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SOME COMMON FIXED POINT THEOREMS FOR SELFMAPPINGS IN UNIFORM SPACE

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Abstract. In this paper, we establish some common fixed point theorems for selfmappings in uniform spaces by employing the concepts of an A-distance, an E-distance as well as the notion of comparison function. A more general contractive condition than that used to establish some of the results of Aamri and El Moutawakil [1] is employed to obtain our results. Our results are generalizations of some of the results of [1].

1. Introduction

A *uniform space* (X, Φ) is a nonempty set X equipped with a nonempty family Φ of subsets of $X \times X$ satisfying the following properties:

(i) if U is in Φ, then U contains the diagonal $\{(x, x)|x \in X\}$;
(ii) if U is in Φ and V is a subset of $X \times X$ which contains U, then V is in Φ;
(iii) if U and V are in Φ, then $U \bigcap V$ is in Φ;
(iv) if U is in Φ, then there exists V in Φ, such that, whenever (x, y) and (y, z) are in V, then (x, z) is in U;
(v) if U is in Φ, then $\{(y, x)|(x, y) \in U\}$ is also in Φ.

Φ is called the *uniform structure* of X and its elements are called *entourages or neighbourhoods or surroundings.*

The space (X, Φ) is called *quasiuniform* if property (v) is omitted. The definition of uniform space is contained in Bourbaki [4], Zeidler [13] as well as available on the internet (by Wikipedia, the free encyclopedia).

The concept of a W-distance on metric space was introduced by Kada et al [6] to generalize some important results in nonconvex minimizations and in fixed point theory for both W-contractive and W-expansive maps. The theory of fixed point or common fixed point for contractive or expansive selfmappings in complete metric space has been well-developed. Interested readers can consult Berinde [2, 3], Jachymski [5], Kada et al [6], Kang [7], Rhoades [8], Rus

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[10], Rus et al [11], Wang et al [12] and Zeidler [13] for further study of fixed point or common fixed point theory.

Using the ideas of Kang [7], Montes and Charris [9] established some results on fixed and coincidence points of maps by means of appropriate W-contractive or W-expansive assumptions in uniform space. Furthermore, Aamri and El Moutawakil [1] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an A-distance and an E-distance.

In Aamri and El Moutawakil [1], the following contractive definition was employed: Let $f, g\colon X \to X$ be selfmappings of X. Then, we have

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))), \forall x, y \in X, \tag{1}$$

where $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$ is a nondecreasing function satisfying

(i) for each $t \in (0, +\infty), 0 < \psi(t)$,
(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0, \forall t \in (0, +\infty)$.

ψ satisfies also the condition $\psi(t) < t$, for each $t > 0$.

In this paper, we shall establish some common fixed point theorems by employing a more general contractive condition than (1).

We shall employ the concepts of an A-distance, an E-distance as well as the notion of comparison function in this work. Berinde [2, 3] extended the Banach's fixed point theorem using different contractive definitions involving the concept of the comparison functions. Rus [10] and Rus et al [11] also contain various generalizations and extensions of the Banach's fixed point theorem in which the contractive conditions involve some comparison functions.

Our results are generalizations of Theorems 3.1–3.3 of [1].

2. Preliminaries

We shall require the following definitions and lemma in the sequel. The Remark 2.1 and Definitions 2.2–2.7 are contained in Aamri and El Moutawakil [1]. Let (X, Φ) be a uniform space.

Remark 2.1. When topological concepts are mentioned in the context of a uniform space (X, Φ), they always refer to the topological space $(X, \tau(\Phi))$.

Definition 2.2. If $V \in \Phi$ and $(x, y) \in V, (y, x) \in V$, x and y are said to be V*-close.* A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is said to be a *Cauchy sequence* for Φ if for any $V \in \Phi$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n, m \geq N$.

Definition 2.3. A function $p\colon X \times X \to \mathbf{R}^+$ is said to be an A*-distance* if for any $V \in \Phi$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

Definition 2.4. A function $p\colon X \times X \to \mathbf{R}^+$ is said to be an E*-distance* if

(p_1) p is an A-distance,
(p_2) $p(x, y) \leq p(x, z) + p(z, y), \forall x, y \in X$.

Definition 2.5. A uniform space (X, Φ) is said to be *Hausdorff* if and only if the intersection of all $V \in \Phi$ reduces to the diagonal $\{(x, x) | x \in X\}$, i.e. if $(x, y) \in V$ for all $V \in \Phi$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $V \in \Phi$ is said to be *symmetrical* if $V = V^{-1} = \{(y, x) | (x, y) \in V\}$.

Definition 2.6. Let (X, Φ) be a uniform space and p be an A-distance on X.

(i) X is said to be *S-complete* if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = 0$.

(ii) X is said to be *p-Cauchy complete* if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\Phi)$.

(iii) $f\colon X \to X$ is said to be *p-continuous* if $\lim_{n\to\infty} p(x_n, x) = 0$ implies that $\lim_{n\to\infty} p(f(x_n), f(x)) = 0$.

(iv) $f\colon X \to X$ is *$\tau(\Phi)$-continuous* if $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\Phi)$ implies $\lim_{n\to\infty} f(x_n) = f(x)$ with respect to $\tau(\Phi)$.

(v) X is said to be *p-bounded* if $\delta_p(X) = \sup\{p(x, y) | x, y \in X\} < \infty$.

Definition 2.7. Let (X, Φ) be a Hausdorff uniform space and p an A-distance on X. Two selfmappings f and g on X are said to be *p-compatible* if, for each sequence $\{x_n\}_{n=0}^{\infty}$ of X such that $\lim_{n\to\infty} p(f(x_n), u) = \lim_{n\to\infty} p(g(x_n), u) = 0$ for some $u \in X$, then we have $\lim_{n\to\infty} p(f(g(x_n)), g(f(x_n))) = 0$.

We shall also state the following definition of a comparison function which is required in the sequel to establish some common fixed point results in uniform space.

Definition 2.8 (Berinde [2,3])**.** A function $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$ is called a *comparison function* if:

(i) ψ is monotone increasing;

(ii) $\lim_{n\to\infty} \psi^n(t) = 0, \forall t \geq 0$.

The definition is also contained in [10, 11].

Remark 2.9. Every comparison function satisfies the condition $\psi(0) = 0$.

Also, both conditions (i) and (ii) imply that $\psi(t) < t, \forall t > 0$.

In this paper, we shall employ the following contractive definition:

Let $f, g\colon X \to X$ be selfmappings of X. There exist $L \geq 0$ and a comparison function $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that $\forall x, y \in X$, we have

$$p(f(x), f(y)) \leq Lp(x, g(x)) + \psi(p(g(x), g(y))). \tag{2}$$

Remark 2.10. The contractive condition (2) is more general than (1) in the sense that if $L = 0$ in (2), then we obtain (1) stated in this paper which was employed by Aamri and El Moutawakil [1].

The following Lemma shall be required in the sequel.

Lemma 2.11. *Let (X, Φ) be a Hausdorff uniform space and p be an A-distance on X. Let $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ be arbitrary sequences in X and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ be sequences in $\mathbf{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following hold:*

(a) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n, \forall n \in \mathbf{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.*
(b) *If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n, \forall n \in \mathbf{N}$, then $\{y_n\}_{n=0}^{\infty}$ converges to z.*
(c) *If $p(x_n, x_m) \leq \alpha_n \forall m > n$, then $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, Φ).*

Remark 2.12. Lemma 2.11 is contained in [1], [7] and [9].

Remark 2.13. A sequence in X is p-Cauchy if it satisfies the usual metric condition. See [1] for this remark.

3. The Main Results

The main results of this paper are the following:

Theorem 3.1. *Let (X, Φ) be a Hausdorff uniform space and p an A-distance on X. Suppose that X is p-bounded and S-complete. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by*

$$x_n = f(x_{n-1}), n = 1, 2, \ldots,$$

with $x_0 \in X$. Let f and g be commuting p-continuous or $\tau(\Phi)$-continuous selfmappings of X such that

(i) *$f(X) \subseteq g(X)$,*
(ii) *$p(f(x_i), f(x_i)) = 0, \forall x_i \in X, i = 0, 1, 2, \ldots$,*
(iii) *$f, g \colon X \to X$ satisfy the contractive condition (2).*

Suppose also that $\psi \colon \mathbf{R}^+ \to \mathbf{R}^+$ is a comparison function. Then, f and g have a common fixed point.

Proof. Let $x_0 \in X$. Choose $x_1 \in X$ such that $f(x_0) = g(x_1)$, choose $x_1 \in X$ such that $f(x_1) = g(x_2)$, and in general, choose $x_n \in X$ such that $f(x_{n-1}) = g(x_n)$.

We recall that $x_n = f(x_{n-1}), n = 1, 2, \ldots$, so that by conditions (ii) and (iii) of the Theorem, we obtain

$$\begin{aligned}
p(f(x_n), f(x_{n+m})) \leq & Lp(x_n, g(x_n)) + \psi(p(g(x_n), g(x_{n+m}))) \\
= & Lp(f(x_{n-1}), f(x_{n-1})) + \psi(p(f(x_{n-1}), f(x_{n+m-1}))) \\
= & \psi(p(f(x_{n-1}), f(x_{n+m-1}))) \\
\leq & \psi(Lp(x_{n-1}, g(x_{n-1})) + \psi(p(g(x_{n-1}), g(x_{n+m-1})))) \\
= & \psi(Lp(f(x_{n-2}), f(x_{n-2})) + \psi(p(f(x_{n-2}), f(x_{n+m-2}))) \\
= & \psi(\psi(p(f(x_{n-2}), f(x_{n+m-2}))) \\
= & \psi^2(p(f(x_{n-2}), f(x_{n+m-2})) \\
\leq & \cdots \leq \psi^n(p(f(x_0), f(x_m)) \leq \psi^n(\delta_p(X)),
\end{aligned}$$

from which we have that

$$p(f(x_n), f(x_{n+m})) \leq \psi^n(\delta_p(X)), \tag{3}$$

where $p(f(x_0), f(x_m)) \leq \delta_p(X)$ and $\delta_p(X) = \sup\{p(x, y)|x, y \in X\} < \infty$.

Therefore, using the definition of comparison function in (3) yields

$$\psi^n(\delta_p(X)) \to 0 \text{ as } n \to \infty,$$

from which it follows that

$$p(f(x_n), f(x_{n+m})) \to 0 \text{ as } n \to \infty.$$

Hence, by applying Lemma 2.11(c), we have that $\{f(x_n)\}_{n=0}^{\infty}$ is a p-Cauchy sequence. Since X is S-complete, $\lim_{n\to\infty} p(f(x_n), u)) = 0$, for some $u \in X$, and therefore $\lim_{n\to\infty} p(g(x_n), u)) = 0$.

Since f and g are p-continuous, then

$$\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0.$$

Also, since f and g are commuting, then $fg = gf$, so that we have

$$\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0,$$

so that by Lemma 2.11(a), we obtain that $f(u) = g(u)$.

Since $f(u) = g(u), fg = gf$, we have $f(f(u)) = f(g(u)) = g(f(u)) = g(g(u))$. Suppose that $p(f(u), f(f(u))) \neq 0$. Using (2) and the condition $\psi(t) < t, \forall t > 0$ in the Remark 2., then, we have

$$\begin{aligned}
p(f(u), f(f(u))) \leq & Lp(u, g(u)) + \psi(p(g(u), g(f(u)))) \\
= & Lp(f(u), f(u)) + \psi(p(f(u), f(f(u)))) \\
= & \psi(p(f(u), f(f(u))) < p(f(u), f(f(u))),
\end{aligned}$$

which is a contradiction. Therefore, $p(f(u), f(f(u))) = 0$.

Condition (ii) of the Theorem yields $p(f(u), f(u)) = 0$. Since $p(f(u), f(f(u))) = 0$ and $p(f(u), f(u)) = 0$, applying Lemma 2.11(a) then yields $f(f(u)) =$

$f(u)$. Thus, we have $g(f(u)) = f(f(u)) = f(u)$. Hence, $f(u)$ is a common fixed point of f and g.

The proof is similar when f and g are $\tau(\Phi)$-continuous as S-completeness implies p-Cauchy completeness. □

Remark 3.2. Theorem 3.1 is a generalization of Theorem 3.1 of Aamri and El Moutawakil [1]

Theorem 3.1 is an existence result for the common fixed point of f and g, while the next two results guarantee the uniqueness of the common fixed point.

Theorem 3.3. *Let (X, Φ) be a Hausdorff uniform space and p an E-distance on X. Suppose that X is p-bounded and S-complete. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by*

$$x_n = f(x_{n-1}), n = 1, 2, \dots,$$

with $x_0 \in X$. Let f and g be commuting p-continuous or $\tau(\Phi)$-continuous selfmappings of X such that

(i) $f(X) \subseteq g(X)$,
(ii) $p(f(x_i), f(x_i)) = 0, \forall x_i \in X, i = 0, 1, 2, \dots$,
(iii) $f, g \colon X \to X$ *satisfy the contractive condition (2).*

Suppose also that $\psi \colon \mathbf{R}^+ \to \mathbf{R}^+$ is a comparison function. Then, f and g have a unique common fixed point.

Proof. f and g have a common fixed point since an E-distance function p is an A-distance. Suppose that there exist $u, v \in X$ such that $f(u) = g(u) = u$ and $f(v) = g(v) = v$.

Let $p(u, v) \neq 0$. Then, we have

$$\begin{aligned} p(u, v) =& p(f(u), f(v)) \leq Lp(u, g(u)) + \psi(p(g(u), g(v))) \\ =& Lp(u, u) + \psi(p(u, v)) = \psi(p(u, v)) < p(u, v), \end{aligned}$$

which is a contradiction. Therefore, we have $p(u, v) = 0$. By carrying out a similar process, we also have that $p(v, u) = 0$.

Using condition (p_2) of Definition 2.4, we have $p(u, u) \leq p(u, v) + p(v, u)$, from which it follows that $p(u, u) = 0$. Since $p(u, u) = 0$ and $p(u, v) = 0$, then by Lemma 2.11(a), we have that $u = v$. □

Remark 3.4. Theorem 3.3 is a generalization of Theorem 3.2 as well as corollaries 3.1 & 3.2 of Aamri and El Moutawakil [1].

Theorem 3.5. *Let (X, Φ) be a Hausdorff uniform space and p an E-distance on X. Suppose that X is p-bounded and S-complete. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by*

$$x_n = f(x_{n-1}), n = 1, 2, \dots,$$

with $x_0 \in X$. Let f and g be p-compatible, p-continuous or $\tau(\Phi)$-continuous selfmappings of X such that

(i) $f(X) \subseteq g(X)$,
(ii) $p(f(x_i), f(x_i)) = 0, \forall x_i \in X, i = 0, 1, 2, \dots,$
(iii) $f, g\colon X \to X$ *satisfy the contractive condition (2).*

Suppose also that $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$ *is a comparison function. Then,* f *and* g *have a unique common fixed point.*

Proof. Just as in the proof of Theorem 3.1, we have for some $u \in X$ that $\lim_{n\to\infty} p(f(x_n, u)) = \lim_{n\to\infty} p(g(x_n, u)) = 0$. Since f and g are p-continuous, we have

$$\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0,$$

while the assumption that f and g are p-compatible implies the following $\lim_{n\to\infty} p(f(g(x_n)), g(f(x_n))) = 0$.

Furthermore, by condition (p_2) of Definition 2.4, we have that

$$p(f(g(x_n)), g(u)) \leq p(f(g(x_n)), g(f(x_n))) + p(g(f(x_n)), g(u)) \tag{3}$$

Taking limits in (3) and applying Lemma 2.11(a), then we have

$$\lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0.$$

Since $\lim_{n\to\infty} p(f(g(x_n)), f(u)) = 0$ and $\lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0$, then by Lemma 2.11(a) we have $f(u) = g(u)$.

The rest of the proof is as in Theorem 3.3. □

Remark 3.6. Theorem 3.5 is a generalization of Theorem 3.3 of Aamri and El Moutawakil [1].

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