Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
23 (2007), 65–69
www.emis.de/journals
ISSN 1786-0091

SECOND ORDER PARALLEL TENSORS ON α – SASAKIAN MANIFOLD

LOVEJOY DAS

Abstract. Levy had proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the metric tensor. Sharma [12] has proved that a second order parallel tensor in a Kaehler space of constant holomorphic sectional curvature is a linear combination with constant coefficients of the Kaehlarian metric and the fundamental 2 – form. In this paper we show that a second order symmetric parallel tensor on an $\alpha - K$ contact ($\alpha \in R_o$) manifold is a constant multiple of the associated metric tensor and we also prove that there is no nonzero skew symmetric second order parallel tensor on an α – Sasakian manifold.

1. Introduction

In 1923, Eisenhart [10] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of the metric tensor is reducible. In 1926, Levy [11] had obtained the necessary and sufficient conditions for the existence of such tensors, Recently Sharma [12] has generalized Levy's result by showing that a second order parallel (not necessarily symmetric and non singular) tensor on an n – dimensional ($n \succ 2$) space of constant curvature is a constant multiple of the metric tensor. Sharma has also proved in [12] that on a Sasakian manifold there is no nonzero parallel 2 – form. In this paper we have considered an almost contact metric manifold and have proved the following two theorems.

Theorem 1.1. *On an $\alpha - K$ contact ($\alpha \in R_o$) manifold a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor.*

2000 *Mathematics Subject Classification.* 53C15, 53C25.
Key words and phrases. Contact metric manifold, second order parallel tensor, K- contact and Sasakian manifold.

Now the question arises whether there is a skew symmetric second order parallel tensor on a $\alpha - k$ contact manifold. We do not have an answer to it. However we do have an answer if the manifold is $\alpha-$ Sasakian where $\alpha \in R_0$.

Theorem 1.2. *On an $\alpha-$ Sasakian manifold there is no nonzero parallel 2 – forms.*

2. Preliminaries

A C^∞ manifold M of dimension $2n+1$ is called a contact manifold if it carries a global 1 – form A such that $A \wedge (dA)^n \neq 0$. On a contact manifold there exists a unique vector field T called the characteristic vector field such that

$$A(T) = 1, \ (dA)(T, X) = 0 \tag{2.1}$$

for any vector field X on M. By polarization we obtain a Riemannian metric g called an associated metric and a $(1,1)$ tensor field ϕ on M such that

$$\begin{gathered} \phi^2 = -I + A \otimes T \\ (dA)(X, Y) = g(X, \phi Y) \\ A(X) = g(X, T) \end{gathered} \tag{2.2}$$

for the arbitrary vector fields X and Y on M. If in addition to (2.1) and (2.2), M^n admits a positive definite Riemannian metric g such that

$$\begin{gathered} g(\phi X, \phi Y) = g(X, Y) - A(X)A(Y) \\ \phi(T) = 0, \ A(\phi(X)) = 0, \forall \ X, Y \in \mathfrak{X}(M) \\ \text{and } \operatorname{rank}(\phi) = 2n \text{ everywhere on } M. \end{gathered} \tag{2.3}$$

Such a manifold satisfying (2.1), (2.2), and (2.3) is called an almost contact metric manifold. The structure endowed in M is called (ϕ, A, T, g) – structure.

For a (ϕ, A, T, g) – structure, the skew symmetric bilinear form

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.4}$$

is called the fundamental 2 – form of the almost contact metric structure.

3. Some Definitions and Theorems

Definition 3.1. An almost contact metric structure is said to be an α – K contact structure if the vector field T is killing with respect to g.

In proving Theorems 1.1 and 1.2, we need the following theorems.

Theorem 3.1. *On an α – K contact structure the following holds.*

$$\nabla_X T = -\alpha\phi x \text{ for all } X \in \mathfrak{X}(M) \tag{3.1}$$

where ∇ is the Riemannian connection of g.

Theorem 3.2. *An almost contact metric structure –* (ϕ, A, T, g) *is* α *– Sasakian iff*

$$(\nabla_x \phi) Y = \alpha\{g(X,Y)T - A(Y)X\} \tag{3.2}$$

where ∇ *denotes the Riemannian connection of* g.

Proof. The proofs of the above theorems follows in a similar fashion as in the Theorem 6.3 by Blair [3]. □

Definition 3.2 ([2])**.** An almost α – Sasakian manifold M is an almost contact metric manifold such that $\phi(X,Y) = \frac{1}{\alpha} d\eta(X,Y), \alpha \in R_0$ and M is a α – Sasakian manifold if the structure is normal.

Theorem 3.3. *An almost contact metric manifold* M *is* α *– Sasakian manifold iff for all* $X, Y \in \mathfrak{X}(M)$

$$R(X,Y)T = \alpha\{A(Y)X - A(X)Y\} \tag{3.3}$$

Proof. The proof of the above theorem follows in view of Lemma 6.1 of Blair [3]

The two conditions of being normal and contact metric may be written as the following:

$$R(T,X)Y = \alpha\{g(X,Y)T - A(Y)X\} \tag{3.4}$$

□

Theorem 3.4. *For an* $\alpha - K$ *contact manifold we have*

$$R(T,X)T = \alpha\{-X + A(X)T\} \tag{3.5}$$

Proof. In view of (3.4), the proof follows immediately. □

For a detailed study on a contact manifold the reader is referred to [2].

4. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let h denote a $(0,2)$ – tensor field on an $\alpha - K$ contact manifold M such that $\nabla h = 0$. Then it follows that

$$h(R(W,X)Y,Z) + h(Y,R(W,X)Z) = 0 \tag{4.1}$$

for arbitrary vector fields X, Y, Z, W on M.

We write (4.1) as follows

$$g(R(W,X)Y,Z) + g(Y,R(W,X),Z) = 0.$$

On substituting $W = Y = Z = T$ in (4.1) we get:

$$g(R(T,X)T,T) + g(T,R(T,X),T) = 0. \tag{4.2}$$

In view of Theorem (3.4), the above equation becomes:

$$g(-\alpha X + \alpha A(X)T,T) + g(T, -\alpha X + \alpha A(X)T) = 0. \tag{4.3}$$

In this equation, using (2.2) we get

$$2\alpha g\left(X,T\right)h\left(T,T\right)-\alpha h\left(X,T\right)-\alpha h\left(T,X\right)=0. \tag{4.4}$$

Differentiating (4.4) covariantly with respect to Y and using Theorem (3.1) we get

$$\begin{aligned}&2\alpha h\left(T,T\right)\ g\left(\nabla_Y X,T\right)-2\alpha^2 h\left(T,T\right)g\left(X,\phi Y\right)\\&-\alpha g\left(\nabla_Y X,T\right)+\alpha^2 g\left(X,\phi Y\right)+\alpha^2 g\left(\phi Y,X\right)-\alpha g\ \left(T,\nabla_Y X\right)=0.\end{aligned} \tag{4.5}$$

Replacing Y by ϕY and using equations (2.2), (2.3) and (4.4) we obtain

$$h\left(X,Y\right)+h\left(Y,X\right)=2h\left(T,T\right)\ g\left(X,Y\right).$$

But h is symmetric, thus on simplifying the above equation we get

$$2h\left(T,T\right)g\left(X,Y\right)=2h\left(X,Y\right). \tag{4.6}$$

In view of the fact that $h\left(T,T\right)$ is constant by differentiating it along any vector on M^{2n+1} we get

$$h\left(T,T\right)g\left(X,Y\right)=h\left(X,Y\right)$$

which completes the proof. □

Proof of Theorem 1.2. Let us consider h to be a parallel $2-$ form on an $\alpha-$ Sasakian manifold M^{2n+1} and let H be a $(1,1)$ tensor field metrically equivalent to h since $h\left(X,Y\right)=g\left(HX,Y\right)$. Now (4.1) can be written as

$$g\left(R\left(W,X\right)Y,Z\right)+g\left(Y,R\left(W,X\right)Z\right)=0. \tag{4.7}$$

Let us put $X=Y=T$ in (4.7) and using the fact that $h\left(X,Y\right)=g\left(HX,Y\right)$ we get

$$g\left(HR\left(W,T\right)T,Z\right)+g\left(HT,R\left(WT\right)Z\right)=0. \tag{4.8}$$

Applying the skew symmetric property of $R\left(X,Y\right)$ and using (3.3) and (3.4) in (4.8) and after simplifying, we obtain

$$\alpha g\left(HZ,T\right)T+\alpha g\left(Z,T\right)HT=\alpha HZ. \tag{4.9}$$

Differentiating (4.9) along ϕX we obtain

$$\begin{aligned}&2\alpha A\left(X\right)A\left(HZ\right)T-\alpha g\left(HZ,X\right)T-\alpha g\left(HZ,T\right)X\\&=\alpha g\left(Z,X\right)HT-2\alpha A\left(X\right)A\left(Z\right)HT+\alpha A\left(Z\right)HX.\end{aligned} \tag{4.10}$$

Let $\{e_i\}, i=1,2,\ldots,2n+1$ be an orthonormal basis of the tangent space. In the above equation (4.10), we substitute $X=e_i$ and take the inner product with e_i and eventually summing over i gives us

$$\alpha\left(2n-1\right)g\left(HZ,T\right)=0.$$

Since $\alpha\left(2n-1\right)\neq 0$, we have $g\left(HZ,T\right)=0$. But $g\left(HZ,T\right)=-g\left(HT,Z\right)$. Thus, $HT=0$ and hence (4.9) shows that $H=0$, which completes the proof. □

References

[1] D. Blair and S. Goldberg. Topology of almost contact manifolds. *J. Differ. Geom.*, 1:347–354, 1967.

[2] D. E. Blair. *Contact manifolds in Riemannian geometry.*, volume 509 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin-Heidelberg-New York., 1976.

[3] D. E. Blair. *Riemannian geometry of contact and symplectic manifolds.*, volume 203 of *Progress in Mathematics.* Birkhäuser, Boston, MA., 2002.

[4] L. Das. Fiberings on almost r-contact manifolds. *Publ. Math.*, 43:1–7, 1993.

[5] L. Das. Prolongation of F-structure to the tangent bundle of order 2. *Int. J. Math. Math. Sci.*, 16(1):201–204, 1993.

[6] L. Das. On CR-structures and F-structure satisfying $F^K + (-)^{K+1} F = 0$. *Rocky Mt. J. Math.*, 36, 2006.

[7] L. Das and R. Nivas. On a differentiable manifold with $[F_1, F_2]\,(K+1,1)$-structure. *Tensor (N.S.)*, 65:29–35, 2004.

[8] L. Das and R. Nivas. On certain structures defined on the tangent bundle. *Rocky Mt. J. Math.*, 36, 2006.

[9] L. S. Das. Invariant submanifolds of the manifold with $\phi(k, -(-)^{k+1})$-structure. *Tensor (N.S.)*, 64(2):189–196, 2003.

[10] L. Eisenhart. Symmetric tensors of the second order whose first covariant deriatives are zero. *Trans. Am. Math. Soc.*, 25:297–306, 1923.

[11] H. Levy. Symmmetric tensors of the second order whose covariant derivatives vanish. *Ann. Math.*, 27:91–98, 1926.

[12] R. Sharma. Second order parallel tensor in real and complex space forms. *Int. J. Math. Math. Sci.*, 12:787–790, 1989.

Received May 2, 2006.

Department of Mathematics,
Kent State University,
New Philadelphia, OH 44663,USA
E-mail address: ldas@kent.edu