Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
30 (2014), 91–101
www.emis.de/journals
ISSN 1786-0091

ON ALMOST EVERYWHERE CONVERGENCE OF SOME SUB-SEQUENCES OF FEJÉR MEANS FOR INTEGRABLE FUNCTIONS ON UNBOUNDED VILENKIN GROUPS

NACIMA MEMIĆ

ABSTRACT. By means of Gát's methods in [2] our aim is to prove the almost everywhere convergence of some sub-sequences of $(\sigma_n f)_n$ to f, for every integrable function f on unbounded Vilenkin groups. These are in fact sub-sequences of the form $(\sigma_{a_n M_n} f)_n$, where the numbers a_n are bounded. This result can be considered as a generalization of Gát's result concerning the almost everywhere convergence of the sequence $(\sigma_{M_n} f)_n$ on unbounded Vilenkin groups for every integrable function f.

1. Introduction

Many results concerning the a.e. convergence of the Fejér means $(\sigma_n f)_n$ have been obtained for Vilenkin groups. On bounded groups, mean convergence holds almost everywhere for integrable functions [4]. However, using different methods on unbounded groups, G. Gát [1] proved this result for L^p functions when $p > 1$, and obtained in [2] that $\sigma_{M_n} f \to f$, a.e. for every integrable function f. The same author [3] established the mean convergence almost everywhere of the full sequence for integrable functions on rarely unbounded groups. In the present paper we establish the almost everywhere convergence of sub-sequences of the form $(\sigma_{a_n M_n} f)_n$, where the numbers a_n are bounded, to the integrable function f.

Let $(m_0, m_1, \dots, m_n, \dots)$ be an unbounded sequence of integers not less than 2. We denote by $\mathbb{P}$ the set of positive integers and let $\mathbb{N} = \mathbb{P} \cup \{0\}$. Let $G := \prod_{n=0}^{\infty} \mathbb{Z}_{m_n}$, where $\mathbb{Z}_{m_n}$ denotes the discrete group of order m_n, with addition *mod* m_n. Each element from G can be represented as a sequence $(x_n)_n$, where $x_n \in \{0, 1, \dots, m_n - 1\}$, for every integer $n \geq 0$. Addition in G is obtained coordinatewise.

2010 *Mathematics Subject Classification.* 42C10.
Key words and phrases. Unbounded Vilenkin groups, Fejér means, almost everywhere convergence.

The topology on G is generated by the subgroups $I_n := \{x = (x_i)_i \in G, x_i = 0 \text{ for } i < n\}$, and their translations $I_n(y) := \{x = (x_i)_i \in G, x_i = y_i \text{ for } i < n\}$. Sometimes we write $I_n(y)$ in the form $I_n(y) = I_n(y_0, \dots, y_{n-1})$.

Define the sequence $(M_n)_n$ as follows: $M_0 = 1$ and $M_{n+1} = m_n M_n$.

If $\mu(I_n)$ denotes the normalized product measure of I_n then it can be easily seen that $\mu(I_n) = M_n^{-1}$.

The generalized Rademacher functions are defined by

$$r_n(x) := e^{\frac{2\pi i x_n}{m_n}}, n \in \mathbb{N}, x \in G,$$

For every non-negative integer n, there exists a unique sequence $(n_i)_i$ so that $n = \sum_{i=0}^{\infty} n_i M_i$.

and the system of Vilenkin functions by

$$\psi_n(x) := \prod_{i=0}^{\infty} r_i^{n_i}(x), \quad n \in \mathbb{N}, x \in G.$$

The Fourier coefficients, the partial sums of the Fourier series, the mean values, the Dirichlet kernels, the Fejér means, and the Fejér kernels with respect to the Vilenkin system are respectively defined as follows

$$\hat{f}(n) = \int f(x)\bar{\psi}_n(x)dx, \quad S_n f = \sum_{k=0}^{n-1} \hat{f}(k)\psi_k, \quad E_n(f) = S_{M_n} f,$$

$$D_n = \sum_{k=0}^{n-1} \psi_k, \quad \sigma_n f = \frac{1}{n}\sum_{k=1}^{n} S_k f, \quad K_n = \frac{1}{n}\sum_{k=1}^{n} D_k,$$

for every $f \in L^1(G)$.

It can be easily seen that

$$S_n f(y) = \int D_n(y - x) f(x)dx, \quad D_{M_n}(x) = M_n 1_{I_n}(x),$$

and

$$E_n f(y) = M_n \int_{I_n(y)} f(x)dx.$$

Let A, s, j be fixed positive integers such that $j \leq A$ and $s < m_A$, then following G.Gát in the definition of the operators $H_{j,A}$ and H_j in [2], we define the operators

$$\tilde{H}^s_{j,A} f(y) = M_{A-j} \left| \int_{\bigcup_{x_{A-j} \neq y_{A-j}} I_A(y_0, \dots, y_{A-j-1}, x_{A-j}, y_{A-j+1}, \dots, y_{A-1})} f(x)\bar{r}^s_A(x)\frac{1}{1 - r_{A-j}(y - x)}dx \right|,$$

and for fixed s, j

$$\tilde{H}_j^s f(y) = \sup_{A: s < m_A} \tilde{H}_{j,A}^s f(y).$$

The notation C will be used for an absolute positive constant that may vary in different contexts.

2. Main results

Lemma 2.1. *For every fixed s the operator $\tilde{H}_1^s$ is bounded on L^2.*

Proof. Let $f \in L^2$. Using the proof of [2, Lemma 2.3.] we can write

$$\tilde{H}_{1,A}^s f = H_{1,A}(f \bar{r}_A^s),$$

from which we get

$$\|\tilde{H}_{1,A}^s f\|_2^2 = \|H_{1,A}(f \bar{r}_A^s)\|_2^2 \le C \|f \bar{r}_A^s\|_2^2 \le C \|f\|_2^2.$$

Since

$$\tilde{H}_{1,A}^s f = \tilde{H}_{1,A}^s (E_{A+1} f),$$

and

$$\tilde{H}_{1,A}^s (E_A f) = 0,$$

it follows

$$\begin{aligned} \| \sup_{A: s < m_A} \tilde{H}_{1,A}^s f\|_2^2 &= \| \sup_{A: s < m_A} \tilde{H}_{1,A}^s (E_{A+1} f - E_A f)\|_2^2 \\ &\le \sum_{A: s < m_A} \|\tilde{H}_{1,A}^s (E_{A+1} f - E_A f)\|_2^2 \\ &\le C \sum_A \|(E_{A+1} f - E_A f)\|_2^2 \le C \|f\|_2^2. \end{aligned}$$

□

Lemma 2.2. *For every fixed s the operator $\tilde{H}_1^s$ is of weak type (L^1, L^1).*

Proof. We proceed as in the proof of [2, Lemma 2.4.]. Namely, let $f \in L^1$ be such that

$$\operatorname{supp}(f) \subset \bigcup_{j=\alpha}^{\beta} I_k(z, j) = I,$$

where $I_k(z, j) = I_{k+1}(z_0, z_1, \ldots, z_{k-1}, j)$, for some fixed $z \in G$ and $j \in \{\alpha, \alpha + 1, \ldots, \beta\} \subset \{0, 1, \ldots, m_k - 1\}$.

If $s < \min(m_k, m_{k+1})$, then suppose that $\int_I f d\mu = 0$, $\int_I f \bar{r}_k^s d\mu = 0$ and $\int_I f \bar{r}_{k+1}^s d\mu = 0$. We construct the set $6I$ as done in [2, Lemma 2.4.].

Take any $y \in I_k(z) \setminus 6I$. Clearly,

$$\tilde{H}_1^s f(y) = \tilde{H}_{1,k+1}^s f(y) = H_{1,k+1}(\bar{r}_{k+1}^s f)(y).$$

From the proof of [2, Lemma 2.4.] it follows that

$$\int_{I_k(z)\setminus 6I} \tilde{H}_1^s f(y)dy = \int_{I_k(z)\setminus 6I} \tilde{H}_{1,k+1}^s f(y)dy$$
$$= \int_{I_k(z)\setminus 6I} H_{1,k+1}(\bar{r}_{k+1}^s f)(y)dy \leq C\|f\|_1.$$

Now if $y \in I_{k-1}(z) \setminus I_k(z)$, we get

$$\tilde{H}_1^s f(y) = \tilde{H}_{1,k}^s f(y) = H_{1,k}(f\bar{r}_k^s)(y) = 0,$$

because $\int_I f\bar{r}_k^s d\mu = 0$.

For $y \in I_{l-1}(z) \setminus I_l(z)$ for any $l \leq k-1$, we get

$$\tilde{H}_1^s f(y) = \tilde{H}_{1,l}^s f(y) = 0,$$

because $\int_I f d\mu = 0$.

It follows that

$$\int_{G\setminus 6I} \tilde{H}_1^s f(y)dy \leq C\|f\|_1.$$

If $m_{k+1} \leq s < m_k$, then we only suppose that f satisfies $\int_I f d\mu = 0$ and $\int_I f\bar{r}_k^s d\mu = 0$.

Then it is easily seen that $\tilde{H}_1^s f(y) = 0$ for every $y \in G \setminus 6I$.

If $m_k \leq s < m_{k+1}$, then for $\int_I f d\mu = 0$ and $\int_I f\bar{r}_{k+1}^s d\mu = 0$, we get in a similar way that

$$\int_{I_k(z)\setminus 6I} \tilde{H}_1^s f(y)dy \leq C\|f\|_1,$$

moreover, $\tilde{H}_1^s f(y) = 0$ for every $y \in G \setminus I_k(z)$.

Finally, if $s \geq \max(m_k, m_{k+1})$, then we only suppose that $\int_I f d\mu = 0$. In this case we also obtain that $\tilde{H}_1^s f(y) = 0$ for every $y \in G \setminus 6I$

We follow the steps in the proof of [2, Lemma 2.4.] and introduce a decomposition lemma but this latter will be the same as the decomposition made in [5, Lemma 2]. For an arbitrary function $f \in L^1$, if $\lambda > 0$ is such that $\|f\|_1 \leq \lambda$ and $(\alpha_k)_k$ is a sequence of integers defined by $\alpha_k = -s$ if $s < m_k$ and $\alpha_k = 0$ otherwise, there exist mutually disjoint intervals $J_j = \bigcup_{l=\alpha_j}^{\beta_j} I_{k_j}(z^j, l)$, $j \in \mathbb{P}$, and integrable functions b and g such that

(1) $f = b + g$,
(2) $\|g\|_\infty \leq C\lambda$,
(3) $\|g\|_1 \leq C\|f\|_1$,
(4) $\text{supp}(b) \subset \bigcup_{j=1}^\infty J_j$,
(5) $\int_{J_j} b d\mu = \int_{J_j} b r_{k_j}^{\alpha_{k_j}} d\mu = 0$, for every $j \in \mathbb{P}$,
(6) $\int_{J_j} |b| d\mu \leq C \int_{J_j} |f| d\mu$, for every $j \in \mathbb{P}$,
(7) $\sum_{j=1}^\infty \mu(J_j) \leq \frac{\|f\|_1}{\lambda}$.

In [5, Lemma 2] it was proved that for every $j \in \mathbb{P}$ there exist constants a_{k_j} and b_{k_j} such that

$$b(x) = f(x) - a_{k_j} - b_{k_j}\bar{r}_{k_j}^{\alpha_{k_j}}(x), \quad \forall x \in J_j.$$

We introduce the functions

$$h_j(x) = [b(x) - (\mu(J_j))^{-1}(\int_{J_j} f\bar{r}^s_{k_j+1}d\mu)r^s_{k_j+1}(x)]1_{J_j}(x), \quad j \in \mathbb{P},$$

if $s < m_{k_j+1}$ and $h_j(x) = b(x)1_{J_j}(x)$, otherwise.

Notice that for $s < m_{k_j+1}$

$$\int_{J_j} h_j d\mu = \int_{J_j} bd\mu - (\mu(J_j))^{-1}\int_{J_j} f\bar{r}^s_{k_j+1}d\mu \int_{J_j} r^s_{k_j+1}d\mu = 0,$$

because $\int_{J_j} r^s_{k_j+1}d\mu = 0$. But also , since $\int_{J_j} r^s_{k_j+1}\bar{r}^s_{k_j}d\mu = 0$, we have

$$\int_{J_j} h_j\bar{r}^s_{k_j}d\mu = \int_{J_j} b\bar{r}^s_{k_j}d\mu - (\mu(J_j))^{-1}\int_{J_j} f\bar{r}^s_{k_j+1}d\mu \int_{J_j} r^s_{k_j+1}\bar{r}^s_{k_j}d\mu = 0.$$

Besides

$$\begin{aligned}\int_{J_j} h_j\bar{r}^s_{k_j+1}d\mu &= \int_{J_j} b\bar{r}^s_{k_j+1}d\mu - \int_{J_j} f\bar{r}^s_{k_j+1}d\mu \\ &= \int_{J_j} f\bar{r}^s_{k_j+1}d\mu - a_{k_j}\int_{J_j} \bar{r}^s_{k_j+1}d\mu \\ &\quad - b_{k_j}\int_{J_j} \bar{r}_{k_j}^{\alpha_{k_j}}\bar{r}^s_{k_j+1}d\mu - \int_{J_j} f\bar{r}^s_{k_j+1}d\mu = 0.\end{aligned}$$

For $s \geq m_{k_j+1}$, we obviously have

$$\int_{J_j} h_j d\mu = \int_{J_j} h_j r_{k_j}^{\alpha_{k_j}} d\mu = 0.$$

In this way we have proved that

$$\int_{G\setminus 6J_j} \tilde{H}^s_1 h_j(y)dy \leq C\|h_j\|_1 \leq C\int_{J_j} |f|d\mu, \quad \forall j \in \mathbb{P}.$$

Following the steps used in [2, Lemma 2.4.], we obtain that

$$\mu(\tilde{H}^s_1\sum_{j=1}^{\infty} h_j > \lambda) \leq \frac{C}{\lambda}\sum_{j=1}^{\infty}\|h_j\|_1 \leq \frac{C}{\lambda}\|f\|_1.$$

From

$$\begin{aligned}f = b + g &= \sum_{j=1}^{\infty} h_j + \sum_{j=1}^{\infty}(\mu(J_j))^{-1}(\int_{J_j} f\bar{r}^s_{k_j+1}d\mu)r^s_{k_j+1}(x)1_{J_j} + g \\ &=: \sum_{j=1}^{\infty} h_j + G.\end{aligned}$$

The mutually disjoint intervals $(J_j)_{j\in\mathbb{P}}$ were constructed such that

$$(\mu(J_j))^{-1}|\int_{J_j} f\bar{r}^s_{k_j+1}d\mu| \leq (\mu(J_j))^{-1}\int_{J_j}|f|d\mu \leq 3\lambda.$$

Therefore, the function G remains bounded. Moreover,

$$\|G\|_1 \leq \|g\|_1 + \sum_{j=1}^{\infty}(\mu(J_j))^{-1}|\int_{J_j} f\bar{r}^s_{k_j+1}d\mu|\int_{J_j} d\mu \leq \|g\|_1 + \|f\|_1 \leq C\|f\|_1.$$

Proceeding as in [2, Lemma 2.4.], we get

$$\mu(\tilde{H}^s_1 G > \lambda) \leq C\frac{\|\tilde{H}^s_1 G\|_2^2}{\lambda^2} \leq C\frac{\|G\|_2^2}{\lambda^2} \leq \frac{C}{\lambda}\|G\|_1 \leq \frac{C}{\lambda}\|f\|_1.$$

Finally,

$$\mu(\tilde{H}^s_1 f > 2\lambda) \leq \mu(\tilde{H}^s_1 \sum_{j=1}^{\infty} h_j > \lambda) + \mu(\tilde{H}^s_1 G > \lambda) \leq \frac{C}{\lambda}\|f\|_1. \qquad \square$$

Lemma 2.3. *There exists an absolute constant $C > 0$ such that for all $j \in \mathbb{P}$, $f \in L^1$ and $\lambda > 0$, we have*

$$\mu(\tilde{H}^s_j f > 2\lambda) \leq \frac{j^2 C}{2^j\lambda}\|f\|_1.$$

Proof. Since

$$\begin{aligned}\tilde{H}^s_j f = \sup_{j\leq A, s<m_A} \tilde{H}^s_{j,A} f &\leq \sum_{k=0}^{j-1} \sup_{\substack{j\leq A, s<m_A\\ A\equiv k \mod j}} \tilde{H}^s_{j,A} f \\ &\leq \sum_{k=0}^{j-1} \sup_{\substack{j\leq A, s<m_A\\ A\equiv k \mod 2j}} \tilde{H}^s_{j,A} f + \sum_{k=0}^{j-1} \sup_{\substack{j\leq A, s<m_A\\ A\equiv k \mod j, A\not\equiv k \mod 2j}} \tilde{H}^s_{j,A} f.\end{aligned}$$

using the properties of the operators $H^N_{j,k}$ introduced in [2, Lemma 2.5.], it is easily seen that we only need to prove that for every $k \in \{0, 1, \ldots, j-1\}$ the operators

$$2^j \sup_{\substack{j\leq A\leq Nj+k\\ A\equiv k \mod 2j}} \tilde{H}^s_{j,A} f,$$

and

$$2^j \sup_{\substack{j\leq A\leq Nj+k\\ A\equiv k \mod j, A\not\equiv k \mod 2j}} \tilde{H}^s_{j,A} f$$

are of weak type (L^1, L^1) uniformly on N.

We use a similar function as the permutation α introduced in [2, Lemma 2.5.]. Put

- $\alpha'(n) = n$, if $n \geq Nj + k$ or $n \neq k + 2lj, k - 1 + (2l + 1)j$, for any $l \in \mathbb{N}$,
- $\alpha'(k + 2jl) = k + (2l + 1)j - 1$, if $2l < N$,

- $\alpha'(k+(2l+1)j-1) = k+2lj$, if $2l < N$.

Let G' be the Vilenkin group generated by the sequence $(m_{\alpha'(i)})_i$. Then, for $A \le Nj+k$, $A \equiv k \mod j$ with $A \ne k \mod 2j$, we have $\alpha'(A-j) = A-1$, $\alpha'(A-1) = A-j$, but $\alpha'(A) = A$.

$$\begin{aligned}\tilde{H}^s_{j,A}f(y) &= M_{A-j}\left|\int_{\bigcup\limits_{x_{A-j}\ne y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},y_{A-j+1},\dots,y_{A-1})} f(x)\bar{r}^s_A(x)\frac{1}{1-r_{A-j}(y-x)}dx\right| \\ &= M_{A-j}\left|\int_{\bigcup\limits_{x'_{A-1}\ne y'_{A-1}} I_A(y'_0,\dots,y'_{A-j-1},y'_{A-j},y'_{A-j+1},\dots,x'_{A-1})} f'(x')(\bar{r}'_A(x'))^s\frac{1}{1-r'_{A-1}(y'-x')}dx'\right| \\ &= \frac{M_{A-j}}{M_{A-1}}\tilde{H}^s_{1,A}f'(y') \le 2^{1-j}\tilde{H}^s_{1,A}f'(y'),\end{aligned}$$

where $(x'_i)_i = (x_{\alpha'(i)})_i \in G'$, for every $x \in G$, $(r'_n)_n$ is the convenient set of Rademacher functions for G' and f' is defined on G' by $f'(x') = f(x)$.

Following the steps of [2, Lemma 2.5.] we get that

$$2^j \sup_{\substack{j\le A\le Nj+k \\ A\equiv k \mod j \\ A\ne k \mod 2j}} \tilde{H}^s_{j,A}f$$

is of weak type (L^1, L^1) uniformly on N.

In a similar way if we introduce the permutation α'':

- $\alpha''(n) = n$, if $n \ge Nj+k$ or $n \ne k+(2l+1)j, k-1+2lj$, for any $l \in \mathbb{N}$,
- $\alpha''(k+(2l+1)j) = k+(2l+2)j-1$, if $2l+1 < N$,
- $\alpha''(k+(2l+2)j-1) = k+(2l+1)j$, if $2l+1 < N$,

let G'' be the Vilenkin group generated by the sequence $(m_{\alpha''(i)})_i$.

If $A = k+2lj$, $l \in \mathbb{P}$, we have $\alpha''(A-j) = A-1$, $\alpha''(A-1) = A-j$, but $\alpha''(A) = A$, then we have

$$\begin{aligned}\tilde{H}^s_{j,A}f(y) &= M_{A-j}\left|\int_{\bigcup\limits_{x_{A-j}\ne y_{A-j}} I_A(y_0,\dots,y_{A-j-1},x_{A-j},y_{A-j+1},\dots,y_{A-1})} f(x)\bar{r}^s_A(x)\frac{1}{1-r_{A-j}(y-x)}dx\right| \\ &= M_{A-j}\left|\int_{\bigcup\limits_{x''_{A-1}\ne y''_{A-1}} I_A(y''_0,\dots,y''_{A-j-1},y''_{A-j},y''_{A-j+1},\dots,x''_{A-1})} f''(x'')(\bar{r}''_A(x''))^s\frac{1}{1-r''_{A-1}(y''-x'')}dx''\right| \\ &= \frac{M_{A-j}}{M_{A-1}}\tilde{H}^s_{1,A}f''(y'') \le 2^{1-j}\tilde{H}^s_{1,A}f''(y''),\end{aligned}$$

where $(x''_i)_i = (x_{\alpha''(i)})_i \in G''$, for every $x \in G$, $(r''_n)_n$ is the convenient set of Rademacher functions for G'' and f'' is defined on G'' by $f''(x'') = f(x)$.

Consequently,

$$2^j \sup_{\substack{j\le A\le Nj+k\\ A\equiv k \mod 2j}} \tilde{H}^s_{j,A} f$$

is of weak type (L^1, L^1) uniformly on N, and the lemma is proved. □

Lemma 2.4. *Let $s \in \mathbb{P}$ be fixed. Then the operator*

$$\sup_{N:s<m_N} |r^s_N D_{M_N} * f|$$

is of weak type (L^1, L^1).

Proof. We first prove that the mentioned operator is bounded on L^2. For $g \in L^2$, we have

$$\begin{aligned}\|r^s_N D_{M_N} * g\|_2^2 &= M_N^2 \int \left| \int_{I_N(y)} \bar{r}^s_N(x) g(x) dx \right|^2 dy \\ &\le M_N^2 \int \left(\int_{I_N(y)} |\bar{r}^s_N(x)|^2 dx \right) \left(\int_{I_N(y)} |g(x)|^2 dx \right) dy \\ &= M_N \int \left(\int_{I_N(y)} |g(x)|^2 dx \right) dy = \|g\|_2^2.\end{aligned}$$

Since $r^s_N D_{M_N} * (E_N f) = 0$ and $r^s_N D_{M_N} * (f) = r^s_N D_{M_N} * (E_{N+1} f)$, the same argument used in Lemma 2.1 gives that $\sup\limits_{N:s<m_N} |r^s_N D_{M_N} * f|$ is bounded on L^2.

Now we use the decomposition mentioned in Lemma 2.2 with the same notations for some fixed function $f \in L^1$ and $\lambda > \|f\|_1$. Put $b_j = b \cdot 1_{J_j}$ for every $j \in \mathbb{P}$. We can write $f = \sum\limits_{j=1}^{\infty} b_j + g$.

Let $y \in G \setminus (\bigcup\limits_{j=1}^{\infty} J_j)$, then for every $j \in \mathbb{P}, N \in \mathbb{N}$ with $s < m_N$,

$$\int_{I_N(y)} \bar{r}^s_N(x) b_j(x) dx = 0.$$

From which we get

$$\int_{I_N(y)} \bar{r}^s_N(x) b(x) dx = 0,$$

consequently,

$$\sup_{N:s<m_N} |(r^s_N D_{M_N} * b)(y)| = 0.$$

Using the boundedness of the operator on L^2 and the argument used in Lemma 2.2, we obtain

$$\mu(\{\sup_{N:s<m_N} |r_N^s D_{M_N} * f| > 2\lambda\} \cap (G \setminus \bigcup_{j=1}^{\infty} J_j))$$

$$\leq \mu(\{\sup_{N:s<m_N} |r_N^s D_{M_N} * b| > \lambda\} \cap (G \setminus \bigcup_{j=1}^{\infty} J_j))$$

$$+ \mu(\sup_{N:s<m_N} |r_N^s D_{M_N} * g| > \lambda)$$

$$= \mu(\sup_{N:s<m_N} |r_N^s D_{M_N} * g| > \lambda) \leq C\frac{\|g\|_1}{\lambda} \leq C\frac{\|f\|_1}{\lambda}. \qquad \square$$

Theorem 2.5. *Let $f \in L^1$, $l \in \mathbb{P}$ fixed. Then $S_{a_N M_N} f \to f$ almost everywhere uniformly on $a_N \in \{1, 2, \dots, \min(l, m_N - 1)\}$.*

Proof. Let $a_N \leq \min(l, m_N - 1)$. We write $D_{a_N M_N}$ in the following form

$$D_{a_N M_N} = \sum_{i=0}^{M_N - 1} \psi_i + \sum_{i=M_N}^{2M_N - 1} \psi_i + \ldots + \sum_{i=(a_N-1)M_N}^{a_N M_N - 1} \psi_i = D_{M_N} + \sum_{s=1}^{a_N - 1} r_N^s D_{M_N}.$$

Since $\sup_{N:s<m_N} |r_N^s D_{M_N} * f|$ is of weak type (L^1, L^1), and from $r_N^s D_{M_N} * g \to 0$, for every polynomial g, repeating the method of [2, Theorem 2.1.], we get that $r_N^s D_{M_N} * f \to 0$, almost everywhere. The result follows from the fact that $D_{M_N} * f \to f$, almost everywhere. $\square$

Theorem 2.6. *Let $f \in L^1$, $l \in \mathbb{P}$ fixed. Then $\sigma_{a_N M_N} f \to f$ almost everywhere uniformly on $a_N \in \{1, 2, \dots, \min(l, m_N - 1)\}$.*

Proof. Let $a_N \leq \min(l, m_N - 1)$.

$$K_{a_N M_N} = \frac{1}{a_N M_N} \sum_{k=1}^{a_N M_N} D_k$$

$$= \frac{1}{a_N M_N} \left(\sum_{k=1}^{M_N} D_k + \sum_{k=M_N+1}^{2M_N} D_k + \ldots + \sum_{k=(a_N-1)M_N+1}^{a_N M_N} D_k \right)$$

$$= \frac{1}{a_N M_N} \left(\sum_{k=1}^{M_N} D_k + \sum_{k=1}^{M_N} D_{M_N+k} + \ldots + \sum_{k=1}^{M_N} D_{(a_N-1)M_N+k} \right)$$

$$= \frac{1}{a_N M_N} \sum_{s=0}^{a_N-1} \sum_{k=1}^{M_N} D_{sM_N+k}$$

$$= \frac{1}{a_N M_N} \left(\sum_{k=1}^{M_N} D_k + \sum_{s=1}^{a_N-1} \sum_{k=1}^{M_N} (D_{sM_N} + r_N^s D_k) \right)$$

$$= \frac{1}{a_N}\sum_{s=1}^{a_N-1} D_{sM_N} + \frac{1}{a_N M_N}\sum_{k=1}^{M_N} D_k + \frac{1}{a_N M_N}\sum_{s=1}^{a_N-1} r_N^s \sum_{k=1}^{M_N} D_k$$
$$= \frac{1}{a_N}\sum_{s=1}^{a_N-1} D_{sM_N} + \frac{1}{a_N} K_{M_N} + \frac{1}{a_N}\sum_{s=1}^{a_N-1} r_N^s K_{M_N}.$$

Using [2, Theorem 2.1.] and Theorem 2.5, in order to prove that $\sigma_{a_N M_N} f = K_{a_N M_N} * f \to f$ almost everywhere uniformly on $a_N \in \{1, 2, \dots, \min(l, m_N - 1)\}$, it suffices to prove that $r_N^s K_{M_N} * f \to 0$, almost everywhere uniformly on $s \in \{1, 2, \dots, \min(l, m_N - 1)\}$.

We use the method of [2, Theorem 2.1.]. Namely, we prove that the operator $\sup_{A: s < m_A} |r_A^s K_{M_A} * f|$ is of weak type (L^1, L^1), then noticing that $r_A^s K_{M_A} * g$ vanishes whenever the polynomial g is constant on I_N-cosets, the result will follow.

In fact, since $K_{M_A}(z) = \frac{M_t}{1 - r_t(z)}$ if $z - z_t e_t \in I_A$, $K_{M_A}(z) = \frac{M_A + 1}{2}$ if $z \in I_A$, and $K_{M_A}(z) = 0$ otherwise, it follows

$$|(r_A^s K_{M_A} * f)(y)| = |\int K_{M_A}(y - x)\bar{r}_A^s(x) f(x)dx|$$
$$\leq \left|\int_{I_A(y)} K_{M_A}(y - x)\bar{r}_A^s(x) f(x)dx\right|$$
$$+ \sum_{t=0}^{A-1}\left|\int_{I_t(y)\setminus I_{t+1}(y)} K_{M_A}(y - x)\bar{r}_A^s(x) f(x)dx\right|$$
$$\leq S_{M_A}|f|(y) + \sum_{t=0}^{A-1} M_t \left|\int_{\bigcup_{x_t \neq y_t} I_A(y_0, \dots, y_{t-1}, x_t, y_{t+1}, \dots, y_{A-1})} f(x)\bar{r}_A^s(x)\frac{1}{1 - r_t(y - x)}dx\right|$$
$$\leq S_{M_A}|f|(y) + \sum_{t=0}^{A-1} \tilde{H}_{A-t,A}^s f(y)$$
$$= S_{M_A}|f|(y) + \sum_{j=1}^{A} \tilde{H}_{j,A}^s f(y).$$

Hence,

$$\sup_{A: s < m_A} |r_A^s K_{M_A} * f|(y) \leq \sup_{A: s < m_A} S_{M_A}|f|(y) + \sum_{j=1}^{\infty} \tilde{H}_j^s f(y).$$

Following the steps in the proof of [2, Theorem 2.1.] and replacing H_j by $\tilde{H}_j^s$, the result follows by applying Lemma 2.3. □

3. Acknowledgement

I would like to thank Professor György Gát for his kind assistance, suggestions and comments during the preparation of this manuscript.

References

[1] G. Gát. Pointwise convergence of the Fejér means of functions on unbounded Vilenkin groups. *J. Approx. Theory*, 101(1):1–36, 1999.
[2] G. Gát. Cesàro means of integrable functions with respect to unbounded Vilenkin systems. *J. Approx. Theory*, 124(1):25–43, 2003.
[3] G. Gát. Almost everywhere convergence of Fejér means of L^1 functions on rarely unbounded Vilenkin groups. *Acta Math. Sin. (Engl. Ser.)*, 23(12):2269–2294, 2007.
[4] J. Pál and P. Simon. On a generalization of the concept of derivative. *Acta Math. Acad. Sci. Hungar.*, 29(1-2):155–164, 1977.
[5] W. S. Young. Mean convergence of generalized Walsh-Fourier series. *Trans. Amer. Math. Soc.*, 218:311–320, 1976.

Received June 1, 2014.

Department of Mathematics,
University of Sarajevo,
Zmaja od Bosne 33-35,
Sarajevo,
Bosnia and Herzegovina
E-mail address: nacima.o@gmail.com