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COHOMOLOGY OF DEFORMATION PARAMETERS OF DIAGONAL NONCOMMUTATIVE NONASSOCIATIVE GRADED ALGEBRAS

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Abstract. We study graded algebras with no monomial in the generators having zero divisors and graded over a finite abelian group. As a vector space over the field, the algebra is generated by a set of algebra elements with as many elements as the grading group, and each generator is graded by a different element of the grading group. Their noncommutativity and nonassociativity turns out to be diagonal and governed by structure constants of any (pure grade) generating basis as a vector space over the field. There are functions q and r coding the noncommutativity and nonassociativity of the algebra. We study the cohomology of such q- and r-functions. We discover that the r-function coding nonassociativity has always trivial cohomology. Quaternions and octonions are constructed in this manner and we study their noncommutativity and nonassociativity using cohomological tools.

1. Introduction

Let G be a finite abelian group. Let A be a G-graded algebra, which as a vector space over the commutative field K can be generated by a set $\{v_a | a \in G\}$. We assume that there are no zero divisors (neither torsion since K is a field) at the level of monomials in the basis elements. We call an algebra (not necessarily associative) with these characteristics a finite *perfect algebra* [5]. This is the subject of the present paper. We explore very immediate properties of such algebras. In particular, we characterize the structures underlying the noncommutativity and nonassociativity by using group cohomological tools. We analyze concretely the quaternion and octonion algebras.

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2. Finite Perfect Algebras

From the definition of a finite perfect algebra it follows that the structure constants $C_{a,b} \in K$ associated with the basis $\{v_a | a \in G\}$,

$$v_a \cdot v_b = C_{a,b}\ v_{a+b}, \tag{1}$$

are all non-zero, i.e.

$$C\colon G \times G \to K^*, \quad (a,b) \mapsto C_{a,b} \neq 0. \tag{2}$$

We want to analyze the noncommutativity and nonassociativity in this type of algebra. We define

$$q_{a,b} = C_{a,b}\ (C_{b,a})^{-1} \tag{3}$$

$$r_{a,b,c} = C_{b,c}\ (C_{a+b,c})^{-1}\ C_{a,b+c}\ (C_{a,b})^{-1}. \tag{4}$$

(the exponents -1 denote the inverses as elements of K). Accordingly,

$$v_a \cdot v_b = q_{a,b}\ v_b \cdot v_a, \tag{5}$$

$$v_a \cdot (v_b \cdot v_c) = r_{a,b,c}\ (v_a \cdot v_b) \cdot v_c. \tag{6}$$

The q- and r-factors encode a very particular type of noncommutativity and nonassociativity, called *diagonal noncommutativity* and *diagonal nonassociativity* since they involve respectively just the exchange of factors or the alteration of parentheses. We want to explore the properties of the C, q and r functions:

$$q\colon G \times G \to K^*,\ (a,b) \mapsto q_{a,b} \neq 0, \tag{7}$$

$$r\colon G \times G \times G \to K^*,\ (a,b,c) \mapsto r_{a,b,c} \neq 0. \tag{8}$$

We consider first a quadratic monomial in the generators (each q-factor results from exchanging factors)

$$v_a \cdot v_b = q_{a,b}\ v_b \cdot v_a = q_{a,b}\ q_{b,a}\ v_a \cdot v_b. \tag{9}$$

Since there are no zero divisors at the level of monomials, we conclude:

$$(\text{for } b = a) \quad q_{a,a} = 1, \tag{10}$$

$$(\text{for } b \neq a) \quad q_{a,b}\ q_{b,a} = 1. \tag{11}$$

We consider now a cubic product of the generators:

$$v_a \cdot (v_b \cdot v_c) = q_{a,b+c}\ (v_b \cdot v_c) \cdot v_a, \tag{12}$$

$$v_a \cdot (v_b \cdot v_c) = r_{a,b,c}\ q_{a,b}\ (r_{b,a,c})^{-1}\ q_{a,c}\ r_{b,c,a}\ (v_b \cdot v_c) \cdot v_a, \tag{13}$$

$$v_a \cdot (v_b \cdot v_c) = r_{a,b,c}\ q_{a+b,c}\ q_{a,b}\ r_{c,b,a}\ q_{c,b}\ (v_b \cdot v_c) \cdot v_a, \tag{14}$$

where we have done the rearrangements in three different manners by exchanging factors (getting extra q-factors) or rearranging parentheses (getting extra r-factors) in the order they appear. In order to have no zero divisors at the level of monomials we obtain

$$q_{b,c}\, q^{-1}_{a+b,c}\, q_{a,b+c}\, q^{-1}_{a,b} = r_{a,b,c}\, r_{c,b,a} \tag{15}$$

$$= q_{a,c}\, q_{b,c}\, q^{-1}_{a+b,c}\, r_{a,b,c} (r_{b,a,c})^{-1}\, r_{b,c,a}. \tag{16}$$

Following Scheunert [3], we call q a *commutation factor on an abelian group* G if following conditions are satisfied:

$$q(a,b)q(b,a) = 1, \tag{17}$$

$$q(a,b+c) = q(a,b)q(a,c), \tag{18}$$

$$q(a+b,c) = q(a,c)q(b,c). \tag{19}$$

We could also call such "commutation factor" a *separated* q–function (in analogy to the separation of variables method), since all identities obtained from monomials with the exception of (10)–which is not enforced by these requirements– are satisfied separately by identities in q-factors alone, or in r-factors alone.

Observe that if q is a "commutation factor" or a "separated" q-function then equations (15) and (16) become ("separated"):

$$1 = r_{a,b,c}\, r_{c,b,a}, \tag{20}$$

$$1 = r_{a,b,c}\, r_{c,a,b}\, r_{b,c,a}. \tag{21}$$

The last identity relates to Jacobi identities and provides a generous source of models for such r-factors [8].

We could consider now a weaker condition than "commutation factor" or "separated" q-function. We call q a *2-cocycle* if

$$q_{b,c}\, q^{-1}_{a+b,c}\, q_{a,b+c}\, q^{-1}_{a,b} = 1 \tag{22}$$

This name will be clear bellow. Using equations (18-19) we verify that every "commutation factor" or a "separated" q-function is a 2-cocycle. Now, if q is a 2-cocycle, then equations (15) and (16) become:

$$1 = r_{a,b,c}\, r_{c,b,a}, \tag{23}$$

$$1 = (q_{a,c}\, q_{b,c}\, q^{-1}_{a+b,c})\, (r_{a,b,c}\, r_{c,a,b}\, r_{b,c,a}). \tag{24}$$

Scheurnert [3] shows that a general "commutation factor" or "separated" q-function (i.e. satisfying (17-19)) over the reals or complex numbers can be generated by factors of the form

$$q_2(a,b) = (-1)^{ab}, \quad \text{for } a,b \in \{0,1\} = Z_2 \ \text{(Supergrading)}, \tag{25}$$

$$q_{N\oplus N}((n,m),(n',m')) = \exp\left\{\frac{2\pi \mathrm{i}}{N}(nm' - n'm)\right\}, \tag{26}$$

$$\text{for } N \geq 2, \qquad (n,m),(n',m') \in Z_N \oplus Z_N.$$

and replications of such factors, where G being finite abelian, it can be decomposed:

$$G = Z_{N_1} \bigoplus \cdots \bigoplus Z_{N_s}. \tag{27}$$

Observe that the function q_2 in (25) does not satisfy (10), and leads to zero divisors. This is actually the main feature in exterior or Grassmann algebras, having nilpotent Grassmann variables. Only functions in (26) and their replications (involving diverse Z_N factors in G) provide models for "commutation factors" or "separated" q-functions in finite perfect algebras.

We now consider a constraint from monomials in the generators of order four:

$$(v_a \cdot v_b) \cdot (v_c \cdot v_d) = r_{a+b,c,d} \left((v_a \cdot v_b) \cdot v_c\right) \cdot v_d, \tag{28}$$

$$(v_a \cdot v_b) \cdot (v_c \cdot v_d) = (r_{a,b,c+d})^{-1}\, r_{b,c,d}\, r_{a,b+c,d}\, r_{a,b,c} \left((v_a \cdot v_b) \cdot v_c\right) \cdot v_d. \tag{29}$$

Again, to avoid zero divisors at the level of monomials we obtain:

$$r_{b,c,d}\, (r_{a+b,c,d})^{-1}\, r_{a,b+c,d}\, (r_{a,b,c+d})^{-1}\, r_{a,b,c} = 1. \tag{30}$$

This identity bears clear similarity with the pentagon identity (and in general with the associahedra) satisfied by the associator [4], and it is remarkable since, much like (9-10) which involve only q-factors, this involves only r-factors (rearrangements of parentheses). We call an r-factor satisfying (31) a *3-cocycle.*

Let f be a function

$$f\colon G \times \cdots \times G \to K^*. \tag{31}$$

We can use an inclusion map L from image(f) into the abelian group G' generated by image(f), which can be finite, and it is clearly a subgroup of the abelian multiplicative group K^*,

$$L\colon \text{image}(f) \to G' = \text{gen}(\text{image}(f)) \subset K^*. \tag{32}$$

In the case of $K^* = C^*$ (the non-zero complex numbers), the additive notation in G' can be obtained by using a logarithm mod 2πi. We can use the map L to convert f into a function $\hat{f}$ between abelian groups (with additive operation), where we can consider cohomological properties of such maps:

$$\hat{f} = L \circ f\colon G \times \cdots \times G \to G'. \tag{33}$$

In this way we define functions $\hat{C}$, $\hat{q}$, and $\hat{r}$. The coboundary of the function $\hat{q}$ is given by [1, 2]:

$$(\delta^{(2)}\hat{q})[a,b,c] = a\,\hat{q}(b,c) - \hat{q}(a+b,c) + \hat{q}(a,b+c) - \hat{q}(a,b). \tag{34}$$

Now, let q be a 2-cocycle. Equation (22) in terms of $\hat{q}$ becomes

$$\hat{q}(b,c) - \hat{q}(a+b,c) + \hat{q}(a,b+c) - \hat{q}(a,b) = 0. \tag{35}$$

Hence, by assuming the trivial action of G on G' (in the term $a\,\hat{q}(b,c)$ in (34)), $\hat{q}$ has vanishing coboundary, and thus we understand why q was labelled a 2-cocycle. The question arises whether the function in equation (26) which gives the possible building elements for the "commutation factor" in a finite perfect algebra has trivial cohomology, i.e. we ask if q itself is a 1-coboundary. We obtain first $\hat{q}_{N\oplus N}$:

$$q_{N\oplus N}((n,m),(n',m')) = \exp\left\{\frac{2\pi \mathrm{i}}{N}\hat{q}_{N\oplus N}((n,m),(n',m'))\right\}, \tag{36}$$

$$\begin{aligned} &\hat{q}_{N\oplus N}\colon (Z_N \oplus Z_N)^2 \to Z_N, \\ &\hat{q}_{N\oplus N}((n,m),(n',m')) = (nm' - n'm) \bmod N. \end{aligned} \tag{37}$$

Let us assume that $\hat{q}_{N\oplus N}$ is a 1-coboundary, i.e. there exists a homomorphism

$$\hat{\phi}\colon Z_N \oplus Z_N \to Z_N \tag{38}$$

such that the function $\hat{q}$ can be written:

$$\hat{q}_{N\oplus N}(a,b) = (\delta^{(1)}\hat{\phi})[a,b] = a\,\hat{\phi}(b) - \hat{\phi}(a+b) + \hat{\phi}(a) = \hat{\phi}(b) - \hat{\phi}(a+b) + \hat{\phi}(a). \tag{39}$$

Again, the action of the group $Z_N \oplus Z_N$ on Z_N is trivial. Using $\hat{q}_{N\oplus N}(a,0) = \hat{q}_{N\oplus N}(0,a) = 0$ we confirm $\hat{\phi}(0) = 0$ (as it should since it is homomorphism). From $\hat{q}_{N\oplus N}(a,a) = 0$ we find $\hat{\phi}(2a) = 2\hat{\phi}(a)$. Continuing the process, from $\hat{q}_{N\oplus N}(a,(n-1)a) = 0$ we find $\hat{\phi}(na) = n\hat{\phi}(a)$. Let $\hat{\phi}((1,0)) = k_1$ and and $\hat{\phi}((0,1)) = k_2$. Now,

$$\begin{aligned} \hat{q}_{N\oplus N}((n,0),(0,m)) &= nm \\ &= \hat{\phi}((0,m)) - \hat{\phi}((n,m)) + \hat{\phi}((n,0)) \\ &= mk_2 - \hat{\phi}((n,m)) + nk_1, \end{aligned} \tag{40}$$

$$\begin{aligned} \hat{q}_{N\oplus N}((0,m),(n,0)) &= -nm \\ &= \hat{\phi}((n,0)) - \hat{\phi}((n,m)) + \hat{\phi}((0,m)) \\ &= nk_1 - \hat{\phi}((n,m)) + mk_2. \end{aligned} \tag{41}$$

From this it follows $2nm = 0 \bmod N$. This is a contradiction for $N > 2$. For N=2 we obtain in this case that $\hat{q}_{2\oplus 2}$ is the 1-coboundary of:

$$\hat{\phi}((n,m)) = nk_1 - nm + mk_2, \tag{42}$$

where $k_1, k_2 \in Z_N$ are arbitrary. Hence, for $N > 2$ the cohomology class associated with $\hat{q}_{N\oplus N}$ is not trivial [6], but for N=2 it has trivial cohomology. We just have proved:

Proposition 1.

$$\begin{aligned} &q_{N\oplus N}\colon (Z_N \oplus Z_N)^2 \to C^*, \\ &q_{N\oplus N}((n,m),(n',m')) = \exp\left\{\frac{2\pi \mathrm{i}}{N}(nm' - n'm)\right\}. \end{aligned} \tag{43}$$

is a 2-cocycle. For $N > 2$ it has nontrivial cohomology (i.e. it is not a 1-coboundary). For $N = 2$, $\hat{q}_{2\oplus 2}$ is the 1-coboundary of the homomorphism in equation (42) for arbitrary $k_1, k_2 \in Z_2$.

The quaternion algebra H is a $Z_2 \oplus Z_2$–graded finite perfect algebra [5], where

$$q_H((n,m),(n',m')) = \exp\{\pi \mathrm{i}(nm' - n'm)\}, \tag{44}$$

$$\begin{aligned} r_H((n,m),(n',m'),(n'',m'')) &= 1 \\ &\forall\ (n,m),(n',m'),(n'',m'') \in Z_2 \otimes Z_2. \end{aligned} \tag{45}$$

Hence, its q-function has trivial cohomology, and clearly its r-function has trivial cohomology as well.

The octonion algebra O is a $Z_2 \oplus Z_2 \oplus Z_2$–graded finite perfect algebra [5] where

$$\begin{aligned} &q_O((n,m,s),(n',m',s')) \\ &\quad = e^{\pi \mathrm{i}\{(nm'-n'm)+(ns'-n's)+(ms'-m's)+n'ms-nm's'+nm's-n'ms'+nms'-n'm's\}}, \end{aligned} \tag{46}$$

$$\begin{aligned} &r_O((n,m,s),(n',m',s'),(n'',m'',s'')) \\ &\qquad = e^{-\pi \mathrm{i}\{nm's''+nm''s'+n'ms''+n'm''s+n''ms'+n''m's\}}, \\ &\qquad\qquad \forall (n,m,s),(n',m',s'),(n'',m'',s'') \in Z_2 \oplus Z_2 \oplus Z_2. \end{aligned} \tag{47}$$

Consider the homomorphism

$$\hat{\phi}_O((n,m,s)) = nm + ns + ms + nms. \tag{48}$$

We can check that $\hat{q}_O$ is the 1-coboundary of $\hat{\phi}_O$. And so, it is also a 2-cocycle. Hence, the q-function of the octonion algebra is a 1-coboundary, i.e. it has trivial cohomology class.

Let us discuss the cohomology of the r-functions in finite perfect algebras. Now, since the algebra is perfect (30) is satisfied, which is equivalent to saying that $\hat{r}$ is a 3-cocycle, since its 3-coboundary vanishes. Now, equation (4) also holds, which just establishes that $\hat{r}$ is the 2-coboundary of $\hat{C}$. Therefore, it always has trivial cohomology. We have just proved:

Proposition 2. *All finite perfect algebras have r-functions with trivial cohomology. In fact, the r-function is the 2-coboundary of the structure constant function for the chosen $\{v_a | a \in G\}$ basis.*

We obtain as a corollary of the previous propositions and the trivial cohomology of $\hat{q}_O$:

Corollary 1. *The quaternion and octonion algebras are finite perfect algebras whose q- and r-functions have trivial cohomology.*

It is remarkable that although the noncommutative features of the quaternion and octonion algebras are nontrivial and the nonassociativity of the octonion algebra is nontrivial, they all result from group homomorphisms with trivial cohomology.

This is the basis of a new exploration [7, 8] to identify algebras with novel gradings and remarkable properties, as the quaternion and octonion algebras are in several respects.

References

[1] M. F. Atiyah and C. T. C. Wall. Cohomology of groups. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 94–115. Thompson, Washington, D.C., 1967.

[2] A. Babakhanian. *Cohomological methods in group theory.* Pure and Applied Mathematics. New York: Marcel Dekker, Inc., 1972.

[3] M. Scheunert. Generalized Lie algebras. *J. Math. Phys.*, 20(4):712–720, 1979.

[4] J. Stasheff. *H-spaces from a homotopy point of view.* Lecture Notes in Mathematics, Vol. 161. Springer-Verlag, Berlin, 1970.

[5] L. Wills-Toro. Noncommutative and nonassociative gradings and their algebras. Master's thesis, Universidad Nacional de Colombia Medellin, 1999.

[6] L. Wills-Toro. *Finite group graded Lie algebraic extensions and Trefoil symmetric relativity, Standard Model, Yang-Mills and gravity theories.* PhD thesis, University of Hawaii Manoa, 2004.

[7] L. Wills-Toro, T. Craven, and J. Velez. Graded Lie algebras and q-commutative and r-associative parameters. Submitted to the São Paulo Journal of Mathematical Sciences, proceedings of the "XVII Latin–American Colloquium of Algebra", 2007.

[8] L. Wills-Toro et al. Work in progress.

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