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ON THE POINTWISE ESTIMATION OF CESARO KERNEL OF NEGATIVE ORDER WITH RESPECT TO WALSH-PALEY SYSTEM

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Abstract. Some pointwise properties of (C, α) kernel $(-1 < \alpha < 0)$ with respect to the Walsh–Paley system are established.

1. Introduction

Let $r_0(x)$ be the function defined by

$$r_0(x) = \begin{cases} 1 & \text{if } x \in \left[0, \frac{1}{2}\right), \\ -1 & \text{if } x \in \left[\frac{1}{2}, 1\right), \end{cases} \qquad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \geq 1, \text{ and } x \in [0, 1).$$

Let $\psi_0(x), \psi_1(x), \psi_2(x), \ldots$ represent the Walsh functions, i.e. $\psi_0(x) = 1$, and if $k = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s$, then

$$\psi_k(x) = r_{n_1}(x) \cdot r_{n_2}(x) \cdots r_{n_s}(x).$$

The idea of using the products of the Rademacher functions is to define the Walsh system originated by Paley [4].

Denote by $K_n^\alpha(t)$ the kernel of the method (C, α) and call it the (C, α) kernel, or the Cesaro kernel:

$$K_n^\alpha(t) = \frac{1}{A_n^\alpha} \sum_{\nu=0}^{n} A_{n-\nu}^\alpha \psi_\nu(t),$$

$$A_k^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!} \quad (\alpha \neq -k).$$

It is well-known ([8], Ch. 3) that

(1) $\quad A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1};$

(2) $\quad A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1};$

(3) $\quad A_n^\alpha \sim n^\alpha.$

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Some properties of the (C,α) kernel $(\alpha > 0)$ have been established by Fine [1], Yano [7], Gát [2], [3] and [5]. Using these properties they studied the problems of pointwise and uniform (C,α) summability of Walsh–Fourier series.

In the present paper we study some pointwise properties of (C,α) kernel $(-1 < \alpha < 0)$ with respect to the Walsh–Paley system. The results of this paper have been published without proof in [6].

2. Main Results

The main results of the paper are presented in the form of the following propositions.

Theorem 1. *The estimation*

$$K_n^{-\alpha}(t) \le c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} \cdot \frac{1}{t^{1-\alpha}}, \quad t \in (0,1), \quad 0 < \alpha < 1,$$

holds.

Theorem 2. *For any $\alpha \in (0,1)$ and $p \ge 2^m$ the equality*

$$\operatorname{Sgn}\left(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\right) = \operatorname{Sgn}\psi_{2^m-1}(t), \quad t \in [0,1),$$

is valid.

3. Auxiliary Results

We shall need the following

Lemma 1. *For any $\alpha > 0$ and $p > 2^m - 1 + \alpha$ the sum*

$$\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)$$

is representable in the form

$$\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) = \sum_{\nu=0}^{2^m-1} \ell_\nu A_{p-q_\nu}^{-\alpha-i}, \tag{1}$$

where ℓ_ν, q_ν, i are nonnegative integers depending only on the point $t \in [0,1]$ and $m \in \mathbb{N}$ (and not depending on p and α); moreover, $i + q_\nu \le 2^m - 1$.

Proof. Using the method of mathematical induction, we can verify that the lemma is valid for $m = 1$. We have

$$\sum_{\nu=0}^{2^1-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) = A_p^{-\alpha}\psi_0(t) + A_{p-1}^{-\alpha}\psi_1(t).$$

Since on the interval $0 \le t < \frac{1}{2}$

$$A_p^{-\alpha}\psi_0(t) + A_{p-1}^{-\alpha}\psi_1(t) = A_p^{-\alpha} + A_{p-1}^{-\alpha},$$

and on the interval $\frac{1}{2} \le t < 1$

$$A_p^{-\alpha}\psi_0(t) + A_{p-1}^{-\alpha}\psi_1(t) = A_p^{-\alpha} - A_{p-1}^{-\alpha} = A_p^{-\alpha-1},$$

our lemma for $m = 1$ is valid. Let the lemma be valid for $m - 1 \in \mathbb{N}$, and we prove that the lemma is valid for $m \in \mathbb{N}$. Indeed, we have

$$\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) = \sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) + \psi_{2^{m-1}}(t)\sum_{\nu=0}^{2^{m-1}-1} A_{p-2^{m-1}-\nu}^{-\alpha}\psi_\nu(t)$$
$$(\alpha > 0, \quad p > 2^m - 1 + \alpha).$$

We consider two cases: (1) $\psi_{2^{m-1}}(t) = 1$; (2) $\psi_{2^{m-1}}(t) = -1$.

Let $\psi_{2^{m-1}}(t) = 1$. By the assumption

$$p - 2^{m-1} > 2^m - 1 + \alpha - 2^{m-1} > 2^{m-1} - 1 + \alpha,$$
$$\sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) = \sum_{\nu=0}^{2^{m-1}-1} c_\nu A_{p-m_\nu}^{-\alpha-i}$$

and

$$\sum_{\nu=0}^{2^{m-1}-1} A_{p-2^{m-1}-\nu}^{-\alpha}\psi_\nu(t) = \sum_{\nu=0}^{2^{m-1}-1} c_\nu A_{p-2^{m-1}-m_\nu}^{-\alpha-i}.$$

Hence we have

$$\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) = \sum_{\nu=0}^{2^{m-1}-1} c_\nu A_{p-m_\nu}^{-\alpha-i} + \sum_{\nu=0}^{2^{m-1}-1} c_\nu A_{p-2^{m-1}-m_\nu}^{-\alpha-i} = \sum_{\nu=0}^{2^m-1} \ell_\nu A_{p-q_\nu}^{-\alpha-i}.$$

Let now $\psi_{2^{m-1}} = -1$. Since $A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}$, we have

$$\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) = \sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) + \psi_{2^{m-1}}(t)\sum_{\nu=0}^{2^{m-1}-1} A_{p-2^{m-1}-\nu}^{-\alpha}\psi_\nu(t)$$

(2) $$= \sum_{\nu=0}^{2^{m-1}-1}\left(A_{p-\nu}^{-\alpha} - A_{p-2^{m-1}-\nu}^{-\alpha}\right)\psi_\nu(t)$$

$$= \sum_{\nu=0}^{2^{m-1}-1}\left(A_{p-\nu}^{-\alpha-1} + A_{p-\nu-1}^{-\alpha-1} + \cdots + A_{p-2^{m-1}-\nu-1}^{-\alpha-1}\right)\psi_\nu(t).$$

By the assumption,

$$\sum_{\nu=0}^{2^{m-1}-1} A_{p-j-\nu}^{-\alpha-1} = \sum_{\nu=0}^{2^{m-1}-1} c_\nu A_{p-j-m_\nu}^{-\alpha-1-i}, \quad j = 0, 1, 2, \dots, 2^{m-1} - 1,$$

hence from (2) we have

$$\sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha-1}\psi_\nu(t) = \sum_{j=0}^{2^{m-1}-1}\sum_{\nu=0}^{2^{m-1}-1} c_\nu A_{p-j-m_\nu}^{-\alpha-1-i} = \sum_{\nu=0}^{2^m-1} \ell_\nu A_{p-q_\nu}^{-\alpha-1-i},$$

i.e. in both cases equation (1) holds. It is evident that in these cases

$$i + q_\nu \le 2^m - 1.$$

Thus the lemma is proved. □

Lemma 2. *For any $\alpha > 0$ and $p > 2^m + \alpha$ the equality*

$$\operatorname{Sgn}\left(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\right) = -\operatorname{Sgn}\left(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha-1}\psi_\nu(t)\right), \quad t \in [0, 1),$$

is valid.

Lemma 2 follows directly from Lemma 1 if we take into account that in the conditions of Lemma 2 $\operatorname{Sgn} A_{p-\nu}^{-\alpha} = -\operatorname{Sgn} A_{p-\nu}^{-\alpha-1}$.

4. Proof of Main Results

Proof of Theorem 1. Let $t \in (0,1)$ and $m \in \mathbb{N}$ ($\mathbb{N}$ is a set of natural numbers), such that $2^{-m} \le t < 2^{-m+1}$. We write $n \ge 1$ in the form $n = p \cdot 2^m + q$, where $0 \le q < 2^m$. We have[1]

$$
\begin{aligned}
K_n^{-\alpha}(t) &= \frac{1}{A_n^{-\alpha}} \sum_{\nu=0}^{n} A_{n-\nu}^{-\alpha} \psi_\nu(t) = \frac{1}{A_n^{-\alpha}} \sum_{\nu=0}^{p\cdot 2^m-1} A_{n-\nu}^{-\alpha} \psi_\nu(t) \\
&\quad + \frac{1}{A_n^{-\alpha}} \sum_{\nu=p\cdot 2^m}^{n} A_{n-\nu}^{-\alpha} \psi_\nu(t) \\
&= \frac{1}{A_n^{-\alpha}} \sum_{r=0}^{p-1} \sum_{\nu=0}^{2^m-1} A_{n-r\cdot 2^m-\nu}^{-\alpha} \psi_{r\cdot 2^m+\nu}(t) \\
&\quad + \frac{1}{A_n^{-\alpha}} \sum_{\nu=0}^{q} A_{n-p\cdot 2^m-\nu}^{-\alpha} \psi_{p\cdot 2^m+\nu}(t) \\
&= \frac{1}{A_n^{-\alpha}} \sum_{r=0}^{p-2} \psi_{r\cdot 2^m} \sum_{\nu=0}^{2^m-1} A_{n-r\cdot 2^m-\nu}^{-\alpha} \psi_\nu(t) \\
&\quad + \frac{1}{A_n^{-\alpha}} \psi_{(p-1)\cdot 2^m}(t) \sum_{\nu=0}^{2^m-1} A_{n-r\cdot 2^m-\nu}^{-\alpha} \psi_\nu(t) \\
&\quad + \frac{1}{A_n^{-\alpha}} \psi_{p\cdot 2^m}(t) \sum_{\nu=0}^{q} A_{q-\nu}^{-\alpha} \psi_\nu(t) = A_1 + A_2 + A_3.
\end{aligned} \tag{3}
$$

Estimate A_1. Using Abelian transformation, we have

$$
\begin{aligned}
A_1 &= \frac{1}{A_n^{-\alpha}} \left| \sum_{r=0}^{p-2} \psi_{r\cdot 2^m}(t) \sum_{\nu=0}^{2^m-1} A_{n-r2^m-\nu}^{-\alpha} \psi_\nu(t) \right| \\
&= \frac{1}{A_n^{-\alpha}} \left| \sum_{r=0}^{p-2} \psi_{r\cdot 2^m}(t) \sum_{\nu=0}^{2^m-2} A_{n-r\cdot 2^m-\nu}^{-\alpha-1} D_\nu(t) \right. \\
&\quad \left. + \sum_{r=0}^{p-2} \psi_{r\cdot 2^m} A_{n-(r+1)2^m+1}^{-\alpha} D_{2^m}(t) \right|,
\end{aligned}
$$

where

$$
D_k(t) = \sum_{i=0}^{k-1} \psi_k(t).
$$

Since (see [8])

$$
c_1(\alpha) n^\alpha \le A_n^\alpha \le c_2(\alpha) n^\alpha, \quad \alpha > -2, \tag{4}
$$

[1] Here the use is made of the equality $\psi_{r+s}(t) = \psi_r(t)\psi_s(t)$, if in the binary expansion $r, s \in \mathbb{N}$ the same terms are omitted.

and $|D_k(t) \leq k$, $t \in [0,1]$, we obtain

$$
\begin{aligned}
|A_1| &\leq \frac{1}{A_n^{-\alpha}} \cdot c(\alpha) \cdot 2^m \sum_{r=0}^{p-2} \sum_{\nu=0}^{2^m-1} (n - r \cdot 2^m - \nu)^{-\alpha-1} \\
&\leq \frac{1}{A_n^{-\alpha}} \cdot c(\alpha) \cdot 2^m (n - (p-1) \cdot 2^m)^{-\alpha} \leq \frac{1}{A_n^{-\alpha}} \cdot c(\alpha) 2^{m(1-\alpha)} \\
&\leq c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} \cdot \frac{1}{t^{1-\alpha}}.
\end{aligned} \tag{5}
$$

For A_2 we have

$$
\begin{aligned}
|A_2| &= \frac{1}{A_n^{-\alpha}} \left| \psi_{(p-1)2^m}(t) \sum_{\nu=0}^{2^m-1} A^{-\alpha}_{n-(p-1)2^m-\nu} \psi_\nu(t) \right| \\
&\leq c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} \sum_{\nu=0}^{2^m-1} (n - (p-1)2^m - \nu)^{-\alpha} \\
&\leq c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} \sum_{\nu=0}^{2^m-1} (2^m + q - \nu)^{-\alpha} \\
&\leq c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} \sum_{\nu=0}^{2^m-1} (2^m - \nu)^{-\alpha} \leq c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} 2^{m(1-\alpha)} \\
&\leq c(\alpha) \cdot \frac{1}{A_n^{-\alpha}} \cdot \frac{1}{t^{1-\alpha}}.
\end{aligned} \tag{6}
$$

Estimate now A_3,

$$
\begin{aligned}
|A_3| &= \frac{1}{A_n^{-\alpha}} \left| \sum_{\nu=0}^{q} A^{-\alpha}_{q-\nu} \psi_\nu(t) \right| \leq c(\alpha) \, \frac{1}{A_n^{-\alpha}} \left(1 + \sum_{\nu=0}^{q} (q-\nu)^{-al} \right) \\
&\leq c(\alpha) \, \frac{1}{A_n^{-\alpha}} \, q^{1-\alpha} \leq c(\alpha) \, \frac{1}{A_n^{-\alpha}} \, 2^{m(1-\alpha)} \leq c(\alpha) \, \frac{1}{A_n^{-\alpha}} \, \frac{1}{t^{1-\alpha}}.
\end{aligned} \tag{7}
$$

Taking into account (5), (6) and (7), from (3) we get

$$
|K_n^{-\alpha}| \leq c(\alpha) \, \frac{1}{A_n^{-\alpha}} \, \frac{1}{t^{1-\alpha}},
$$

which was to be proved. □

Proof of Theorem 2. We use the method of mathematical induction. For $m = 1$ the lemma is valid. Indeed,

$$
\sum_{\nu=0}^{2^1-1} A^{-\alpha}_{p-\nu} \psi_\nu(t) = A_p^{-\alpha} \psi_0(t) + A^{-\alpha}_{p-1} \psi_1(t);
$$

on the interval $0 \leq t < \frac{1}{2}$

$$
A_p^{-\alpha} \psi_0(t) + A^{-\alpha}_{p-1} \psi_1(t) = A_p^{-\alpha} + A^{-\alpha}_{p-1} > 0,
$$

and on the interval $\frac{1}{2} \leq t < 1$

$$
A_p^{-\alpha} \psi_0(t) + A^{-\alpha}_{p-1} \psi_1(t) = A_p^{-\alpha} - A^{-\alpha}_{p-1} = A_p^{-\alpha-1} < 0.
$$

Since $\psi_1(t) = 1$ if $0 \le t < \frac{1}{2}$, and $\psi_1(t) = -1$ if $\frac{1}{2} \le t < 1$, therefore

$$\operatorname{Sgn}\left(\sum_{\nu=0}^{2^1-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\right) = \operatorname{Sgn}\psi_1(t).$$

Let the lemma be valid for $m-1 \in \mathbb{N}$ and let us prove that the lemma is valid for $m \in \mathbb{N}$. We have

$$\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) = \sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) + \psi_{2^{m-1}}(t)\sum_{\nu=0}^{2^{m-1}-1} A_{p-2^{m-1}-\nu}^{-\alpha}\psi_\nu(t). \tag{8}$$

Let $\psi_{2^{m-1}} = 1$. Since $p - 2^{m-1} \ge 2^{m-1}$, by virtue of the assumption,

$$\operatorname{Sgn}\left(\sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\right) = \operatorname{Sgn}\psi_{2^{m-1}-1}(t),$$

$$\operatorname{Sgn}\left(\sum_{\nu=0}^{2^{m-1}-1} A_{p-2^{m-1}-\nu}^{-\alpha}\psi_\nu(t)\right) = \operatorname{Sgn}\psi_{2^{m-1}-1}(t).$$

Hence from (8) it follows that

$$\begin{aligned}\operatorname{Sgn}\left(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\right) &= \operatorname{Sgn}\psi_{2^{m-1}-1}(t)\\ &= \operatorname{Sgn}\psi_{2^{m-1}-1}(t)\cdot\operatorname{Sgn}\psi_{2^{m-1}}(t)\\ &= \operatorname{Sgn}\left(\psi_{2^{m-1}-1}(t)\psi_{2^{m-1}}(t)\right) = \operatorname{Sgn}\psi_{2^m-1}(t).\end{aligned}$$

If, however, $\psi_{2^{m-1}}(t) = -1$, equality (8) yields

$$\begin{aligned}\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t) &= \sum_{\nu=0}^{2^{m-1}-1}\left(A_{p-\nu}^{-\alpha} - A_{p-2^{m-1}-\nu}^{-\alpha}\right)\psi_\nu(t)\\ &= \sum_{\nu=0}^{2^{m-1}-1}\left(A_{p-\nu}^{-\alpha-1} + A_{p-\nu-1}^{-\alpha-1} + \cdots + A_{p-2^{m-1}-\nu+1}^{-\alpha-1}\right)\psi_\nu(t).\end{aligned} \tag{9}$$

Taking into account Lemma 1, for all $j = 1, 2, \dots, 2^{m-1}-1$ we obtain

$$\operatorname{Sgn}\left(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\right) = \operatorname{Sgn}\left(\sum_{\nu=0}^{2^{m-1}-1} A_{p-j-\nu}^{-\alpha-1}\psi_\nu(t)\right),$$

and consequently, using Lemma 2, from (9) it follows that

$$\begin{aligned}\operatorname{Sgn}\left(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\right) &= \operatorname{Sgn}\left(\sum_{\nu=0}^{2^{m-1}-1} A_{p-\nu}^{-\alpha-1}\psi_\nu(t)\right)\\ = -\operatorname{Sgn}\left(\sum_{\nu=0}^{2^m-1} A_{p-\nu}^{-\alpha}\psi_\nu(t)\right) &= -\operatorname{Sgn}\psi_{2^{m-1}-1}(t)\\ = \operatorname{Sgn}\psi_{2^{m-1}}(t)\operatorname{Sgn}\psi_{2^{m-1}-1}(t) &= \operatorname{Sgn}\psi_{2^m-1}(t).\end{aligned}$$

Thus the theorem is proved. □

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