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NEW TYPE OF SEQUENCE SPACES OF NON-ABSOLUTE TYPE AND SOME MATRIX TRANSFORMATIONS

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Abstract. In this paper, we introduce the sequence space $e_u^r(p)$. We show it posses BK-property, prove that the space $e_u^r(p)$ and $l(p)$ are linearly isomorphic to each other and also compute the α-, β- and γ-duals of $e_u^r(p)$ and discuss some of its inclusion properties. Furthermore, we construct the basis of $e_u^r(p)$. Moreover, we characterize the classes $(e_u^r(p)\colon l_p)$ and $(e_u^r(p)\colon f)$ of infinite matrices.

1. Preliminaries, Background and Notation

Let ω denote the space of all sequences(real or complex). The family under pointwise addition and scalar multiplication forms a linear(vector)space over real of complex numbers. Any subspace of ω is called the sequence space. So the sequence space is the set of scalar sequences(real of complex) which is closed under co-ordinate wise addition and scalar multiplication. Throughout the paper $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denotes the set of non-negative integers, the set of real numbers and the set of complex numbers, respectively. Let l_∞, c and c_0, respectively, denotes the space of all bounded sequences , the space of convergent sequences and the sequences converging to zero. Also, by cs, l_1 and $l(p)$ we denote the spaces of all convergent, absolutely and p-absolutely convergent series, respectively.

A linear topological space X over the field of real numbers $\mathbb{R}$ is said to be a paranormed space if there is a sub-additive function $h\colon X \to R$ such that $h(\theta) = 0$, $h(-x) = h(x)$ and scalar multiplication is continuous, that is, $|\alpha_n - \alpha| \to 0$ and $h(x_n - x) \to 0$ imply $h(\alpha_n x_n - \alpha x) \to 0$ for all $\alpha' s$ in $\mathbb{R}$ and $x's$ in X , where θ is a zero vector in the linear space X. For the sequence space X and Y, define the set

$$S(X : Y) = \{z = (z_k) : xz = (x_k z_k) \in Y\}. \tag{1}$$

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With the notation of (1) , the α-, β- and γ- duals of a sequence space X, which are respectively denoted by X^{α}, X^{β} and X^{γ} and are defined by

$$X^{\alpha} = S(X : l_1),\ X^{\beta} = S(X : cs) \text{ and } X^{\gamma} = S(X : bs).$$

If a sequence space X paranormed by h contains a sequence (b_n) with the property that for every $x \in X$ there is a unique sequence of scalars (α_n) such that

$$\lim_n h(x - \sum_{k=0}^{n} \alpha_k b_k) = 0$$

then (b_n) is called a Schauder basis (or briefly basis) for X. The series $\sum \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_n) and written as $x = \sum \alpha_k b_k$.

Let X, Y be two sequence spaces and let $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in \mathbb{N}$. Then, the matrix A defines the A-transformation from X *into* Y, if for every sequence $x = (x_k) \in X$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x exists and is in Y; where $(Ax)_n = \sum_k a_{nk} x_k$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. By $A \in (X : Y)$ we mean the characterizations of matrices from X to Y i.e. $A \colon X \to Y$. A sequence x is said to be A-summable to l if Ax converges to l which is called as the A-limit of x.

For a sequence space X, the matrix domain X_A of an infinite matrix A is defined as

$$X_A = \{x = (x_k) : x = (x_k) \in \omega\}. \tag{2}$$

An approach of constructing of a new sequence space by means of the matrix domain of a particular limitation method was used by Stieglitz and Tietz [21] , Ng and Lee [16], Malkowsky [10, 11], Başar et al [1, 14, 3, 4], Hamid et al [5, 6, 18, 19]. The matrix domain of Ceşaro mean C_1 of order 1, Reisz mean R_p and Nörlund mean N_q in the sequence space l_p, l_∞, c and c_0 are examined and characterized by some matrix transformations related to those spaces by several authors viz.[4, 13, 12, 16, 17, 18, 19]. Recently, Malkowsky and Savas [11] have generalized those results by the weighted mean method which is extended to the paranormed case by Altay and Başar [1, 14]. Although, the Euler mean of order r which is not a special case of weighted mean method is one of the important methods in summability theory, the matrix domains of those methods has not been studied until now.

In this paper, we introduce the Euler sequence space $e_u^r(p)$ of non absolute type which are the matrix domains of Euler mean E^r of order r, $0 < r < 1$ in the sequence space l_p and l_∞ respectively. Also, we give some inclusion relations and determine its α-,β- and γ-duals of $e_u^r(p)$. Furthermore, we construct the basis of $e_u^r(p)$. In the final section, we characterize the classes the classes $(e_u^r(p) : l_p)$ and $(e_u^r(p) : f)$ of infinite matrices.

Let l_∞ and c be Banach spaces of bounded and convergent sequences $x = \{x_n\}_{n=0}^{\infty}$ with supremum norm $\|x\| = \sup_n |x_n|$. Let T denote the shift operator on ω, that is, $Tx = \{x_n\}_{n=1}^{\infty}$, $T^2x = \{x_n\}_{n=2}^{\infty}$ and so on. A Banach limit L is defined on l_∞ as a non-negative linear functional such that L is invariant *i.e.*, $L(Sx) = L(x)$ and $L(e) = 1,\ e = (1, 1, 1, \ldots)$ [2, 7].

Lorentz [7], called a sequence $\{x_n\}$ almost convergent if all Banach limits of x, $L(x)$, are same and this unique Banach limit is called F-limit of x. In his paper, Lorentz proved the following criterion for almost convergent sequences.

A sequence $x = \{x_n\} \in l_\infty$ is almost convergent with F-limit $L(x)$ if and only if

$$\lim_{m\to\infty} t_{mn}(x) = L(x)$$

where

$$t_{mn}(x) = \frac{1}{m}\sum_{j=0}^{m-1} T^j x_n,\ (T^0 = 0)$$

uniformly in $n \geq 0$.

We denote the set of almost convergent sequences by f.

2. Euler sequence spaces of non absolute type

In this section, we define the Euler sequence space $e_u^r(p)$ of non absolute type and show it is linearly isomorphic to the space l_p. We also determine the α-,β- and γ-duals of $e_u^r(p)$ and finally in this section we give basis for the space $e_u^r(p)$, where $u = (u_k) \neq 0$ for all $k \in \mathbb{N}$.

The sequence space $e_u^r(p)$ is defined as follows:

$$e_u^r(p) = \left\{ x = (x_k) \in \omega : \sum_n \left| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k u_k x_k \right| < \infty \right\}$$

where E_u^r denotes the method of Euler means of order r, defined by the matrix $E_u^r = (e_{nk}^r)$,

$$e_{nk}^r = \begin{cases} \binom{n}{k} (1-r)^{n-k} r^k u_k, & \text{if } 0 \leq k \leq n, \\ 0, & \text{if } k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$. It is known that the method E^r is regular for $0 < r < 1$ [15] and we assume unless otherwise stated $0 < r < 1$.

With the definition of matrix domain (2), we can redefine the space $e_u^r(p)$ as:

$$e_u^r(p) = (l_p)_{E_u^r};\ (1 \leq p < \infty). \tag{3}$$

Let us define the sequence $y = \{y_k(r)\}$, which will be frequently used as the E_u^r-transform of a sequence $x = \{x_k\}$, i.e.,

$$y_k(r) = \sum_{j=0}^{k} \binom{n}{k} (1-r)^{k-j} r^j u_j x_j, \ k \in \mathbb{N}. \tag{4}$$

Now, we may begin with the following theorem which is essential in the text.

Theorem 1. *The set $e_u^r(p)$ becomes a linear space with the coordinate wise addition and scalar multiplication which is the BK-space with the norm*

$$\|x\|_{e_u^r(p)} = \|E_u^r x\|_{l_p}, \ \textit{where } 1 \leq p \leq \infty.$$

Proof. The first part of the theorem is a routine verification and so we omit the detail. Furthermore, since (3) holds and l_p and l_∞ are BK- spaces with respect to their usual norms and the matrix $E_u^r = (e_{nk}^r)$ is a triangle matrix, i.e., $e_{nn}^r \neq 0$ and $e_{nk}^r = 0$ for $k > n$ for all $n, k\mathbb{N}$. Therefore, the space $e_u^r(p)$ is a BK-space [21], hence the proof is complete. □

Remark 1. The absolute property does not hold on the space $e_u^r(p)$, i.e., there exists a sequence x such that $\|x\|_{e_u^r(p)} \neq \||x|\|_{e_u^r(p)}, \ 1 \leq p \leq \infty$.

Theorem 2. *The Euler sequence space $e_u^r(p)$ of non absolute type is linearly isomorphic to the space l_p, i.e., $e_u^r(p) \cong l_p$; where $1 \leq p < \infty$.*

Proof. To prove the result we should show the existence of a linear bijection between the space $e_u^r(p)$ and l_p for $1 \leq p < \infty$. Consider the transformation T defined with the notation of (4) from $e_u^r(p)$ to l_p by $x \to y = Tx$. The linearity of T is trivial. Further, it is trivial that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let $y = l_p$ for $1 \leq p < \infty$ and define the sequence $x = \{x_k(r)\}$ by

$$x_k(r) = \sum_{j=0}^{k} \binom{k}{j} \frac{(1-r)^{k-j} r^{-k} y_j}{u_k}, \ k \in \mathbb{N}.$$

Then, we respectively obtain in the cases of $1 \leq p < \infty$ that

$$\|x\|_{e_u^r(p)} = \left[\sum_n \left| \sum_{k=0}^{n} \binom{n}{k} (1-r)^{n-k} r^k u_k x_k(r) \right|^p \right]^{\frac{1}{p}}$$

Thus, we have that $x \in e_u^r(p)$. Consequently, T is surjective and is norm preserving, where $1 \leq p < \infty$. Hence, T is linear bijection which implies that the spaces $e_u^r(p)$ and l_p are linearly isomorphic for $1 \leq p < \infty$. Hence the result is proved. □

Theorem 3. *Except the case $p = 2$, the space $e_u^r(p)$ is not an inner product space, hence not a Hilbert space for $1 \leq p < \infty$.*

Proof. We wish to prove that the space $e_u^r(p)$ is the only Hilbert space among the $e_u^r(p)$ spaces for $1 \leq p < \infty$. Since, the space $e_u^r(2)$ is a BK-space with the norm $\|x\|_{e_u^r(p)} = \|E_u^r x\|_{l_2}$, by Theorem 1 and its norm can be obtained from an inner product, i.e.,

$$\|x\|_{e_u^r(p)} = \langle E_u^r x, E_u^r x\rangle^{\frac{l}{2}}$$

holds, the space $e_u^r(2)$ is a Hilbert space.

To prove this, let us take $u = e = (1, 1, \ldots)$ and consider the sequence

$$v = \{v_k(r)\} = \left\{\frac{r+k-1}{r}\left(1-\frac{1}{r}\right)^{k-1}\right\}$$

and

$$w = \{w_k(r)\} = \left\{\frac{r-k-1}{r}\left(1-\frac{1}{r}\right)^{k-1}\right\}.$$

Then, one can easily see that

$$\|v+w\|^2_{e_u^r(p)} + \|v-w\|^2_{e_u^r(p)} = 8 \neq 4(2^{\frac{2}{p}}) = 2\left(\|v\|^2_{e_u^r(p)} + \|w\|^2_{e_u^r(p)}\right); p \neq 2,$$

i.e., the norm of the space $e_u^r(p)$ does not satisfy the parallelogram equality which means that the norm cannot be obtained from an inner product. Hence, the space $e_u^r(p)$ with $p \neq 2$ is a Banach space which is not a Hilbert space. This completes the proof. □

3. The inclusion relations

In this section, we prove some inclusion relations concerning the space $e_u^r(p)$.

Theorem 4. *The inclusion $l_p \subset e_u^r(p)$ strictly holds for $1 \leq p < \infty$.*

Proof. To prove the validity of the inclusion $l_p \subset e_u^r(p)$ for $1 \leq p < \infty$, it suffices to show the existence of a number $M > 0$ such that

$$\|x\|_{e_u^r(p)} \leq M\|x\|_{l_p} \text{ for every } x \in l_p.$$

Let us take any $x \in l_p$. Then we obtain, with the notation of (4), by applying the Hölders inequality for $1 < p < \infty$ that

(5)

$$\begin{aligned} |y_k(r)|^p &= \left|\sum_{j=0}^{k}\binom{k}{j}(1-r)^{k-j}r^j u_j x_k\right|^p \\ &\leq \left[\sum_{j=0}^{k}\binom{k}{j}(1-r)^{k-j}r^j u_j\,|x_k|^p\right] \times \left[\sum_{j=0}^{k}\binom{k}{j}(1-r)^{k-j}r^j u_j\,|x_k|^{p-1}\right] \\ &= \sum_{j=0}^{k}\binom{k}{j}(1-r)^{k-j}r^j u_j\,|x_k|^p. \end{aligned}$$

Thus, we have by using the fact in (5) that

$$\sum_k |y_k(r)|^p \leq \sum_k \sum_{j=0}^{k} \binom{k}{j} (1-r)^{k-j} r^j u_j \, |x_k|^p$$
$$= \sum_k |x_k|^p \sum_{k=j}^{\infty} \binom{k}{j} (1-r)^{k-j} r^j u_j = \frac{1}{r} \sum_k |x_k|^p \, ,$$

which yields that

$$\|x\|_{e_u^r(p)} \leq \left(\frac{1}{r}\right)^{\frac{1}{p}} \|x\|_{l_p}, \tag{6}$$

for $1 < p < \infty$, as expected. Besides, let us consider the sequence $v = \{v_k(r)\}$ defined by $v_k(r) = (-r)^{-k}$ for all $k \in \mathbb{N}$. Then, since $E_u^r v = \{(-r)^k\} \in l_p$, we immediately observe that v is in $e_u^r(p)$ but not in l_p. Because, there is at least one sequence in $e_u^r(p) - l_p$, the inclusion $l_p \subset e_u^r(p)$ is strict. By the similar discussion, it may be easily proved that the inequality (6) also holds in the case $p = 1$ and so we omit the detail. This completes the proof. □

Theorem 5. *Neither of the spaces $e_u^r(p)$ and l_∞ includes the other one, where $1 \leq p < \infty$.*

Proof. It is trivial by Theorem 5 that the sequences spaces $e_u^r(p)$ and l_∞ are not disjoint. Let us now consider the sequences $v = \{v_k(r)\}$ defined as in the proof of Theorem 5 above, and $x = e = (1, 1, \ldots)$. Then $v \in e_u^r(p)$- l_∞ and $x \in l_\infty$ - $e_u^r(p)$. Hence the sequence spaces $e_u^r(p)$ and l_∞ overlap but neither contains the other. This completes the proof. □

4. Duals and basis for the space $e_u^r(p)$

In this section, we state and prove theorems determining α-, β- and γ-duals of the Euler sequence spaces $e_u^r(p)$ of non-absolute type and then we give the basis for the space $e_u^r(p)$.

We begin with quoting lemmas, due to Stieglitz and Tietz [21], that are needed in proving the theorems dealt with in sections (4) and (5). We denote the collection of all finite subsets of $\mathbb{N}$ by $\mathbb{F}$.

Lemma 1. *$A \in (l_p : l_1)$ if and only if*
(i)

$$\sup_{N \in \mathbb{F}} \sum_{k \in \mathbb{N}} \left| \sum_{n \in N} a_{nk} \right|^q < \infty, \ (1 < p \leq \infty). \tag{7}$$

(ii)

$$\sup_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} |a_{nk}| < \infty, \ (p = 1). \tag{8}$$

Lemma 2. *$A \in (l_p : c)$ if and only if*
(i) for $1 < p < \infty$,

$$\lim_{k \in \infty} a_{nk} \text{ exists for each } k \in \mathbb{N}, \tag{9}$$

and

$$\sup_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} |a_{nk}|^q < \infty. \tag{10}$$

(ii) For $p = 1$, (9) holds and

$$\sup_{n,k \in \mathbb{N}} |a_{nk}| < \infty. \tag{11}$$

Lemma 3. *$A \in (l_p : l_\infty)$ if and only if (10) holds.*

Lemma 4. *$A \in (l_\infty : c)$ if and only if (9) holds and*

$$\lim_{n \to \infty} \sum_{k \in \mathbb{N}} |a_{nk}| = \sum_{k \in \mathbb{N}} \left| \lim_{n \to \infty} a_{nk} \right|. \tag{12}$$

Lemma 5. *$A \in (l_1 : l_p)$ if and only if*

$$\sup_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} |a_{nk}|^p < \infty, \ (1 \leq p < \infty). \tag{13}$$

Now we give the theorems determining the α-, β- and γ-duals of the Euler sequence spaces.

Theorem 6. *Define the sets $a_q^r(u)$ and $a_\infty^r(u)$ as follows:*

$$a_q^r(u) = \left\{ a = (a_k) \in \omega : \ \sup_{N \in F} \sum_k \left| \sum_{n \in N} \binom{n}{k} \frac{(r-1)^{n-k} r^{-n} a_n}{u_k} \right|^q < \infty \right\}$$

and

$$a_\infty^q(u) = \left\{ a = (a_k) \in \omega : \ \sup_k \sum_n \left| \binom{n}{k} \frac{(r-1)^{n-k} r^{-n} a_n}{u_k} \right|^q < \infty \right\}.$$

Then

$$(e_u^r(1))^\alpha = a_q^r(u) \text{ and } (e_u^r(p))^\alpha = a_\infty^r(u) \text{ where } 1 < p < \infty.$$

Proof. We first prove the part second. Let us define the matrix B_u^r whose rows are the products of the rows of the matrix $E_u^{\frac{1}{r}}$ with the sequence $a = (a_n)$. Therefore, we easily obtain by bearing in mind the relation (4) that

$$a_n x_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(r-1)^{n-k} r^{-n} a_n y_k}{u_n} = (B_u^r y)_n, \ n \in \mathbb{N}. \tag{14}$$

Thus, we observe by (14) that $ax = (a_n x_n) \in l_1$ whenever $x \in e_u^r(p)$ if and only if $B_u^r y \in l_1$ whenever $y \in l_p$. This means that $a = (a_n) \in (e_u^r(p))^\alpha$ if and only if $B_u^r \in (l_p : l_1)$. Then, we derive by lemma (7)-(i) with B_u^r- instead of A that

$$\sup_{N\in F}\sum_k \left|\sum_{n\in N}\binom{n}{k}\frac{(r-1)^{n-k}r^{-n}a_n}{u_k}\right|^q < \infty.$$

This yields $(e_u^r(p))^\alpha = a_q^r(u)$.

Now to prove part first, we observe by (14) that $ax = (a_kx_k) \in l_1$ whenever $x \in e_u^r(1)$ if and only if $B_u^r y \in l_1$ whenever $y \in l_1$. This means that $a = (a_n) \in (e_u^r(1))^\alpha$ if and only if $B_u^r \in (l_1 : l_1)$. Then, we derive by lemma (7)-(i) with B_u^r- instead of A that

$$\sup_k \sum_n \left|\binom{n}{k}\frac{(r-1)^{n-k}r^{-n}a_n}{u_n}\right| < \infty.$$

This yields $(e_u^r(p))^\alpha = a_\infty^r(u)$. □

Theorem 7. *Define the sets $d_1^r(u)$, $d_2^r(u)$, $d_3^r(u)$ and $d_4^r(u)$ as follows:*

$$d_1^r(u) = \left\{a=(a_k)\in\omega : \sum_{j=k}^{\infty}\binom{j}{k}\frac{(r-1)^{j-k}r^{-j}a_j}{u_j} \text{ exists for each } k\right\},$$

$$d_2^r(u) = \left\{a=(a_k)\in\omega : \sup_{n,k}\left|\sum_{j=k}^{n}\binom{j}{k}\frac{(r-1)^{j-k}r^{-j}a_j}{u_j}\right| < \infty\right\},$$

$$d_3^r(u) = \{a=(a_k)\in\omega : \lim_n \sum_k \left|\sum_{j=k}^{n}\binom{j}{k}\frac{(r-1)^{j-k}r^{-j}a_j}{u_j}\right| = \sum_k \left|\sum_{j=k}^{\infty}\binom{j}{k}\frac{(r-1)^{j-k}r^{-j}a_j}{u_j}\right| < \infty\},$$

$$d_4^r(u) = \left\{a=(a_k)\in\omega : \sup_n \sum_{j=0}^{n}\left|\sum_{j=k}^{n}\binom{j}{k}\frac{(r-1)^{j-k}r^{-j}a_j}{u_j}\right| < \infty\right\},$$

and

$$b_q^r(u) = \left\{a=(a_k)\in\omega : \sup_n \sum_{j=0}^{n}\left|\sum_{j=k}^{n}\binom{j}{k}\frac{(r-1)^{j-k}r^{-j}a_j}{u_j}\right|^q < \infty\right\},\ (1<q<\infty)$$

Then
(i) $(e_u^r(1))^\beta = d_1^r(u)\cap d_2^r(u)$,
(ii) $(e_u^r(p))^\beta = d_1^r(u)\cap b_q^r(u)$, $(1<q<\infty)$.

Proof. (ii) Consider the equation

$$\sum_{k=0}^{n} a_kx_k = \sum_{k=0}^{n}\left[\sum_{j=0}^{k}\binom{k}{j}\frac{(r-1)^{j-k}r^{-j}y_j}{u_k}\right]a_k$$

$$= \sum_{k=0}^{n}\left[\sum_{j=k}^{n}\binom{n}{k}\frac{(r-1)^{j-k}r^{-j}a_j}{u_j}\right]y_k = (T_u^r y)_n \tag{15}$$

where $T_u^r = (t_{nk}^r)$ is defined by

$$t_{nk}^r = \begin{cases} \sum_{j=k}^{n}\binom{j}{k}\frac{(r-1)^{j-k}r^{-j}a_j}{u_j}, & \text{if } 0 \le k \le n, \\ 0, & \text{if } k > n \end{cases} \tag{16}$$

for all $k, n \in \mathbb{N}$. Thus, we deduce from lemma (8) with (15) that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in e_u^r(p)$ if and only if $T_u^r y \in c$ whenever $y = (y_k) \in l_p$. That is to say that $a = (a_k) \in (e_u^r(p))^\beta$ if and only if $T_u^r \in (l_p : c)$. Therefore, we derive from (9) and (10) that

$$\sum_{j=k}^{n}\binom{j}{k}\frac{(r-1)^{j-k}r^{-j}a_j}{u_j} \text{ exists for each } k \in \mathbb{N}$$

and

$$\sup_n \sum_{k=0}^{n}\left|\sum_{j=k}^{n}\binom{j}{k}\frac{(r-1)^{j-k}r^{-j}a_j}{u_j}\right|^q < \infty,$$

which shows that

$$(e_u^r(p))^\beta = d_1^r(u) \cap b_q^r(u), \ (1 < p < \infty).$$

(i) By combining lemma (8)-(ii) with (15) that () $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in e_u^r(1)$ if and only if $T_u^r y \in c$ whenever $y = (y_k) \in l_1$. That is to say that $a = (a_k) \in (e_u^r(1))^\beta$ if and only if $T_u^r \in (l_1 : c)$. Therefore, we derive from (9) and (11) that

$$\sum_{j=k}^{\infty}\binom{j}{k}\frac{(r-1)^{j-k}r^{-j}a_j}{u_j} \text{ exists for each } k \in \mathbb{N}$$

and

$$\sup_{n,k}\left|\sum_{j=k}^{n}\binom{j}{k}\frac{(r-1)^{j-k}r^{-j}a_j}{u_j}\right| < \infty,$$

which shows that

$$(e_u^r(1))^\beta = d_1^r(u) \cap d_2^r(u).$$

□

Theorem 8. *The γ-duals of the spaces $e_u^r(p)$ are as:*

$$(e_u^r(1))^\gamma = d_2^r(u) \text{ and } (e_u^r(p))^\gamma = b_q^r(u), \ (1 < q < \infty).$$

Proof. It is natural that the present theorem may be proved by the same technique used n the proofs of Theorems 12 and 13 above. □

We now construct the basis for the space $e_u^r(p)$. Since the isomorphism T defined in the proof of the Theorem 3 is onto; the inverse image of the basis $\{e^(k)\}_k$ of the space l_p is the basis of the new space $e_u^r(p)$. Therefore, we have the following theorem:

Theorem 9. *Define the sequence $b_u^k(r) = \{b_{n,u}^(k)(r)\}_n$ of the elements of the space $e_u^r(p)$ for every fixed $k \in \mathbb{N}$ by*

$$b_{n,u}^{(k)} = \begin{cases} \binom{n}{k}\frac{(r-1)^{n-k}r^{-n}}{u_n}, & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases} \tag{17}$$

Then, the sequence $\{b_u^{(k)}(r)$ is a basis for the space $e_u^r(p)$ and any $x \in e_u^r(p)$ has a unique representation of the form

$$x = \sum_k \lambda_k b_u^{(k)} \tag{18}$$

where $\lambda_k(r) = (E^r x)_k$ for all $k \in \mathbb{N}$ and $1 \leq p < \infty$.

Proof. It is a clear $b_u^{(k)}(r) \subset e_u^r(p)$ because

$$E_u^r b_u^{(r)}(r) = e^{(k)} \in l_p, \tag{19}$$

where $e^{(k)}$ is the sequence whose only non-zero term is a 1 at the kth place for each $k \in \mathbb{N}$.

Let $x \in e_u^r(p)$ be given. For every non-negative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(q) b_u^{(k)}(r). \tag{20}$$

Then, we obtain by applying R_u^r to (20) with (19) that

$$E_u^r x^{[m]} = \sum_{k=0}^{m} \lambda_k(r) E_u^r b^{(k)}(r) = \sum_{k=0}^{m} (E_u^r x)_k e^{(k)}$$

and

$$\left(E_u^r(x - x^{[m]})\right)_i = \begin{cases} 0, & \text{if } 0 \leq i \leq m, \\ (E_u^r x)_i\,, & \text{if } i > m, \end{cases}$$

where $i, m \in \mathbb{N}$. Given $\varepsilon > 0$, there exists an integer m_0 such that

$$\left(\sum_{i=m}^{\infty} |(E_u^r x)_i|^p\right)^{\frac{1}{p}} < \frac{\varepsilon}{2},$$

for all $m \geq m_0$. Hence,

$$g\left\|x - x^{[m]}\right\|_{e_u^r(p)} = \left(\sum_{i=m}^{\infty} |(E_u^r x)_i|^p\right)^{\frac{1}{p}}$$

$$\leq \left(\sum_{i=m_0}^{\infty} |(E_u^r x)_i|^p \right)^{\frac{1}{p}} < \frac{\varepsilon}{2} < \varepsilon,$$

for all $m \geq m_0$, which proves that $x \in r^q(u,p)$ is represented as (18).

Let us show the uniqueness of the representation for $x \in e_u^r(p)$ given by (18). Suppose, on the contrary; that there exists a representation $x = \sum_k \mu_k(r) b^k(r)$. Since the linear transformation T from $e_u^r(p)$ to l_p used in the Theorem 3 is continuous, we have

$$\begin{aligned}(E_u^r x)_n &= \sum_k \mu_k(r) \left(E_u^r b^k(r) \right)_n \\ &= \sum_k \mu_k(r) e_n^{(k)} = \mu_n(r)\end{aligned}$$

for $n \in \mathbb{N}$, which contradicts the fact that $(E_u^r x)_n = \lambda_n(r)$ for all $n \in \mathbb{N}$. Hence, the representation (18) is unique. This completes the proof. □

5. Matrix mappings related to Euler sequence spaces

In this section, we characterize the matrix mappings on the Euler sequence spaces $e_u^r(p)$ into some known sequence spaces.

For brevity in notation, we shall write

$$\widehat{a}_{nk} = \sum_{j=k}^{\infty} \binom{j}{k} \frac{(r-1)^{j-k} r^{-j} a_{nj}}{u_j}, \;\; n,k \in \mathbb{N}.$$

Theorem 10. *(i) : $A \in (e_u^r(1) : l_\infty)$ if and only if*

$$\sup_{n,k} |\widehat{a}_{nk}| < \infty. \tag{21}$$

(ii): Let $1 < p < \infty$. Then $A \in (e_u^r(p) : l_\infty)$ if and only if

$$\sup_{n,k} |\widehat{a}_{nk}|^q < \infty, \tag{22}$$

and

$$\{a_{nk}\} \in b_q^r(u), \;\; n \in \mathbb{N}. \tag{23}$$

Proof. (ii) Suppose that the conditions (22) and (23) hold and take any $x \in e_u^r(p)$. Then, the sequence $\{a_{nk}\}_k \in (e_u^r(p))^\beta$ for all $n \in \mathbb{N}$ and this implies the existence of the A-transform of x.

Let us consider the following equality derived by using the relation (4) from the m^{th} partial sum of a the series $\sum_k a_{nk} x_k$.

$$\sum_{k=0}^{m} a_{nk} x_k = \sum_{k=0}^{m} \sum_{j=k}^{m} \binom{j}{k} \frac{(r-1)^{j-k} r^{-j} a_{nj}}{u_j} y_k; \;\; n,m \in \mathbb{N}. \tag{24}$$

Therefore, we obtain from (24) as $m \to \infty$ that

$$\sum_k a_{nk}x_k = \sum_k \widehat{a}_{nk}y_k; \ n \in \mathbb{N}. \tag{25}$$

Now, by passing to the supremum over n in (25), we derive by applying Hölder's inequality that

$$\begin{aligned}\|Ax\|_{l_\infty} &= \sup_n \left| \sum_k a_{nk}x_k = \sum_k \widehat{a}_{nk}y_k \right| \\ &\leq \sup_n \left(\sum_k |\widehat{a}_{nk}|^q \right)^{\frac{1}{q}} \left(\sum_k |y_k|^p \right)^{\frac{1}{p}} < \infty,\end{aligned}$$

which shows the sufficiency of the conditions (22) and (23).

Conversely, we suppose that $A \in (e_u^r(p) : l_\infty)$. Then, since $\{a_{nk}\}_k \in (e_u^r(p))^\beta$ for all $n \in \mathbb{N}$, the necessity of (23) is immediate and $\{\widehat{a}_{nk}\}$ exists. Since, $\{a_{nk}\}_k \in (e_u^r(p))^\beta$, (25) holds and the sequences $a_n = \{a_{nk}\}_k$ define the continuous linear functionalf_n on $e_u^r(p)$ by

$$f_n(x) = \sum_k a_{nk}x_k; \ n, k \in \mathbb{N}.$$

Since, $e_u^r(p)$ and l_p are norm isomorphic, Th. (3), it should follow with (25) that

$$\|f_n\| = \|\widehat{a}_n\|_q.$$

This just says that the functional defined by the rows of A on $e_u^r(p)$ are point wise bounded. Hence, by Banach-Steinhaus Theorem, they are uniformly bounded, which yields that there exists a constant $M > 0$ such that

$$\|f_n\| \leq M, \ \forall \ n \in \mathbb{N}.$$

It therefore follows, using the complete identification just referred to, that

$$\left(\sum_k |\widehat{a}_{nk}| \right)^{\frac{1}{q}} = \|f_n\| \leq M$$

holds for all $n \in \mathbb{N}$, which shows the necessity of the condition (23). This completes the proof. □

Proof of (i) follows similarly as part (ii) above.

Theorem 11. *$A \in (e_u^r(1) : l_p)$ if and only if*

$$\sup_k \left| \sum_k \widehat{a}_{nk} \right|^p < \infty. \tag{26}$$

Proof. Suppose that the condition (26) hold and $x \in e_u^r(p)$. Then, $y \in l_1$ and $\{a_{nk}\} \in (e_u^r(p))^{\beta}$ for all $n \in \mathbb{N}$, i.e., the A-transform of x exists. Hence, the series $\sum_k \widehat{a}_{nk} y_k$ converges absolutely for every fixed $n \in \mathbb{N}$ and each $y \in l_1$. Therefore, we obtain from (25) by applying Minkowski's inequality that

$$\left[\sum_n |(Ax)_n|^p\right]^{\frac{1}{p}} \leq \sum_k |y_k| \left[\sum_n |\widehat{a}_{nk}|^p\right]^{\frac{1}{p}} < \infty,$$

which shows that $Ax \in l_p$.

Conversely, suppose that $A \in (e_u^r(p) : l_p)$ and $1 < p < \infty$. Then, since Ax exists and is in the space l_p for all $x \in e_u^r(p)$, $\{a_{nk}\} \in (e_u^r(p))^{\beta}$ for all $n \in \mathbb{N}$, which leads us to the relation (25). Let us define the matrix $B = (b_{nk})$ by $b_{nk} = \widehat{a}_{nk}$ for all $n, k \in \mathbb{N}$. Therefore, one can easily see that $B \in (l_1 : l_p)$ and hence B satisfies the condition (13) of lemma () which yields the necessity of (26). This completes the proof. □

Theorem 12. *(i): $A \in (e_u^r(p) : f)$ if and only if (22) and*

$$f - \lim \widehat{a}_{nk} = \alpha_k, \ (n \in \mathbb{N}). \tag{27}$$

(ii): Let $1 < p < \infty$. Then, $A \in (e_u^r(p) : f)$ if and only if (23), (24) and (27) hold.

Proof. (ii): Suppose the conditions (23), (24) and (27) hold and $x \in e_u^r(p)$. Then Ax exists and we have by (27) that

$$|\widehat{a}(n,k,m)|^q \to |\alpha_k|^q \ as \ m \to \infty,$$

uniformly in n and for each $k \in \mathbb{N}$, which leads us with (23) to the inequality

$$\begin{aligned} \sum_{j=0}^{k} |\alpha_j|^q &= \lim_{m\to\infty} \sum_{j=0}^{k} |\widehat{a}(n,j,m)|^q, \ (uniformly \ in \ n) \\ &\leq \sup_{m,n} \sum_j |\widehat{a}(n,j,m)|^q = M < \infty \end{aligned}$$

holding for every $k \in \mathbb{N}$. This gives that $(\alpha_k) \in l_q$. Since, $x \in e_u^r(p)$ by the hypothesis and $e_u^r(p) \cong l_p$, we have $y \in l_p$. Therefore, we derive by applying Hölder's inequality that $(\alpha_k y_k) \in l_1$ for each $y \in l_p$. For any given $\epsilon > 0$, choose a fixed $k_0 \in \mathbb{N}$ such that

$$\left(\sum_{k=k_0+1}^{\infty} |y_k|^p\right)^{\frac{1}{p}} < \frac{\epsilon}{4M^{\frac{1}{q}}}.$$

Then, there is some $m_0 \in \mathbb{N}$ by () such that

$$\left|\sum_{k=0}^{k_0} (\widehat{a}(n,k,m) - \alpha_k)\, y_k\right| < \frac{\epsilon}{2},$$

for every $m \geq m_0$, uniformly in n. Therefore, we have

$$\begin{aligned}
\left|\frac{1}{m+1}\sum_{i-0}^{m}(Ax)_{n+i} - \sum_k \alpha_k y_k\right| &= \left|\sum_k \left(\widehat{a}(n,k,m) - \alpha_k\right) y_k\right| \\
&\leq \left|\sum_{k=0}^{k_0} \left(\widehat{a}(n,k,m) - \alpha_k\right) y_k\right| + \left|\sum_{k=k_0+1}^{\infty} \left(\widehat{a}(n,k,m) - \alpha_k\right) y_k\right| \\
&< \frac{\epsilon}{2} + \left(\sum_{k=k_0+1}^{\infty} \left[|\widehat{a}(n,k,m)| + |\alpha_k|\right]^q\right)^{\frac{1}{q}} \left(\sum_{k=k_0+1}^{\infty} |y_k|^p\right)^{\frac{1}{p}} \\
&< \frac{\epsilon}{2} + 2M^{\frac{1}{q}} \frac{\epsilon}{4M^{\frac{1}{q}}} = \epsilon,
\end{aligned}$$

for all sufficiently large m uniformly in n. Hence, $Ax \in f$. This proves the sufficiency.

Conversely, suppose that $A \in (e_u^r(p) : f)$. Then, since $f \subset l_\infty$, the necessities of (23) and (24) are immediately obtained from Theorem 16. To prove the necessity of (27), consider the sequence $b^{(k)}(r) = \{b_n^{(k)}(r)\}_n \in e_u^r(p)$ for every $k \in \mathbb{N}$, where

$$b_n^k(r) = \begin{cases} \binom{n}{k}\frac{(r-1)^{n-k}r^{-n}}{u_n}, & \text{if } n \geq k, \\ 0, & \text{if } n < k. \end{cases}$$

Since Ax exists and is in f for each $x \in e_u^r(p)$, one can easily see that

$$Ab^{(k)}(r) = \left\{\sum_{j=k}^{\infty} \binom{n}{k} \frac{(r-1)^{j-k} r^{-j} a_{nj}}{u_j}\right\} \in f$$

for all $k \in \mathbb{N}$, which proves the necessity of (27). This concludes the proof of part (ii). Proofs of (i) can be proved in a similar fashion and the proof is complete. □

If f- limit is replaced by the ordinary limit in the Theorem 18, then we have:

Corollary 1. *(i): $A \in (e_u^r(1) : c)$ if and only if (22) and*

$$\lim_n \widehat{a}_{nk} = \alpha_k. \tag{28}$$

(ii) Let $1 < p < \infty$. Then, $A \in (e_u^r(p) : c)$ if and only if (23), (24) and (28) hold.

(iii) Let $1 < p < \infty$. Then, $A \in (e_u^r(p) : c_0)$ if and only if (23), (24) and (28) hold along with $\alpha_k = 0$ for each $k \in \mathbb{N}$.

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