi_{k_1}(y^1) \psi_{k_2}(y^2) \bar{\psi}_{k_1}(x^1) \bar{\psi}_{k_2}(x^2),$$
$$\sigma_{n_1,n_2} f := \frac{1}{n_1 n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} S_{k_1,k_2} f,$$
$$K_{n_1,n_2}(y, x) = K_{n_1,n_2}(y - x) := \frac{1}{n_1 n_2} \sum_{k_1=0}^{n_1-1} \sum_{k_2=0}^{n_2-1} D_{k_1,k_2}(y - x),$$
$$(y = (y^1, y^2), x = (x^1, x^2) \in G_m \times G_{\tilde{m}}).$$

It is also well-known that

$$\sigma_{n_1,n_2} f(y) = \int_{G_m \times G_{\tilde{m}}} f(x) K_{n_1,n_2}(y - x) d\mu(x),$$
$$S_{M_{n_1}, \tilde{M}_{n_2}} f(x) = M_{n_1} \tilde{M}_{n_2} \int_{I_{n_1}(x^1) \times I_{n_2}(x^2)} f = (E^1_{n_1} \otimes E^2_{n_2}) f(x).$$

One of the most celebrated problems in dyadic harmonic analysis is the pointwise convergence of the Fejér (or $(C,1)$) means of functions on one and two-dimensional unbounded Vilenkin groups.

Fine [4] proved every Walsh-Fourier series (in the Walsh case $m_j = 2$ for all $j \in \mathbb{N}$) is a.e. (C, α) summable for $\alpha > 0$. His argument is an adaptation of the older trigonometric analogue due to Marcinkiewicz [12]. Schipp [19] gave a simpler proof for the case $\alpha = 1$, i.e. $\sigma_n f \to f$ a.e. $(f \in L^1(G_m))$. He proved that $\sigma^* = \sup |\sigma_n|$, $h \in \mathbb{N}$ is of weak type (L^1, L^1). That σ^* is bounded from H^1 to L^1 was discovered by Fujii [6].

The theorem of Schipp are generalized to the p-series fields ($m_j = p$ for all $j \in \mathbb{N}$) by Taibleson [24], and later to bounded Vilenkin systems by Pál and Simon [16].

Now, what about the Vilenkin groups with unbounded generating sequences? The methods known in the trigonometric or in the Walsh, bounded Vilenkin case are not powerful enough. One of the main problems is that the proofs on the bounded Vilenkin groups (or in the trigonometric case) heavily use the fact that the L^1 norm of the Fejér kernels are uniformly bounded. This is not the case if the group G_m is an unbounded one [17]. From this it follows that the original theorem of Fejér does not hold on unbounded Vilenkin groups. Namely, Price proved [17] that for an arbitrary sequence m ($\sup_n m_n = \infty$) and $a \in G_m$ there exists a function f continuous on G_m and $\sigma_n f(a)$ does not converge to $f(a)$. Moreover, he proved [17] that if $\frac{\log m_n}{M_n} \to \infty$, then there exists a function f continuous on G_m whose Fourier series are not $(C, 1)$ summable on a set $S \subset G_m$ which is non-denumerable. On the other hand, Nurpeisov gave [15] a necessary and sufficient condition of the uniform convergence of the Fejér means $\sigma_{M_n} f$ of continuous functions on unbounded Vilenkin groups. Namely, define the uniform modulus of continuity as

$$\omega_n(f) := \sup_{h \in I_n(0), x \in G_m} |f(x + h) - f(x)|.$$

Nurpeisov proved [15]: A necessary and sufficient condition that the means $\sigma_{M_n} f$ of the Fourier series of the continuous function f converge uniformly to f on an unbounded Vilenkin group for all such an f is that

$$\omega_{n-1}(f) \log(m_n) = o(1).$$

Since the uniform modulus of continuity can be any nonincreasing real sequence which converges to zero (for the proof see [18, 5]), then as a consequence of this it is possible to give a sequence m increasing enough fast, and a function even in the Lipschitz class Lip(1), such that the M_nth Fejér means do not converge to the function uniformly.

So, it seems that it is impossible to give a (Hölder) function class such that the uniform convergence of the Fejér means would hold for all functions in this class if there is no condition on sequence m at all.

On the other hand, mean convergence of the full partial sums for $L^p, p > 1$, is known for the unbounded case. For the proof see [20]. This trivially implies the norm convergence $\sigma_n f \to f$ for all $f \in L^p$, where $1 < p < \infty$.

What about the a.e. convergence? Simon proved [22] that the maximal operator σ^* is of type (H, L^1) if and only if the Vilenkin group is a bounded one. It does not sound good in the point of view of the a.e. convergence, since in the bounded case, the method of the proof of the a.e. relation $\sigma_n f \to f$ $(f \in L^1)$ implies that σ^* is of type (H, L^1). However, below we prove that if we take the maximal function of the partial sequence of the Fejér means σ_{M_n} in place of the whole sequence, then we get an operator of type (H, L^1) regardless of the boundedness of m.

Concerning the a.e. convergence we can say a bit more. Namely, in 1999 the author [8] proved that if $f \in L^p(G_m)$, where $p > 1$, then $\sigma_n f \to f$ almost everywhere. This was the very first "positive" result with respect to the a.e. convergence of the Fejér means of functions on unbounded Vilenkin groups.

In 2001 Simon proved [23] the following theorem with respect to the Fejér means of L^1 functions. A sequence m is said to be strong quasi-bounded if

$$\frac{1}{M_{n+1}} \sum_{j=0}^{n-1} M_{j+1} < C \log m_n.$$

Then every bounded m is quasi-bounded, and there are also some unbounded ones. Let m is strong quasi-bounded. Then for all $f \in L^1(G_m)$

$$\sigma_{M_n} f(x) - f(x) = o(\max(\log m_0, \ldots, m_{n-1})).$$

Later, in 2003, the author of this paper improved [10] this result, and gave a partial answer for L^1 case. He discussed this partial sequence of the sequence of the Fejér means. Namely, if $f \in L^1(G_m)$, then he proved (see [10]) that $\sigma_{M_n} f \to f$ almost everywhere, where m is any sequence. In my opinion, it is highly likely that the methods of the papers [8, 10] can be applied and improved in order to prove the a.e. relation $\sigma_n f \to f$ for all $f \in L \log^+ L$ and m. Anyway, it is not an easy task...

What can be said in the case of two-dimensional functions? This is "another story". For double trigonometric Fourier series Marcinkiewicz and Zygmund [13] proved that $\sigma_{m,n} f \to f$ a.e. as $m, n \to \infty$ provided the integral lattice points (m, n) remain in some positive cone, that is provided $\beta^{-1} \leq m/n \leq \beta$ for some fixed parameter $\beta \geq 1$. It is known that the classical Fejér means are dominated by decreasing functions whose integrals are bounded but this fails to hold for the one-dimensional Walsh-Fejér kernels. This growth difference is exacerbated in higher dimensions so that the trigonometric techniques are not powerful enough for the Walsh case.

In 1992 Móricz, Schipp and Wade [14] proved that $\sigma_{2^{n_1},2^{n_2}} f \to f$ a.e. for each $f \in L^1([0,1)^2)$, when $n_1, n_2 \to \infty$, $|n_1 - n_2| \leq \alpha$ for some fixed α. Later, Gát and Weisz proved (independently, in the same year) this for the whole sequence, that is, the theorem of Marcinkiewicz and Zygmund with respect to the Walsh-Paley system (see [7] and [27]). In 2000 Blahota and the author of this paper generalized this theorem with respect to two-dimensional bounded Vilenkin systems [2].

If we do not provide a "cone restriction" for the indices in $\sigma_{n,k} f$ that is, we discuss the convergence of this two-dimensional Fejér means in the Pringsheim sense, then the situation changes. In 1992 Móricz, Schipp and Wade [14] proved with respect to the Walsh-Paley system that $\sigma_{n,k} f \to f$ a.e. for each $f \in L \log^+ L([0,1)^2)$, when $\min\{n, k\} \to \infty$. Later, in 2002 Weisz generalized [28] this with respect to two-dimensional bounded Vilenkin systems. In 2000 Gát proved [9] that the theorem of Móricz, Schipp and Wade above can not be improved. Namely, let $\delta : [0, +\infty) \to [0, +\infty)$ be a measurable function with property $\lim_{t\to\infty} \delta(t) = 0$. Gát proved the existence of a function $f \in L^1([0,1)^2)$ such that $f \in L \log^+ L\delta(L)$, and $\sigma_{n,k} f$ does not converge to f a.e. as $\min\{n, k\} \to \infty$.

What can be said in the two-dimensional case with respect to unbounded Vilenkin systems? In 1997 Wade proved [26] the following. Let

$$\beta_{k,j} := \max\{m_0, \ldots, m_{k-1}, \tilde{m}_0, \ldots, \tilde{m}_{j-1}\}.$$

The sequence m is called δ-quasi bounded, $0 \leq \delta < 1$, if the sums

$$\sum_{j=0}^{n-1} m_j/(m_{j+1}\dots m_n)^\delta$$

are (uniformly) bounded. Let the generating sequences $m, \tilde{m}$ be δ-quasi bounded. Then for all $f \in L^1(G_m \times G_{\tilde{m}})$ we have

$$\sigma_{M_n,\tilde{M}_k} f(x) - f(x) = o(\beta_{n,k}\beta^{2r}_{n+r,k+r}),$$

as $n, k \to \infty$, provided that $|n-k| < \alpha$, where $\alpha, r \in \mathbb{N}$ are some constants for almost every $x \in G_m \times G_{\tilde{m}}$.

On the other hand, there was nothing concerning the pointwise convergence before the following manuscript of the author. In [11] he proved the following theorem. Let $f \in (L\log^+ L)(G_m \times G_{\tilde{m}})$. Then we have $\sigma_{M_{n_1},\tilde{M}_{n_2}} f \to f$ almost everywhere, where $\min\{n_1, n_2\} \to \infty$ provided that the distance of the indices is bounded, that is, $|n_1 - n_2| < \alpha$ for some fixed constant $\alpha > 0$. Here it is necessary to emphasize that in this paper $m, \tilde{m}$ can be any sequences.

At last, we prove a (H, L) type inequality with respect to the one-dimensional Fejér means of integrable function on unbounded Vilenkin groups. Define the maximal operator $\sigma^\dagger f := \sup_{n\in\mathbb{N}} |\sigma_{M_n} f|$, where f is an integrable function.

Theorem 1. *Let $f \in H(G_m)$. Then we have*

$$\|\sigma^\dagger f\|_1 \leq C\|f\|_H.$$

In order to prove this theorem we need a modified Calderon-Zygmund decomposition lemma due to Simon: on unbounded Vilenkin groups (see [22]). For $z \in G_m, k \in \mathbb{N}, j \in \{0, \dots, m_k - 1\}$ we use the notation

$$I_k(z, j) = I_{k+1}(z_0, \dots, z_{k-1}, j).$$

Lemma 2. *Let $f \in L^1(G_m)$, and $\lambda > \|f\|_1 > 0$ arbitrary. Then the function f can be decomposed in the following form:*

$$f = f_0 + \sum_{j=1}^{\infty} f_j, \quad \|f_0\|_\infty \leq C\lambda, \quad \|f_0\|_1 \leq C\|f\|_1,$$

$$\operatorname{supp} f_j \subset \bigcup_{l=\alpha_j}^{\beta_j} I_{k_j}(z^j, l) = J_j, \quad \int_{G_m} f_j \, d\mu = 0 \quad (j \in \mathbb{P}),$$

and for

$$F = \bigcup_{j\in\mathbb{P}} J_j, \quad \mu(F) \leq C\frac{\|f\|_1}{\lambda}.$$

Moreover, the sets J_j are disjoint ($j \in \mathbb{P}$).

Proof of Theorem 1. Basically, the proof is a kind of application of the results in [10]. Namely, for an integrable function f we define the following operator:

$$H_1 f(y) := \sup_{A\in\mathbb{N}} \left| M_{A-1} \int_{\bigcup_{x_{A-1}\neq y_{A-1}} I_A(y_0,\dots,y_{A-2},x_{A-1})} f(x) \frac{1}{1 - r_{A-1}(y-x)} d\mu(x) \right|.$$

In [10] we proved that the operator H_1 is of type (L^2, L^2). Let $f \in L^1(G_m)$ such that

$$\int_{G_m} f d\mu = 0, \quad \operatorname{supp} f \subset \bigcup_{j=\alpha}^{\beta} I_k(z,j) =: I,$$

where $I_k(z,j) = I_{k+1}(z_0,\dots,z_{k-1},j)$, $z \in G_m$, and

$$j \in \{\alpha, \alpha+1, \dots, \beta\} \subset \{0,1,\dots,m_k-1\}.$$

Let $\gamma := \lfloor(\alpha+\beta)/2\rfloor$. Define the distance of $j,k \in \{0,1,\dots,m_k-1\} = Z_{m_k}$ as

$$\rho(j,k) := \begin{cases} |j-k|, & \text{if } |j-k| \le \frac{m_k}{2}, \\ m_k - |j-k|, & \text{if } |j-k| > \frac{m_k}{2}. \end{cases}$$

In other words, Z_{m_k} is considered as a circle. Define the set $6I$ in the following way: If $\beta - \alpha + 1 \ge m_k/6$, then $6[\alpha,\beta] := \{0,\dots,m_k-1\}$,

$$6I := \bigcup_{j \in 6[\alpha,\beta]} I_k(z,j) = I_k(z).$$

On the other hand, if $\beta - \alpha + 1 < m_k/6$, then

$$\begin{aligned} 6[\alpha,\beta] &:= \{j \in Z_{m_k} : \rho(j,\gamma) \le 3(\beta-\alpha+1)\}, \\ 6I &:= \bigcup_{j \in 6[\alpha,\beta]} I_k(z,j). \end{aligned}$$

It is obvious that $\mu(I) \le \mu(6I) \le 6\mu(I)$. In [10, Lemma 2.4] Gát proved:

$$\int_{G_m \setminus 6I} |H_1 f(y)|\, d\mu(y) \le C\|f\|_1.$$

More or less, H_1 seems like a quasi-local operator (for the exact definition of quasi-local operators see e.g. [21]). Then, by standard argument, with application of the theorem of Cauchy and Buniakovskii, we have for an atom a,

$$\operatorname{supp} a \subset \bigcup_{j=\alpha}^{\beta} I_k(z,j) =: I,$$

that

$$\begin{aligned} \|H_1 a\|_1 &= \int_{G_m \setminus 6I} |H_1 a y)|\, d\mu(y) + \int_{6I} |H_1 a(y)|\, d\mu(y) \\ &\le C\|a\|_1 + \|1_{6I}\|_2 \|H_1 a\|_2 \le C\|a\|_1 + C\sqrt{\mu(6I)}\|a\|_2 \le C. \end{aligned}$$

This immediately gives for all $f \in H$ that

$$\|H_1 f\|_1 \le C\|f\|_H. \tag{1}$$

For any $1 \le j \in \mathbb{N}$ define the operator H_j in the following way

$$\begin{aligned} H_j f(y) := \sup_{j \le A \in \mathbb{N}} \Bigg| M_{A-j} \int_{\bigcup_{x_{A-j} \ne y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},\dots,y_{A-1})} & f(x) \\ \times \frac{1}{1 - r_{A-j}(y-x)}\, d\mu(x) \Bigg|, \end{aligned}$$

where $y \in G_m$. Hereinafter, we prove for the operator H_j:

$$\|H_j f\|_1 \leq C \frac{j}{2^j} \|f\|_H \tag{2}$$

for all $f \in H$. The proof applies inequality (1) for a modified Vilenkin group. We apply a finite permutation for the coordinate groups of the Vilenkin group G_m such that for all $A \geq j, A \in \mathbb{N}$ the $A-j$th coordinate group and the $A-1$st coordinate group will be adjacent. Then we use inequality (1) for the modified group. Introduce the operators $H_{j,k}$ and $H_{j,k}^N$ as follows:

$$H_j f(y) \leq \sum_{k=0}^{j-1} \sup_{\substack{j \leq A \in \mathbb{N} \\ A \equiv k \mod j}} \left| M_{A-j} \int_{\bigcup_{x_{A-j} \neq y_{A-j}} I_A(y_0, \ldots, y_{A-j-1}, x_{A-j}, \ldots, y_{A-1})} f(x) \right.$$
$$\left. \times \frac{1}{1 - r_{A-j}(y-x)} \, dx \right| =: \sum_{k=0}^{j-1} H_{j,k} f(y),$$

and

$$H_{j,k}^N f(y) := \sup\{ |M_{A-j} \int_{\bigcup_{x_{A-j} \neq y_{A-j}} I_A(y_0, \ldots, y_{A-j-1}, x_{A-j}, \ldots, y_{A-1})} f(x)$$
$$\times \frac{1}{1 - r_{A-j}(y-x)} \, dx| : j \leq A \leq Nj + k, A \equiv k \mod j \}.$$

Since $H_{j,k}^N f$ is monotone increasing as N gets larger, then by the Beppo-Levi theorem we get that if we prove that the operators $2^j H_{j,k}^N$ satisfy (1), uniformly in N (it means that the constant C does not depend on N, j, k), that is, $2^j \|H_{j,k}^N f\|_1 \leq C\|f\|_H$, then $2^j H_{j,k}$ is also of this type. This would imply

$$\|H_j f\|_1 \leq \sum_{k=0}^{j-1} \|H_{j,k} f\|_1 \leq \sum_{k=0}^{j-1} \frac{C}{2^j} \|f\|_H \leq \frac{Cj}{2^j} \|f\|_H.$$

That is, the proof of inequality (2) would be complete.

Recall that the Vilenkin group G_m is the complete direct product of its coordinate groups Z_{m_l}, that is, $G_m = \mathop{\times}\limits_{l=0}^{\infty} Z_{m_l}$. We define another Vilenkin group. Its coordinate groups will be the same, but with certain rearrangement. Let the function $\alpha : \mathbb{N} \to \mathbb{N}$ be defined in the following way. If $n \geq k + Nj$, or $n \not\equiv k, k-1 \mod j$, then

$$\alpha(n) := n,$$

and

$$\alpha(k + lj) := k + (l+1)j - 1, \quad \alpha(k + (l+1)j - 1) := k + lj,$$

for all $l < N, l \in \mathbb{N}$. Then define the Vilenkin group $G_m^{j,k}$ as:

$$G_m^{j,k} = \mathop{\times}\limits_{l=0}^{\infty} Z_{m_{\alpha(l)}}.$$

We give a measure preserving bijection between the two Vilenkin groups. We denote it by β, or more precisely (if it is needed) by $\beta_{j,k}$. It will not cause any confusion. That is,

$$\beta = \beta_{j,k} : G_m \to G_m^{j,k},$$

and let the n^{th} coordinate of the sequence $\beta_{j,k}(x) \in G_m^{j,k}$ be $x_{\alpha(n)}$. That is,

$$(\beta(x))_n = x_{\alpha(n)} \quad (n \in \mathbb{N}).$$

Consequently, we have a finite permutation of the coordinates. This is very important for us, since when we discuss the operator H_1 on the Vilenkin group $G_m^{j,k}$, then we can apply the result given (H_1 is of type (1)) for the operator $H_{j,k}^N$ on the Vilenkin group G_m.

Denote by $\tilde{m}$ the sequence for which $\tilde{m}_l = m_{\alpha(l)}$. Introduce the notations $\tilde{x} := \beta(x)\,(x \in G_m)$, $\tilde{r}_l := r_{\alpha(l)}\,(l \in \mathbb{N})$. Recall that $A \equiv k \mod j$. Then we have

$$\begin{aligned}
1 - r_{A-j}(y-x) &= 1 - \exp\left(2\pi\imath \frac{y_{A-j} - x_{A-j}}{m_{A-j}}\right) \\
&= 1 - \exp\left(2\pi\imath \frac{\tilde{y}_{A-1} - \tilde{x}_{A-1}}{m_{A-j}}\right) \\
&= 1 - \exp\left(2\pi\imath \frac{\tilde{y}_{A-1} - \tilde{x}_{A-1}}{\tilde{m}_{A-1}}\right) \\
&= 1 - \tilde{r}_{A-1}(\tilde{y} - \tilde{x}).
\end{aligned}$$

Moreover, denote by $\tilde{M}$ the sequence of the generalized powers with respect to the sequence $\tilde{m}$. This gives

$$\begin{aligned}
\tilde{M}_{A-1} &= \tilde{m}_0 \dots \tilde{m}_{A-2} \\
&= m_0 m_1 \dots m_{A-j-1} m_{A-j+1} \dots m_{A-1} \\
&= \frac{m_0 \dots m_{A-1}}{m_{A-j}} \\
&= M_{A-j} \frac{m_{A-j} m_{A-j+1} \dots m_{A-1}}{m_{A-j}} \\
&= M_{A-j} m_{A-j+1} \dots m_{A-1}.
\end{aligned}$$

This gives $M_{A-j} \le \tilde{M}_{A-1}/2^{j-1}$. By the above written we get

$$\begin{aligned}
&\left| M_{A-j} \int_{\bigcup_{x_{A-j} \neq y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},\dots,y_{A-1})} f(x) \frac{1}{1 - r_{A-j}(y-x)}\, dx \right| \\
&= \left| M_{A-j} \int_{\bigcup_{\tilde{x}_{A-1} \neq \tilde{y}_{A-1}} I_A(\tilde{y}_0,\dots,\tilde{y}_{A-2},\tilde{x}_{A-1})} \tilde{f}(\tilde{x}) \frac{1}{1 - \tilde{r}_{A-1}(\tilde{y}-\tilde{x})}\, d\tilde{x} \right| \\
&\le \frac{1}{2^{j-1}} H_1 \tilde{f}(\tilde{y}),
\end{aligned}$$

where the function $\tilde{f}$ is defined on $G_m^{j,k}$ by $f(x) = \tilde{f}(\tilde{x})$ for all $x \in G_m$. The definition of $H_{j,k}^N$ gives

$$H_{j,k}^N f(y) \le \frac{1}{2^{j-1}} H_1 \tilde{f}(\tilde{y}).$$

So, let the function a be an atom on G_m, the interval corresponding the support of this atom is denoted by I. We give an upper bound for the Hardy norm of $\tilde{a}$. Since the supremum norms of a and $\tilde{a}$ equals, that is, finite, then $\tilde{a}$ is in the Hardy space $H(G_m^{j,k})$, and consequently, belongs to the Lebesgue space L^p for all $p > 1$. The L^p norms of the function a and $\tilde{a}$ also equals, so:

$$\|\tilde{a}\|_{H(G_m^{j,k})} \le \|\tilde{a}\|_{L^p(G_m^{j,k})} = \|a\|_{L^p(G_m)} \le \mu^{1/p-1}(I)$$

for all $1 < p < \infty$. This gives $\|\tilde{a}\|_{H(G_m^{j,k})} \leq 1$. By the above we have

$$\|H_{j,k}^N a\|_1 \leq \frac{C}{2^j} \int_{G_m^{j,k}} H_1 \tilde{a}(\tilde{y}) d\mu(\tilde{y}) \leq \frac{C}{2^j} \|\tilde{a}\|_{H(G_m^{j,k})} \leq \frac{C}{2^j}.$$

This implies that (2) for any $f \in H(G_m)$.

In [10, Proof of Theorem 2.1] one can read

$$|\sigma_{M_A} f(y)| \leq |f|^*(y) + \sum_{j=1}^{\infty} H_j f(y),$$

where f is any integrable function, and $f^* := \sup_n |E_n f|$. That is, for the maximal operator $\sigma^\dagger := \sup_A |\sigma_{M_A}|$ we have

$$\sigma^\dagger f \leq |f|^* + \sum_{j=1}^{\infty} H_j f.$$

So, by (2) we have that the proof of the Theorem 1 is complete. □

References

[1] G. Agaev, N. Vilenkin, G. Dzhafarli, and A. Rubinstein. *Multiplicative systems of functions and harmonic analysis on 0-dimensional groups.* Izd.("ELM"), Baku, 1981.

[2] I. Blahota and G. Gát. Pointwise convergence of double Vilenkin-Fejér means. *Stud. Sci. Math. Hungar.*, 36:49–63, 2000.

[3] D. Burkholder. Distribution function inequalities for martingales. *Ann. Probability*, 1:19–42, 1973.

[4] N. Fine. Cesàro summability of Walsh-Fourier series. *Proc. Nat. Acad. Sci. U.S.A.*, 41:558–591, 1955.

[5] S. Fridli. On the modulus of continuity with respect to functions defined on Vilenkin groups. *Acta Mat. Hungar.*, 45(3-4):393–396, 1985.

[6] N. Fujii. A maximal inequality for H^1 functions on the generalized Walsh-Paley group. *Proc. Amer. Math. Soc.*, 77:111–116, 1979.

[7] G. Gát. Pointwise convergence of the Cesàro means of double Walsh series. *Ann. Univ. Sci. Budap. Rolando Eoetvoes, Sect. Comput.*, 16:173–184, 1996.

[8] G. Gát. Pointwise convergence of the Fejér means of functions on unbounded Vilenkin groups. *Journal of Approx. Theory*, 101(1):1–36, 1999.

[9] G. Gát. On the divergence of the $(C, 1)$ means of double Walsh-Fourier series. *Proc. Am. Math. Soc.*, 128(6):1711–1720, 2000.

[10] G. Gát. Cesàro means of integrable functions with respect to unbounded Vilenkin systems. *Journal of Approximation Theory*, 124(1):25–43, 2003.

[11] G. Gát. On the pointwise convergence of Cesàro means of two-variable functions with respect to unbounded Vilenkin systems. *Journal of Approx. Theory*, 128(1):69–99, 2004.

[12] J. Marcinkiewicz. Quelques théorèmes sur les séries orthogonales. *Ann Soc. Polon. Math.*, 16:85–96, 1937.

[13] J. Marcinkiewicz and J. A. Zygmund. On the summability of double Fourier series. *Fund. Math.*, 32:122–132, 1939.

[14] F. Móricz, F. Schipp, and W. Wade. Cesàro summability of double Walsh-Fourier series. *Trans Amer. Math. Soc.*, 329:131–140, 1992.

[15] R. Nurpeisov. On the uniform $(C, 1)$ summability of Fourier series and integrability of series with respect to multiplicative orthonormal systems. *Analysis Math.*, 15:127–143, 1989.

[16] J. Pál and P. Simon. On a generalization of the concept of derivative. *Acta Math. Acad. Sci. Hungar.*, 29:155–164, 1977.

[17] J. Price. Certain groups of orthonormal step functions. *Canadian J. Math.*, 9:413–425, 1957.

[18] A. Rubinstein. On the modulus of continuity of functions defined on 0-dimensional groups. *Mat. Zametki*, 23:379–388, 1978.

[19] F. Schipp. Über gewiessen Maximaloperatoren. *Annales Univ. Sci. Budapestiensis , Sectio Math.*, 18:189–195, 1975.

[20] F. Schipp. On L^p-norm convergence of series with respect to product systems. *Analysis Math.*, 2:49–63, 1976.

[21] F. Schipp, W. Wade, P. Simon, and J. Pál. *Walsh series: an introduction to dyadic harmonic analysis*. Adam Hilger, Bristol and New York, 1990.

[22] P. Simon. Investigations with respect to the Vilenkin system. *Annales Univ. Sci. Budapestiensis , Sectio Math.*, 27:87–101, 1985.

[23] P. Simon. Cesàro means of Vilenkin-Fourier series. *Publ. Math.*, 59(1–2):203–219, 2001.

[24] M. Taibleson. Fourier Series on the Ring of Integers in a p-series Field. *Bull. Amer. Math. Soc.*, 73:623–629, 1967.

[25] N. Vilenkin. On a class of complete orthonormal systems. *Izv. Akad. Nauk. SSSR, Ser. Math.*, 11:363–400, 1947.

[26] W. Wade. Growth of Cesàro means of double Vilenkin-Fourier series of unbounded type. *Analysis of divergence. Control and management of divergent processes. Proceedings of the 7th international workshop in analysis and its applications, IWAA, Orono, ME, USA June 1-6, 1997. Boston, MA: Birkhauser. Applied and Numerical Harmonic Analysis*, pages 41–50, 1999.

[27] F. Weisz. Cesàro summability of two-dimensional Walsh-Fourier series. *Trans. Amer. Math. Soc.*, 348:2169–2181, 1996.

[28] F. Weisz. Summability results of Walsh- and Vilenkin-Fourier series. *Functions, Series, Operators – Alexits Memorial Conference, Budapest (Hungary) 1999*, pages 443–464, 2002.

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