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A STUDY ON THE GENERALIZATION OF JANOWSKI FUNCTIONS IN THE UNIT DISC

Y. POLATOĞLU, M. BOLCAL, A. ŞEN AND E. YAVUZ

Abstract. Let Ω be the class of functions $w(z)$, $w(0) = 0$, $|w(z)| < 1$ regular in the unit disc $D = \{z : |z| < 1\}$. For arbitrarily fixed numbers $A \in (-1, 1]$, $B \in [-1, A)$, $0 \leq \alpha < 1$ let $P(A, B, \alpha)$ be the class of regular functions $p(z)$ in D such that $p(0) = 1$, and which is $p(z) \in P(A, B, \alpha)$ if and only if $p(z) = \frac{1+[(1-\alpha)A+\alpha B]w(z)}{1+Bw(z)}$ for some function $w(z) \in \Omega$ and every $z \in D$.

In the present paper we apply the principle of subordination ([1], [3], [4], [5]) to give new proofs for some classical results concerning the class $S^*(A, B, \alpha)$ of functions $f(z)$ with $f(0) = 0$, $f'(0) = 1$, which are regular in D satisfying the condition: $f(z) \in S^*(A, B, \alpha)$ if and only if $z\frac{f'(z)}{f(z)} = p(z)$ for some $p(z) \in P(A, B, \alpha)$ and for all z in D.

1. Introduction

Let Ω be the family of functions $w(z)$ regular in the unit disc D and satisfying the conditions $w(0) = 0, |w(z)| < 1$, for $z \in D$.

For arbitrary fixed numbers $A, B, \alpha, -1 \leq B < A \leq 1, 0 \leq \alpha < 1$, let $P(A, B, \alpha)$ denote the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots + p_n z^n + \cdots \tag{1}$$

regular in D and such that $p(z)$ is in $P(A, B, \alpha)$ if and only if

$$p(z) \prec \frac{1 + [(1-\alpha)A + \alpha B]\, z}{1 + Bz} \Leftrightarrow p(z) = \frac{1 + [(1-\alpha)A + \alpha B]\, w(z)}{1 + Bw(z)} \tag{2}$$

for some function $w(z) \in \Omega$ and every $z \in D$.

Furthermore, let $S^*(A, B, \alpha)$ denote the family of functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{3}$$

regular in D and such that $f(z)$ is in $S^*(A, B, \alpha)$ if and only if

$$z\frac{f'(z)}{f(z)} = p(z) \tag{4}$$

for some $p(z)$ in $P(A, B, \alpha)$ and for all z in D.

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2. New Results On The Class $S^*(A,B,\alpha)$

In this section we shall give representation theorems, distortion theorems and establish the radius of starlikeness for the class $S^*(A,B,\alpha)$. Our proofs are based on I.S. Jack's Lemma [2].

Lemma 1. *Let $w(z)$ be a non-constant and analytic function in the unit disc D with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r$ at the point z_1, then $z_1 w'(z_1) = kw(z_1)$ and $k \geq 1$.*

From the definition of the class $P(1,-1,0)$ called the Caratheodory class and $P(A,B,\alpha)$ we easily obtain the following lemma.

Lemma 2. *If $p(z) \in P(A,B,\alpha)$ if and only if*

$$p(z) = \frac{[1+(1-\alpha)A+\alpha B]q(z) + [(1-\alpha)(A+B)]}{[1+B]q(z)+[1-B]} \tag{5}$$

for some $q(z) \in P(1,-1,0)$.

Let ζ be an arbitrary fixed point of D. We consider the functional

$$F(p) = p(\zeta), p(z) \in P(A,B,\alpha). \tag{6}$$

Lemma 3. *The set of the values of the functional (6) is the closed disc with centered at $C(r)$ and having the radius $\rho(r)$, where*

$$\begin{cases} C(r) = \left(\frac{1-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}, 0\right), & \rho(r) = \frac{(1-\alpha)(A-B)r}{1-B^2r^2}, \ B \neq 0, \\ C(r) = (1,0), & \rho(r) = (1-\alpha)\,|A|\,r, \ B = 0. \end{cases}$$

Proof. Every boundary function $p_0(z)$ of $P(A,B,\alpha)$ with respect to the functional (6) can be written in the form (5), where

$$q(z) = \frac{1+\varepsilon z}{1-\varepsilon z}, |\varepsilon| = 1.$$

Hence

$$p_0(z) = \frac{1+[(1-\alpha)A+\alpha B]z}{1+Bz}. \tag{7}$$

Since $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$,

$$\begin{gathered} p_0(z) = C(r) + \rho\eta, \\ \eta = \varepsilon e^{i\theta}\frac{1+Br\bar{\varepsilon}e^{-i\theta}}{1+Br\varepsilon e^{i\theta}}, \end{gathered} \tag{8}$$

which completes the proof. $\square$

Lemma 4. *The function*

$$w = w(z) = \begin{cases} \frac{(1-\alpha)(A-B)z}{1+Bz}, & B \neq 0, \\ (1-\alpha)Az, & B = 0, \end{cases}$$

maps $|z|=r$ onto the disc centered at $C(r)$, and having the radius $\rho(r)$

$$\begin{cases} C(r) = \left(-\frac{B(1-\alpha)(A-B)r^2}{1-B^2r^2}, 0, \right), & \rho(r) = \frac{(1-\alpha)(A-B)r}{1-B^2r^2}, B \neq 0, \\ C(r) = (0,0), & \rho(r) = (1-\alpha)\,|A|\,r, B = 0. \end{cases}$$

Proof. This is immediate from

$$
\begin{aligned}
&w = \frac{(1-\alpha)(A-B)z}{1+Bz} \Rightarrow \\
&u^2+v^2+\frac{2B(1-\alpha)(A-B)r^2}{1-B^2r^2}u-\frac{(1-\alpha)^2(A-B)^2r^2}{1-B^2r^2}=0, B\neq 0, \\
&w=(1-\alpha)Az \Rightarrow u^2+v^2-(1-\alpha)^2A^2r^2=0, B=0.
\end{aligned} \tag{9}
$$

□

Theorem 1. *Let $f(z) = z + a_2z^2 + \cdots$ be an analytic function in the unit disc D. If $f(z)$ satisfying*

$$
\left(z\frac{f'(z)}{f(z)}-1\right) \prec \begin{cases} \frac{(1-\alpha)(A-B)z}{1+Bz} = F_1(z), & B\neq 0, \\ (1-\alpha)Az = F_2(z), & B=0, \end{cases} \tag{10}
$$

then $f(z) \in S^*(A,B,\alpha)$ and this result is as sharp as the function

$$
\left(\frac{1+[(1-\alpha)A+B\alpha]z}{1+Bz}\right).
$$

Proof. We define the function $w(z)$ by

$$
\frac{f(z)}{z} = \begin{cases} (1+Bw(z))^{\frac{(1-\alpha)(A-B)}{B}}, & B\neq 0, \\ e^{(1-\alpha)Aw(z)}, & B=0, \end{cases} \tag{11}
$$

where $(1+Bw(z))^{\frac{(1-\alpha)(A-B)}{B}}$ and $e^{(1-\alpha)Aw(z)}$ have the value 1 at the origin. Then $w(z)$ is analytic in D and $w(0)=0$. If we take the logarithmic derivate of equality (11), simple calculations yield

$$
\left(z\frac{f'(z)}{f(z)}-1\right) = \begin{cases} \frac{(1-\alpha)(A-B)zw'(z)}{1+Bw(z)}, & B\neq 0, \\ (1-\alpha)Azw'(z), & B=0. \end{cases} \tag{12}
$$

Now it is easy to realize that the subordination (10) is equivalent to $|w(z)|<1$ for all $z\in D$ indeed assume the contrary. There exist $z_1 \in D$ such that $|w(z_1)|=1$. Then by I.S. Jack's Lemma $z_1w'(z_1) = kw(z_1)$ and $k\geq 1$, for such $z_1\in D$ and using Lemma 4 we have

$$
\left(z_1\frac{f'(z_1)}{f(z_1)}-1\right) = \begin{cases} \frac{(1-\alpha)(A-B)kw(z_1)}{1+Bw(z_1)} = F_1(w(z_1)) \notin F_1(D), & B\neq 0, \\ (1-\alpha)Akw(z_1) = F_2(w(z_1)) \notin F_2(D), & B=0, \end{cases} \tag{13}
$$

because $|w(z_1)|=1$ and $k\geq 1$. But this contradicts condition (10) of this theorem and so $|w(z)|<1$ for all $z\in D$. By using condition (10) we get

$$
z\frac{f'(z)}{f(z)} = \begin{cases} \frac{1+[(1-\alpha)A+\alpha B]w(z)}{1+Bw(z)}, & B\neq 0, \\ 1+(1-\alpha)Aw(z), & B=0, \end{cases}
$$

which ends the proof. □

Corollary 1. *Let $f(z)\in S^*(A,B,\alpha)$. Then $f(z)$ can be written in the form*

$$
f(z) = \begin{cases} z(1+Bw(z))^{\frac{(1-\alpha)(A-B)}{B}}, & B\neq 0, \\ ze^{(1-\alpha)Aw(z)}, & B=0. \end{cases}
$$

Theorem 2. *If $f(z)\in S^*(A,B,\alpha)$, then*

$$
\begin{cases} r(1-Br)^{\frac{(1-\alpha)(A-B)}{B}} \leq |f(z)| \leq r(1+Br)^{\frac{(1-\alpha)(A-B)}{B}}, & B\neq 0, \\ re^{-(1-\alpha)|A|r} \leq |f(z)| \leq re^{(1-\alpha)|A|r}, & B=0. \end{cases} \tag{14}
$$

These bounds are sharp with the extremal function

$$f_*(z) = \begin{cases} z(1+Bz)^{\frac{(1-\alpha)(A-B)}{B}}, & B \neq 0, \\ ze^{(1-\alpha)Az}, & B = 0. \end{cases} \tag{15}$$

Proof. The set of the values of $\left(z\frac{f'(z)}{f(z)}\right)$ is the closed disc with centered at $C(r) = \frac{1-B[A(1-\alpha)+B\alpha]r^2}{1-B^2r^2}$ and having the radius $\rho(r) = \frac{(1-\alpha)(A-B)r}{1-B^2r^2}$ by using Lemma 3, that is

$$\left| z\frac{f'(z)}{f(z)} - \frac{1-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2} \right| \leq \frac{(1-\alpha)(A-B)r}{1-B^2r^2}. \tag{16}$$

After simple calculations from (16) we get

$$\begin{cases} \frac{1-(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2} \leq Re\left(z\frac{f'(z)}{f(z)}\right) \\ \leq \frac{1+(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}, & B \neq 0, \\ 1-(1-\alpha)\left|A\right|r \leq Re\left(z\frac{f'(z)}{f(z)}\right) \leq 1+(1-\alpha)\left|A\right|r, & B = 0. \end{cases} \tag{17}$$

On the other hand we have

$$Re\left(z\frac{f'(z)}{f(z)}\right) = r\frac{\partial}{\partial r}\log\left|f(z)\right|, \left|z\right| = r. \tag{18}$$

If we substitute (18) into the (17) we get

$$\begin{cases} \frac{1}{r} - \frac{(1-\alpha)(A-B)}{1-Br} \leq \frac{\partial}{\partial r}\log\left|f(z)\right| \leq \frac{1}{r} + \frac{(1-\alpha)(A-B)}{1+Br}, & B \neq 0, \\ \frac{1}{r} - (1-\alpha)\left|A\right| \leq \frac{\partial}{\partial r}\log\left|f(z)\right| \leq \frac{1}{r} + (1-\alpha)\left|A\right|, & B = 0. \end{cases} \tag{19}$$

Integrating both sides (19) we obtain (14). □

Corollary 2. *The radius of starlikeness of the class $S^*(A, B, \alpha)$ is*

$$r_s = \frac{2}{(1-\alpha)(A-B) + \sqrt{(1-\alpha)^2(A-B)^2 + 4B[(1-\alpha)A+\alpha B]}}. \tag{20}$$

This radius is sharp because the extremal function is given in (15).

Proof. From (17) we have

$$Re\left(z\frac{f'(z)}{f(z)}\right) \geq \frac{1-(1-\alpha)(A-B)r-B[(1-\alpha)A+\alpha B]r^2}{1-B^2r^2}. \tag{21}$$

Hence for $r < r_s$ the first hand side of the preceding inequality is positive this implies that

$$r_s = \frac{2}{(1-\alpha)(A-B) + \sqrt{(1-\alpha)^2(A-B)^2 + 4B[(1-\alpha)A+\alpha B]}}.$$

Also note that the inequality (20) becomes an equality for the function which is given in (15). It follows that

$$r_s = \frac{2}{(1-\alpha)(A-B) + \sqrt{(1-\alpha)^2(A-B)^2 + 4B[(1-\alpha)A+\alpha B]}}.$$

and the proof is complete. □

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Department of Mathematics and Computer Sciences,
Faculty of Science and Arts,
İstanbul Kültür University,
İstanbul, Turkey
E-mail address: `y.polatoglu@iku.edu.tr`
E-mail address: `m.bolcal@iku.edu.tr`
E-mail address: `a.sen@iku.edu.tr`
E-mail address: `e.yavuz@iku.edu.tr`