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FINSLERIAN LIE DERIVATIVE AND LANDSBERG MANIFOLDS

D. CS. KERTÉSZ

ABSTRACT. In this paper we take a close look at Lie derivatives on a Finsler bundle and give a geometric meaning to the vanishing of the mixed curvature of certain covariant derivatives on a Finsler bundle. As an application, we obtain some characterizations of Landsberg manifolds.

INTRODUCTION

Geometric objects on a Finsler vector bundle can be 'Lie differentiated' with respect to projectable vector fields. This Lie derivative has mostly been used to investigate infinitesimal transformations of Finsler manifolds and their generalizations, see, e.g., [3, 5, 6, 9, 10]. In these cases, the projectable vector field is in particular the complete lift of the vector field on the base manifold, which represents the infinitesimal transformation. We show that the 'Finslerian Lie derivative' has a natural dynamical interpretation for general projectable vector fields (Lemma 2). This Lie derivative is closely related to the Berwald derivative induced by an Ehresmann connection: the Lie derivative along a horizontal lift of a vector field is the same as the horizontal Berwald derivative with respect to the vector field. In fact the Berwald derivative can be characterized with this dynamical interpretation [2].

In his paper [4], Ichijyō considered covariant derivatives on a Finsler bundle, that differs from the Berwald derivative arising from an Ehresmann connection only in its vertical part. He called a manifold endowed with such a covariant derivative on its Finsler bundle an (N, C)-manifold, referring with N to the Ehresmann connection, and with C to the tensor that modifies the vertical part of the Berwald derivative. We refer to such structures as C-Ehresmann

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manifolds. Ichijyō investigated the geometric meaning of the vanishing of their mixed curvature. We give simple coordinate-free proofs of his results using the Lie derivative.

In sections 2 and 3 we make some general remarks on the Lie derivative and parallel translations. Then we can easily connect the parallelness of a geometric object with respect to the Berwald derivative to their invariance under parallel translations. We show in section 4 that the mixed curvature of a C-Ehresmann manifold can be obtained as the Lie derivative of the vertical part of the covariant derivative along horizontal lifts, and hence it vanishes if, and only if, the parallel translations (with respect to the Ehresmann connection) preserve the induced covariant derivatives on the tangent spaces (Proposition 8). These results provide some characterizations of Landsberg manifolds, which we collect in Proposition 9.

1. Preliminaries

We follow the notations and conventions of [8], however, for the reader's convenience, here we briefly recall some of them.

Throughout this paper, M is a smooth manifold. We denote by $C^\infty(M)$ the real algebra of smooth functions on M. Every mapping is assumed to be smooth unless otherwise stated. The $C^\infty(M)$-module of vector fields on M is denoted by $\mathfrak{X}(M)$. The tangent bundle of M is $\tau\colon TM \to M$, and $\mathring{T}M$ is the manifold of nonzero tangent vectors to M called the *slit tangent manifold*. The *slit Finsler bundle* is $\mathring{\pi}\colon \mathring{T}M \times_M TM \to \mathring{T}M$, where

$$\mathring{T}M \times_M TM := \{(u,v) \in \mathring{T}M \times TM \mid \tau(u) = \tau(v)\},$$

and $\mathring{\pi}$ is the restriction of the first projection of $\mathring{T}M \times TM$. Sometimes we treat mappings defined on TM as mappings defined on $\mathring{T}M$, without emphasizing the restriction.

Sections of $\mathring{\pi}$ are called *Finsler vector fields*, we denote their $C^\infty(\mathring{T}M)$-module by $\Gamma(\mathring{\pi})$. Elements of the tensor algebra of $\Gamma(\mathring{\pi})$ are called *Finsler tensors*. The type (k,l) Finsler tensors form a $C^\infty(\mathring{T}M)$-module denoted by $\mathcal{F}^k_l(\mathring{\pi})$. General Finsler vector fields will be denoted by $\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}$. A vector field X on M induces a Finsler vector field

$$\widehat{X}\colon u \in \mathring{T}M \mapsto (u, X(\tau(u))) \in \mathring{T}M \times_M TM,$$

called the *basic lift* of X. We have the *canonical Finsler vector field* which acts by $\delta(u) := (u,u)$.

A Finsler vector field $\widetilde{X}$ restricted to a single slit tangent space $\mathring{T}_pM$ gives a vector field X^p on the manifold $\mathring{T}_pM$, if we identify $T\mathring{T}_pM$ with $\mathring{T}_pM \times T_pM$. The same 'restriction' works for tensors as well. For example, from a Finsler tensor $g \in \mathcal{F}^0_2(\mathring{\pi})$ we get a tensor g^p on $\mathring{T}_pM$ given by $g^p(X^p, Y^p) := g(\widetilde{X}, \widetilde{Y}) \restriction \mathring{T}_pM$.

We have the canonical surjection

$$\mathbf{j}\colon T\mathring{T}M \to \mathring{T}M, \quad \mathbf{j}(w) := (\tau_{\mathring{T}M}(w), \tau_*(w)), \tag{1}$$

where τ_* is the derivative (or tangent map) of τ. We also have the canonical injection $\mathbf{i}\colon \mathring{T}M \times_M TM \to T\mathring{T}M$, where $\mathbf{i}(u,v)$ is the velocity of the curve

$$t \in \mathbb{R} \mapsto u + tv \in TM$$

at 0. These form the exact sequence

$$0 \longrightarrow \mathring{T}M \times_M TM \xrightarrow{\mathbf{i}} T\mathring{T}M \xrightarrow{\mathbf{j}} \mathring{T}M \times_M TM \longrightarrow 0.$$

The vertical subbundle of $T\mathring{T}M$ is $V\mathring{T}M := \mathrm{Ker}(\tau_*) = \mathrm{Im}(\mathbf{i}) = \mathrm{Ker}(\mathbf{j})$.

A bundle map $\mathcal{H}\colon \mathring{T}M \times_M TM \to T\mathring{T}M$ is called an Ehresmann connection in $\mathring{T}M$, if

$$\mathbf{j} \circ \mathcal{H}(u,v) = (u,v) \quad \text{for all} \quad (u,v) \in \mathring{T}M \times_M TM. \tag{2}$$

Then the horizontal subbundle $H\mathring{T}M := \mathrm{Im}(\mathcal{H})$ is complementary to $V\mathring{T}M$. We have the projections $\mathbf{h}$ and $\mathbf{v}$ on $T\mathring{T}M$ with $\mathrm{Im}(\mathbf{h}) = H\mathring{T}M$, $\mathrm{Ker}(\mathbf{h}) = V\mathring{T}M$, $\mathrm{Im}(\mathbf{v}) = V\mathring{T}M$, $\mathrm{Ker}(\mathbf{v}) = H\mathring{T}M$. These can be given explicitly as $\mathbf{h} = \mathcal{H} \circ \mathbf{j}$ and $\mathbf{v} = 1 - \mathbf{h}$. Finally, we have the vertical map $\mathcal{V} := \mathbf{i}^{-1} \circ \mathbf{v}$, which is a left inverse for $\mathbf{i}$.

Whenever it is convenient, we treat the mappings above as module homomorphisms. For example, $\mathcal{H}$ and $\mathbf{i}$ can be regarded as module homomorphisms from $\Gamma(\mathring{\pi})$ to $\mathfrak{X}(\mathring{T}M)$ in a natural manner.

We speak of a horizontal or vertical vector field on $\mathring{T}M$, if it takes values only in $H\mathring{T}M$ or $V\mathring{T}M$, respectively. A vector field $X \in \mathfrak{X}(M)$ has a horizontal lift $X^{\mathsf{h}} := \mathcal{H}\widehat{X}$ and a vertical lift $X^{\mathsf{v}} := \mathbf{i}\widehat{X}$.

Let D be a covariant derivative on $\mathring{\pi}$ and $\mathcal{H}$ an Ehresmann connection in $\mathring{T}M$. The *horizontal* and *vertical parts* of D are given by

$$D^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y} = D_{\mathcal{H}\widetilde{X}}\widetilde{Y}, \quad D^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y} = D_{\mathbf{i}\widetilde{X}}\widetilde{Y}.$$

Of course the horizontal and vertical parts determine D by the formula

$$D_\xi\widetilde{Y} = D_{\mathbf{h}\xi}\widetilde{Y} + D_{\mathbf{v}\xi}\widetilde{Y} = D^{\mathsf{h}}_{\mathbf{j}\xi}\widetilde{Y} + D^{\mathsf{v}}_{\mathcal{V}\xi}\widetilde{Y}.$$

The vertical part D^{v} of a covariant derivative can be interpreted naturally as a family $(D^p)_{p\in M}$, where D^p is a covariant derivative on $\mathring{T}_pM$. If we identify $T\mathring{T}_pM$ with $\mathring{T}_pM \times T_pM$, we can define D^p conveniently as

$$D^p_{X^p}Y^p = (D^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y})^p.$$

An Ehresmann connection $\mathcal{H}$ in $\mathring{T}M$ induces the *Berwald derivative* ∇ on $\mathring{\pi}$ such that $\nabla^{\mathsf{h}}_{\widetilde{X}}\widetilde{Y} := \mathcal{V}[\mathcal{H}\widetilde{X}, \mathbf{i}\widetilde{Y}]$ and $\nabla^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y} := \mathbf{j}[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}]$ for all $\widetilde{X}, \widetilde{Y} \in \Gamma(\mathring{\pi})$. For details see, e.g., [8, section 7.10].

Let $(\mathcal{U},(u^i)_{i=1}^n)$ be a chart of M with induced chart $(\tau^{-1}(\mathcal{U}),(x^i)_{i=1}^n,(y^i)_{i=1}^n)$ on TM. The horizontal and vertical parts of ∇ are just

$$\nabla^{\mathsf{h}}_{\widehat{\frac{\partial}{\partial u^j}}}\widetilde{Y} = \left(\left(\frac{\partial}{\partial u^j}\right)^{\mathsf{h}} Y^i + Y^k\frac{\partial N^i_j}{\partial y^k}\right)\widehat{\frac{\partial}{\partial u^i}}, \quad \nabla^{\mathsf{v}}_{\widehat{\frac{\partial}{\partial u^j}}}\widetilde{Y} = \frac{\partial Y^i}{\partial y^j}\widehat{\frac{\partial}{\partial u^i}},$$

where $\widetilde{Y} = Y^i\widehat{\frac{\partial}{\partial u^i}}$ and the functions $N^i_j \in C^\infty(\mathring{T}M)$ are the Christoffel symbols of $\mathcal{H}$ given by

$$\mathcal{H}\widehat{\frac{\partial}{\partial u^j}} = N^i_j\frac{\partial}{\partial y^i}.$$

We note that these local formulae hold only on $\tau^{-1}(\mathcal{U})\cap\mathring{T}M$.

2. Lie derivatives on a Finsler bundle

If $\varphi\colon W \subset \mathbb{R}\times M \to M$ is the flow of a vector field on M, we use the notation $\varphi_s(p) := \varphi(s,p)$.

Recall that given a smooth mapping $\Phi\colon M \to N$ between manifolds M and N, two vector fields $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are called Φ-related, if $\Phi_*\circ X = Y\circ\Phi$. Then we also write $X \sim_\Phi Y$. Equivalently, if φ^X and φ^Y are the flows of X and Y, then $X\sim_\Phi Y$ if, and only if, $\varphi^Y_t\circ\Phi = \Phi\circ\varphi^X_t$.

A vector field $\xi \in \mathfrak{X}(TM)$ is called *projectable* if it is τ-related to a vector field on M. In the next lemma we collect some useful characterizations of projectable vector fields.

Lemma 1. *Let $\xi\in\mathfrak{X}(TM)$ and let $\varphi^\xi\colon W\subset\mathbb{R}\times TM\to TM$ be the flow of ξ. The following are equivalent:*

(i) ξ is projectable;
(ii) φ^ξ preserves the fibres of TM;
(iii) φ^ξ_{t*} maps vertical vectors into vertical ones;
(iv) in any induced chart $(\tau^{-1}(\mathcal{U}),(x,y))$ on TM, ξ takes the form

$$\xi \underset{(\mathcal{U})}{=} (X^i\circ\tau)\frac{\partial}{\partial x^i} + \xi^{n+i}\frac{\partial}{\partial y^i},$$

where $X^i\in C^\infty(\mathcal{U})$, $\xi^{n+i}\in C^\infty(\tau^{-1}(\mathcal{U}))$;
(v) $[\xi,\eta]$ is vertical for every vertical vector field η;

Proof. Notice that a vector field on TM is vertical exactly if it is τ-related to the zero vector field $o\in\mathfrak{X}(M)$.

(i)⇒(ii) If ξ is τ-related to a vector field X on M, then $\varphi^X_t\circ\tau = \tau\circ\varphi^\xi_t$ wherever both sides are defined, thus φ^ξ_t indeed preserves fibres.

(ii)⇒(iii) We show that for any fibre-preserving map ϕ on TM, $\phi_*\circ\mathbf{i}(u,v)$ $((u,v)\in\mathring{T}M\times_M TM)$ is vertical. Indeed, $\mathbf{i}(u,v)=\dot\alpha(0)$ where $\alpha(t)=u+tv$. Since ϕ preserves fibres, $\tau\circ\phi\circ\alpha$ is constant, thus

$$0 = (\tau\circ\phi\circ\alpha)\dot{}(0) = \tau_*\circ\phi_*\circ\dot\alpha(0) = \tau_*\circ\phi_*\circ\mathbf{i}(u,v).$$

(iii)⇒(v) Using the dynamical interpretation of the Lie bracket, for a vertical vector field η on TM we have

$$[\xi,\eta](u)=\lim_{s\to 0}\frac{1}{s}\big(\varphi^{\xi}_{-s*}\circ\eta\circ\varphi^{\xi}_{s}(u)-\eta(u)\big),\quad u\in TM.$$

Each term in the parenthesis on the right-hand side are vertical by (iii), hence so is the limit.

(v)⇒(iv): Let $(\mathcal{U},(u^i)_{i=1}^n)$ be a chart on M, $(\tau^{-1}(\mathcal{U}),(x^i)_{i=1}^n,(y^i)_{i=1}^n)$ its induced chart. We can set $\xi=\xi^i\frac{\partial}{\partial x^i}+\xi^{n+i}\frac{\partial}{\partial y^i}$, $\eta=\eta^i\frac{\partial}{\partial y^i}$, where $\xi^i,\xi^{n+i},\eta^i\in C^\infty(TM)$. Then

$$[\xi,\eta]\underset{(\mathcal{U})}{=}\xi(\eta^i)\frac{\partial}{\partial y^i}-\eta(\xi^i)\frac{\partial}{\partial x^i}-\eta(\xi^{n+i})\frac{\partial}{\partial y^i}.$$

Thus, if $[\xi,\eta]$ is vertical, the functions $\eta(\xi^i)$ need to be zero for any vertical vector field η. Hence ξ^i are fibrewise constant, i.e., there are functions $X^i\in C^\infty(\mathcal{U})$ such that $\xi^i=X^i\circ\tau$.

(iv)⇒(i) Let Y be an arbitrary vector field on M and set $X:=\tau_*\circ\xi\circ Y$. If ξ is of the form given in (iv), then since $\frac{\partial}{\partial y^i}$ is vertical and $\frac{\partial}{\partial x^i}\sim_\tau\frac{\partial}{\partial u^i}$, we have

$$X\underset{(\mathcal{U})}{=}\tau_*\circ\Big((X^i\circ\tau)\frac{\partial}{\partial x^i}\Big)\circ Y=\Big(X^i\frac{\partial}{\partial u^i}\Big)\circ\tau\circ Y=X^i\frac{\partial}{\partial u^i}.$$

Thus X is independent of the choice of Y, and $\xi\sim_\tau X$. □

Property (v) enables us to define the Lie derivative of a Finsler vector field $\widetilde{Y}$ along a projectable vector field ξ on TM as $\widetilde{\mathcal{L}}_\xi\widetilde{Y}:=\mathbf{i}^{-1}[\xi,\mathbf{i}\widetilde{Y}]$. We can extend the Lie derivative $\widetilde{\mathcal{L}}_\xi$ to Finsler tensors on $\mathring{\pi}$. For example, if $A\in\mathcal{F}^1_2(\mathring{\pi})$,

$$\widetilde{\mathcal{L}}_\xi A(\widetilde{X},\widetilde{Y})=\widetilde{\mathcal{L}}_\xi(A(\widetilde{X},\widetilde{Y}))-A(\widetilde{\mathcal{L}}_\xi\widetilde{X},\widetilde{Y})-A(\widetilde{X},\widetilde{\mathcal{L}}_\xi\widetilde{Y}).\tag{3}$$

Similarly, the vertical (or horizontal) part of a covariant derivative D on $\mathring{\pi}$ can be Lie differentiated along ξ:

$$(\widetilde{\mathcal{L}}_\xi D^{\mathsf{v}})_{\widetilde{X}}\widetilde{Y}=\widetilde{\mathcal{L}}_\xi(D^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y})-D^{\mathsf{v}}_{\widetilde{\mathcal{L}}_\xi\widetilde{X}}\widetilde{Y}-D^{\mathsf{v}}_{\widetilde{X}}\widetilde{\mathcal{L}}_\xi\widetilde{Y}.$$

Now we turn to the dynamical interpretation of the Lie derivative. Let $\phi\colon\mathring{T}M\to\mathring{T}M$ be a fibre preserving diffeomorphism. As we have seen in the proof of Lemma 1, the derivative of ϕ maps vertical vectors into vertical ones, so we can define the *push-forward of a Finsler vector field* $\widetilde{Y}$ by

$$\phi_\#\widetilde{Y}:=\mathbf{i}^{-1}\circ\phi_*\circ\mathbf{i}\widetilde{Y}\circ\phi^{-1}.$$

Similarly, we can pull back a covariant Finsler tensor along ϕ. For example, if $g\in\mathcal{F}^0_2(\mathring{\pi})$, then

$$\phi^*g(\widetilde{X},\widetilde{Y})(u)=g(\mathbf{i}^{-1}\circ\phi_*\circ\mathbf{i}\widetilde{X},\mathbf{i}^{-1}\circ\phi_*\circ\mathbf{i}\widetilde{Y})(u).$$

Lemma 2. *Let ξ be a projectable vector field on $\mathring{T}M$ and φ its flow. Let $\widetilde{Y} \in \Gamma(\mathring{\pi})$, $A \in \mathcal{T}^1_2(\mathring{\pi})$, $g \in \mathcal{T}^0_2(\mathring{\pi})$ and D a covariant derivative on $\mathring{\pi}$. Then*

$$\widetilde{\mathcal{L}}_\xi \widetilde{Y}(u) = \lim_{s\to 0} \frac{1}{s}\big(\varphi_{-s\#}\widetilde{Y}(u) - \widetilde{Y}(u)\big);) \tag{4}$$

$$\widetilde{\mathcal{L}}_\xi \widetilde{Y}(u) = \lim_{s\to 0} \frac{1}{s}\big(\widetilde{Y}\circ\varphi_s(u) - \mathbf{i}^{-1}\circ\varphi_{s*}\circ\mathbf{i}\widetilde{Y}(u)\big); \tag{5}$$

$$(\widetilde{\mathcal{L}}_\xi A)(\widetilde{X},\widetilde{Y})(u) = \lim_{s\to 0} \frac{1}{s}\big(\varphi_{-s\#}A(\varphi_{s\#}\widetilde{X},\varphi_{s\#}\widetilde{Y})(u) - A(\widetilde{X},\widetilde{Y})(u)\big); \tag{6}$$

$$(\widetilde{\mathcal{L}}_\xi D^{\mathsf{v}})_{\widetilde{X}}\widetilde{Y}(u) = \lim_{s\to 0} \frac{1}{s}\big(\varphi_{-s\#}D^{\mathsf{v}}_{\varphi_{s\#}\widetilde{X}}\varphi_{s\#}\widetilde{Y}(u) - D^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}(u)\big); \tag{7}$$

$$(\widetilde{\mathcal{L}}_\xi g)(\widetilde{X},\widetilde{Y})(u) = \lim_{s\to 0} \frac{1}{s}\big(\varphi_s^* g(\widetilde{X},\widetilde{Y})(u) - g(\widetilde{X},\widetilde{Y})(u)\big); \tag{8}$$

for all $u \in \mathring{T}M$.

Proof. From the dynamical interpretation of the usual Lie bracket,

$$[\xi, \mathbf{i}\widetilde{Y}](u) = \lim_{s\to 0} \frac{1}{s}\big(\varphi_{-s*}\circ\mathbf{i}\widetilde{Y}\circ\varphi_s(u) - \mathbf{i}\widetilde{Y}(u)\big).$$

Since φ_{-s*} maps vertical vectors into vertical ones by Lemma 1(iii), we can act on both sides by $\mathbf{i}^{-1}$, and we obtain (4). Then (5) follows from the continuity of the mapping $(s,v)\mapsto\varphi_{s*}(v)$.

The expression in the parenthesis in the right-hand side of (6) can be decomposed as

$$\begin{aligned}\varphi_{-s\#}A(\varphi_{s\#}\widetilde{X},\varphi_{s\#}\widetilde{Y}) &- \varphi_{-s\#}A(\widetilde{X},\varphi_{s\#}\widetilde{Y})\\ &+ \varphi_{-s\#}A(\widetilde{X},\varphi_{s\#}\widetilde{Y}) - \varphi_{-s\#}A(\widetilde{X},\widetilde{Y}) + \varphi_{-s\#}A(\widetilde{X},\widetilde{Y}) - A(\widetilde{X},\widetilde{Y}).\end{aligned}$$

Then from the $\mathbb{R}$-bilinearity of A, from the continuity of φ and $(s,v)\mapsto\varphi_{s*}(v)$, and from (4) it follows that the limit in (6) is the right-hand side of (3) at u. Since D^{v} is also $\mathbb{R}$-bilinear, the proof of (7) is analogous. Formula (8) can be proved through similar decomposition and by applying (5). □

Remark 3. The vanishing of the Lie derivative of a 'geometric object' (Finsler vector field, tensor, covariant derivative) on $\mathring{T}M$ along ξ means that the 'geometric object' is invariant under the flow of ξ. This is clear if we consider the parentheses in the right side of (4), (6), (7), (8) as $T_{\tau(u)}M$ or $\mathbb{R}$-valued curves with parameter s. Thus we have the following:

$$\begin{aligned}\widetilde{\mathcal{L}}_\xi\widetilde{Y} = 0 &\Leftrightarrow \varphi_{s\#}\widetilde{Y} = \widetilde{Y};\\ \widetilde{\mathcal{L}}_\xi g = 0 &\Leftrightarrow \varphi_s^* g = g;\\ \widetilde{\mathcal{L}}_\xi A = 0 &\Leftrightarrow A(\varphi_{s\#}\widetilde{X},\varphi_{s\#}\widetilde{Y}) = \varphi_{s\#}A(\widetilde{X},\widetilde{Y});\\ \widetilde{\mathcal{L}}_\xi D^{\mathsf{v}} = 0 &\Leftrightarrow D^{\mathsf{v}}_{\varphi_{s\#}\widetilde{X}}\varphi_{s\#}\widetilde{Y} = \varphi_{s\#}D^{\mathsf{v}}_{\widetilde{X}}\widetilde{Y}.\end{aligned}$$

Notice that a horizontal lift X^{h} is projectable by (1) and (2), and the Lie derivative along it is the horizontal Berwald derivative with respect to $\widehat{X}$:

$$\widetilde{\mathcal{L}}_{X^{\mathsf{h}}}\widetilde{Y} = \mathbf{i}^{-1}[X^{\mathsf{h}}, \mathbf{i}\widetilde{Y}] = \mathcal{V}[\mathcal{H}\widehat{X}, \mathbf{i}\widetilde{Y}] = \nabla^{\mathsf{h}}_{\widehat{X}}\widetilde{Y}. \tag{9}$$

3. Parallel translation

Let $\mathcal{H}$ be a Ehresmann connection in $\mathring{T}M$. We assume that $\mathcal{H}$ is *homogeneous*: $[X^{\mathsf{h}}, C] = \mathbf{i}\nabla^{\mathsf{h}}_{\widehat{X}}\delta = 0$ for all $X \in \mathfrak{X}(M)$, where $C := \mathbf{i}\delta$ is the *Liouville vector field*.

Let I be an open interval and let $\gamma : I \to M$ be a curve. A vector field X along γ is said to be *parallel* if

$$\dot{X}(t) = \mathcal{H}(X(t), \dot{\gamma}(t)) \quad \text{for all } t \in I. \tag{10}$$

Such a vector field uniquely exists for all $t \in I$, $v \in \mathring{T}_{\gamma(t)}M$ (see [8, p. 390]), and we denote it by γ_v^{h}. The parallel translation along γ (w.r.t. $\mathcal{H}$) from $\gamma(t)$ to $\gamma(s)$ is the mapping

$$P_s^t(\gamma) : \mathring{\pi}_{\gamma(s)}M \longrightarrow \mathring{\pi}_{\gamma(t)}M, \quad v \longmapsto P_s^t(\gamma)(v) := \gamma_v^{\mathsf{h}}(t).$$

For piecewise smooth curves, the parallel translation can be defined by composing the parallel translations along the smooth segments of the curve.

If $\gamma : I \to M$ is an integral curve of a vector field $X \in \mathfrak{X}(M)$, then γ_v^{h} is an integral curve of X^{h}, because $(\gamma_v^{\mathsf{h}})^{\cdot} = \mathcal{H} \circ (\gamma_v^{\mathsf{h}}, \dot{\gamma}) = \mathcal{H} \circ (\gamma_v^{\mathsf{h}}, X \circ \gamma) = X^{\mathsf{h}} \circ \gamma_v^{\mathsf{h}}$. As a result, the flow of X^{h} determines the parallel translations along the integral curves of X:

Lemma 4. *If $\gamma \colon I \to M$ is the integral curve of a vector field $X \in \mathfrak{X}(M)$, then the parallel translation along γ with respect to a homogeneous Ehresmann connection $\mathcal{H}$ on $\mathring{\pi}$ acts by the rule*

$$P_s^t(\gamma) = \varphi^{\mathsf{h}}_{t-s} \restriction \mathring{T}_{\gamma(s)}M, \quad s, t \in I, \tag{11}$$

where $\varphi^{\mathsf{h}} \colon W \subset \mathbb{R} \times \mathring{T}M \to \mathring{T}M$ is the flow of X^{h}.

Remark 5. Later, we will need the derivative of $P_s^t(\gamma)$ expressed with the flow of X^{h}, thus we need to differentiate (11).

Consider the canonical embedding $i_p \colon T_pM \to TM$ and, for a fixed $u \in T_pM$, the canonical isomorphism from T_pM to T_uT_pM, given by

$$\iota_u \colon v \in T_pM \mapsto \dot{\alpha}(0) \in T_uT_pM, \quad \text{where } \alpha \colon t \in \mathbb{R} \mapsto u + tv \in T_pM.$$

Then

$$\mathbf{i}(u, v) = (i_p \circ \alpha)^{\cdot}(0) = (i_p)_* \circ \dot{\alpha}(0) = (i_p)_*\iota_u(v).$$

We agreed to identify $\mathring{T}_pM \times T_pM$ with $T\mathring{T}_pM$, thus we omit ι_u, and we may consider $\mathbf{i}(u, v)$ as $(i_p)_*(u, v)$. Then we can write (11) as $i_{\gamma(t)} \circ P_s^t(\gamma) = \varphi^{\mathsf{h}}_{t-s} \circ i_{\gamma(s)}$. Taking the derivative of both sides, we get

$$\mathbf{i} \circ P_s^t(\gamma)_* = \varphi^{\mathsf{h}}_{t-s*} \circ \mathbf{i}. \tag{12}$$

Remark 6. Since locally every curve is an integral curve of a vector field, the parallel translation along an arbitrary curve can be obtained by composing flows of horizontal lifts. More precisely, if $\gamma : I \to M$ is a curve, then for all $s, t \in I$, $s < t$ there exists a partition $s = \tau_0 < \cdots < \tau_k = t$ such that $\dot{\gamma} \upharpoonright [\tau_{i-1}, \tau_i] = X_i \circ \gamma \upharpoonright [\tau_{i-1}, \tau_i]$ for some vector fields $X_1, \dots, X_k$, and

$$P_s^t(\gamma) = \phi_k \circ \cdots \circ \phi_1,$$

where $\phi_i := (\varphi_i)^{\mathsf{h}}_{\tau_i - \tau_{i-1}} \upharpoonright \mathring{T}_{\gamma(\tau_{i-1})}M$.

4. The vanishing of the mixed curvature

Let D be a covariant derivative on $\mathring{\pi}$ and $\mathcal{H}$ an Ehresmann connection in $\mathring{T}M$. Then the *mixed curvature* of D is the Finsler tensor $\mathbf{P} \in \mathcal{F}^1_3(\mathring{\pi})$ given by

$$\begin{aligned}\mathbf{P}(\widetilde{X}, \widetilde{Y})\widetilde{Z} &:= R^D(\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y})\widetilde{Z} = D_{\mathbf{i}\widetilde{X}} D_{\mathcal{H}\widetilde{Y}} \widetilde{Z} - D_{\mathcal{H}\widetilde{Y}} D_{\mathbf{i}\widetilde{X}} \widetilde{Z} - D_{[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}]} \widetilde{Z}, \\ &= D^{\mathsf{v}}_{\widetilde{X}} D^{\mathsf{h}}_{\widetilde{Y}} \widetilde{Z} - D^{\mathsf{h}}_{\widetilde{Y}} D^{\mathsf{v}}_{\widetilde{X}} \widetilde{Z} - D_{[\mathbf{i}\widetilde{X}, \mathcal{H}\widetilde{Y}]} \widetilde{Z},\end{aligned}$$

where R^D is the 'classical curvature tensor' of D. In case of the Berwald derivative ∇ determined by $\mathcal{H}$, the mixed curvature is called the *Berwald tensor*, and is denoted by $\mathbf{B}$.

By a *C-Ehresmann manifold* we mean a manifold endowed with an Ehresmann connection $\mathcal{H}$ and a covariant derivative D such that $D^{\mathsf{h}} = \nabla^{\mathsf{h}}$ and $C \in \mathcal{F}^1_2(\mathring{\pi})$ is the difference tensor given by $C(\widetilde{X}, \widetilde{Y}) = D^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y} - \nabla^{\mathsf{v}}_{\widetilde{X}} \widetilde{Y}$.

Lemma 7. *In a C-Ehresmann manifold, for all $X \in \mathfrak{X}(M)$, $\widetilde{Y}, \widetilde{Z} \in \Gamma(\mathring{\pi})$, we have*

$$(\mathcal{L}_{X^{\mathsf{h}}} D^{\mathsf{v}})_{\widetilde{Y}} \widetilde{Z} = -\mathbf{P}(\widetilde{Y}, \widehat{X}, \widetilde{Z}).$$

Proof. An immediate calculation shows that

$$\begin{aligned}(\mathcal{L}_{X^{\mathsf{h}}} D^{\mathsf{v}})_{\widetilde{Y}} \widetilde{Z} &\overset{(9)}{=} \nabla^{\mathsf{h}}_{\widehat{X}} D^{\mathsf{v}}_{\widetilde{Y}} \widetilde{Z} - D^{\mathsf{v}}_{\nabla^{\mathsf{h}}_{\widehat{X}} \widetilde{Y}} \widetilde{Z} - D^{\mathsf{v}}_{\widetilde{Y}} \nabla^{\mathsf{h}}_{\widehat{X}} \widetilde{Z} \\ &= \nabla^{\mathsf{h}}_{\widehat{X}} D^{\mathsf{v}}_{\widetilde{Y}} \widetilde{Z} - D^{\mathsf{v}}_{\mathcal{V}[X^{\mathsf{h}}, \mathbf{i}\widetilde{Y}]} \widetilde{Z} - D^{\mathsf{v}}_{\widetilde{Y}} \nabla^{\mathsf{h}}_{\widehat{X}} \widetilde{Z} \\ &\overset{(*)}{=} \nabla^{\mathsf{h}}_{\widehat{X}} D^{\mathsf{v}}_{\widetilde{Y}} \widetilde{Z} - D_{[X^{\mathsf{h}}, \mathbf{i}\widetilde{Y}]} \widetilde{Z} - D^{\mathsf{v}}_{\widetilde{Y}} \nabla^{\mathsf{h}}_{\widehat{X}} \widetilde{Z} \\ &= -D^{\mathsf{v}}_{\widetilde{Y}} \nabla^{\mathsf{h}}_{\widehat{X}} \widetilde{Z} + \nabla^{\mathsf{h}}_{\widehat{X}} D^{\mathsf{v}}_{\widetilde{Y}} \widetilde{Z} + D_{[\mathbf{i}\widetilde{Y}, X^{\mathsf{h}}]} \widetilde{Z} \\ &= -\mathbf{P}(\widetilde{Y}, \widehat{X}, \widetilde{Z}).\end{aligned}$$

In step $(*)$ we used that $[X^{\mathsf{h}}, \mathbf{i}\widetilde{Y}]$ is vertical, and hence

$$\mathbf{v}[X^{\mathsf{h}}, \mathbf{i}\widetilde{Y}] = [X^{\mathsf{h}}, \mathbf{i}\widetilde{Y}]. \qquad \square$$

Proposition 8. *For a C-Ehresmann manifold, the following are equivalent:*

(i) The mixed curvature of D vanishes;
(ii) $\mathcal{L}_{X^{\mathsf{h}}} D^{\mathsf{v}} = 0$ for all $X \in \mathfrak{X}(M)$;
(iii) $\mathbf{B} = \nabla^{\mathsf{h}} C$;

(iv) The flow φ^{h} *of the horizontal lift* X^{h} *of any vector field* $X \in \mathfrak{X}(M)$ *satisfies*

$$\varphi^{\mathsf{h}}_{s\#} D^{\mathsf{v}}_{\widetilde{Y}} \widetilde{Z}(u) = D_{\varphi^{\mathsf{h}}_{s\#}\widetilde{Y}} \varphi^{\mathsf{h}}_{s\#} \widetilde{Z}(u)$$

for all $u \in \mathring{T}M$ *and parameter* s *such that both sides are defined;*

(v) If $\Phi\colon \mathring{T}_pM \to \mathring{T}_qM$ *is the parallel translation along a piecewise smooth curve connecting* p *and* q*, then it is an automorphism between the affinely connected manifolds* $(\mathring{T}_pM, D^p)$ *and* $(\mathring{T}_qM, D^q)$*, that is*

$$D^q_{\Phi_\# Y} \Phi_\# Z = \Phi_\# D^p_Y Z, \quad Y, Z \in \mathfrak{X}(\mathring{T}_pM).$$

(Here $\Phi_\#$ *is the usual push-forward given by* $\Phi_\# Y := \Phi_* \circ Y \circ \Phi^{-1}$*.)*

Proof. Since basic lifts locally generate $\Gamma(\mathring{\pi})$, Lemma 7 proves (i)⇔(ii). Applying Lemma 7 to the Berwald derivative, we get $(\mathcal{L}_{X^{\mathsf{h}}} \nabla^{\mathsf{v}})_{\widetilde{Y}} \widetilde{Z} = -\mathbf{B}(\widetilde{Y}, \widehat{X}, \widetilde{Z})$. As a result,

$$\begin{aligned} -\mathbf{P}(\widetilde{Y}, \widehat{X}, \widetilde{Z}) &= (\mathcal{L}_{X^{\mathsf{h}}} D^{\mathsf{v}})_{\widetilde{Y}} \widetilde{Z} = (\mathcal{L}_{X^{\mathsf{h}}} \nabla^{\mathsf{v}})_{\widetilde{Y}} \widetilde{Z} + (\mathcal{L}_{X^{\mathsf{h}}} C)(\widetilde{Y}, \widetilde{Z}) \\ &= -\mathbf{B}(\widetilde{Y}, \widehat{X}, \widetilde{Z}) + \nabla^{\mathsf{h}} C(\widehat{X}, \widetilde{Y}, \widetilde{Z}), \end{aligned}$$

and we have (i)⇔(iii).

In Remark 3 we have explained (ii)⇔(iv). Finally (v) implies (iv) by the relations (11) and (12) we established between parallel translations and flows of horizontal lifts. The converse holds because any curve consists of segments that are integral curves, see Remark 6. □

5. Landsberg manifolds

Let (M, F) be a Finsler manifold and let $\mathcal{H}$ be the canonical Ehresmann connection (see [8, Th. 9.3.5] or [7, Ch.3/D]). We have the *fundamental tensor* $g := \frac{1}{2} \nabla^{\mathsf{v}} \nabla^{\mathsf{v}} F^2$, the *Cartan tensor* $\mathcal{C}_\flat = \nabla^{\mathsf{v}} g$ and the *vector-valued Cartan tensor* given by $g(\mathcal{C}(\widetilde{X}, \widetilde{Y}), \widetilde{Z}) = \mathcal{C}_\flat(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$. These have the following properties:

$$(13) \qquad (a)\ \mathcal{C}_\flat(\delta, \widetilde{X}, \widetilde{Y}) = 0, \quad (b)\ \nabla^{\mathsf{h}} g(\delta, \widetilde{X}, \widetilde{Y}) = 0, \quad (c)\ \nabla^{\mathsf{h}}_\delta \mathcal{C}_\flat = -\nabla^{\mathsf{h}} g,$$

see Def. and Lemma 9.2.25, Cor. 9.3.16, and Prop. 9.3.17 in [8], or formula (53), Cor. 6.4 and Prop. 6.5 in [1]. We have the *mixed Ricci formula*

$$(14) \quad \nabla^{\mathsf{v}} \nabla^{\mathsf{h}} g(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}) - \nabla^{\mathsf{h}} \nabla^{\mathsf{v}} g(\widetilde{Y}, \widetilde{X}, \widetilde{Z}, \widetilde{U}) = - g(\mathbf{B}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}), \widetilde{U}) - g(\widetilde{Z}, \mathbf{B}(\widetilde{X}, \widetilde{Y}, \widetilde{U}))$$

(see [8, Lemma 7.13.10] or [1, Lemma 3.7]), and the relation

$$(15) \qquad g(\nabla^{\mathsf{h}} \mathcal{C}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}), \widetilde{U}) = \nabla^{\mathsf{h}} \mathcal{C}_\flat(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{U}) - \nabla^{\mathsf{h}} g(\widetilde{X}, \mathcal{C}(\widetilde{Y}, \widetilde{Z}), \widetilde{U}),$$

which can be checked by a simple calculation. The Berwald tensor is completely symmetric and satisfies $\mathbf{B}(\delta, \widetilde{Y}, \widetilde{Z}) = 0$.

Proposition 9. *Let (M,F) be a Finsler manifold, ∇ the Berwald derivative induced by the canonical Ehresmann connection $\mathcal{H}$, and $\mathcal{C}$ the vector-valued Cartan tensor. Let D be the Hashiguchi derivative on $\mathring{\pi}$ given by $D^{\mathsf{h}} = \nabla^{\mathsf{h}}$, $D^{\mathsf{v}} = \nabla^{\mathsf{v}} + \frac{1}{2}\mathcal{C}$. Then the following are equivalent:*

(i) $\nabla^{\mathsf{h}} g = 0$*, i.e., (M,F) is a Landsberg manifold;*
(ii) $\nabla^{\mathsf{h}}\mathcal{C}_\flat$ *is completely symmetric;*
(iii) $\mathbf{B} = \frac{1}{2}\nabla^{\mathsf{h}}\mathcal{C}$*;*
(iv) The mixed curvature of D vanishes;
(v) $\mathcal{L}_{X^{\mathsf{h}}} D^{\mathsf{v}} = 0$ *for all $X \in \mathfrak{X}(M)$;*
(vi) The flow φ^{h} of the horizontal lift X^{h} of any vector field $X \in \mathfrak{X}(M)$ satisfies

$$\varphi^{\mathsf{h}}_{s\#} D^{\mathsf{v}}_{\widetilde{Y}} \widetilde{Z}(u) = D^{\mathsf{v}}_{\varphi^{\mathsf{h}}_{s\#}\widetilde{Y}} \varphi^{\mathsf{h}}_{s\#} \widetilde{Z}(u)$$

for all $u \in \mathring{T}M$ and parameter s such that both sides are defined;
(vii) With the same notation, $\varphi^{\mathsf{h}*}_s g = g$*.*
(viii) If $\Phi \colon \mathring{T}_pM \to \mathring{T}_qM$ is the parallel translation along a piecewise smooth curve connecting p and q, then it is an automorphism between the affinely connected manifolds $(\mathring{T}_pM, D^p)$ and $(\mathring{T}_qM, D^q)$, that is

$$D^q_{\Phi_\# Y} \Phi_\# Z = \Phi_\# D^p_Y Z, \quad Y, Z \in \mathfrak{X}(\mathring{T}_pM).$$

(ix) With the notation of (viii), Φ is an isometry between the Riemannian manifolds $(\mathring{T}_pM, g^p)$ and $(\mathring{T}_qM, g^q)$.

Proof. The equivalence of (iii)–(vi) and (viii) is immediate from Proposition 8 applied to the $\frac{1}{2}\mathcal{C}$-Ehresmann manifold.

We prove that (i),(ii) and (iii) are also equivalent. If we suppose (i), we have

$$(*) \qquad \nabla^{\mathsf{h}}\mathcal{C}_\flat(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) := \nabla^{\mathsf{h}}\nabla^{\mathsf{v}} g(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) \overset{(14),(\mathrm{i})}{=} g(\mathbf{B}(\widetilde{X},\widetilde{Y},\widetilde{Z}),\widetilde{U}) + g(\widetilde{Z},\mathbf{B}(\widetilde{X},\widetilde{Y},\widetilde{U})).$$

Since g and $\mathbf{B}$ are both symmetric, so is $\nabla^{\mathsf{h}}\mathcal{C}_\flat$, and we get (ii). Conversely, if $\nabla^{\mathsf{h}}\mathcal{C}_\flat$ is symmetric, then

$$-\nabla^{\mathsf{h}} g(\widetilde{X},\widetilde{Y},\widetilde{Z}) \overset{(13)(c)}{=} \nabla^{\mathsf{h}}\mathcal{C}_\flat(\delta,\widetilde{X},\widetilde{Y},\widetilde{Z}) = \nabla^{\mathsf{h}}\mathcal{C}_\flat(\widetilde{X},\delta,\widetilde{Y},\widetilde{Z}) \overset{(13)(\mathrm{a})}{=} \mathcal{C}_\flat(\nabla^{\mathsf{h}}_{\widetilde{X}}\delta,\widetilde{Y},\widetilde{Z}).$$

Since $\mathcal{H}$ is homogeneous, $\nabla^{\mathsf{h}}\delta = 0$, and we obtain (i).

Next we show that (iii) is a consequence of (i) and (ii). First, we have

$$g(\nabla^{\mathsf{h}}\mathcal{C}(\widetilde{X},\widetilde{Y},\widetilde{Z}),\widetilde{U}) \overset{(15)}{=} \nabla^{\mathsf{h}}\mathcal{C}_\flat(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) - \nabla^{\mathsf{h}} g(\widetilde{X},\mathcal{C}(\widetilde{Y},\widetilde{Z}),\widetilde{U}) \overset{(\mathrm{i})}{=} \nabla^{\mathsf{h}}\mathcal{C}_\flat(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}).$$

Applying Christoffel's trick on $\widetilde{Y}$, $\widetilde{Z}$, $\widetilde{U}$ and using the symmetry of $\nabla^{\mathsf{h}}\mathcal{C}_\flat$, $\mathbf{B}$ and g, we obtain

$$
\begin{aligned}
g(\nabla^{\mathsf{h}}\mathcal{C}(\widetilde{X},\widetilde{Y},\widetilde{Z}),\widetilde{U}) &= \nabla^{\mathsf{h}}\mathcal{C}_\flat(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) \\
&= \nabla^{\mathsf{h}}\mathcal{C}_\flat(\widetilde{X},\widetilde{Y},\widetilde{Z},\widetilde{U}) + \nabla^{\mathsf{h}}\mathcal{C}_\flat(\widetilde{X},\widetilde{Z},\widetilde{U},\widetilde{Y}) - \nabla^{\mathsf{h}}\mathcal{C}_\flat(\widetilde{X},\widetilde{U},\widetilde{Y},\widetilde{Z}) \\
&\overset{(*)}{=} g(\mathbf{B}(\widetilde{X},\widetilde{Y},\widetilde{Z}),\widetilde{U}) + g(\widetilde{Z},\mathbf{B}(\widetilde{X},\widetilde{Y},\widetilde{U})) \\
&\quad + g(\mathbf{B}(\widetilde{X},\widetilde{Z},\widetilde{U}),\widetilde{Y}) + g(\widetilde{U},\mathbf{B}(\widetilde{X},\widetilde{Z},\widetilde{Y})) \\
&\quad - g(\mathbf{B}(\widetilde{X},\widetilde{U},\widetilde{Y}),\widetilde{Z}) - g(\widetilde{Y},\mathbf{B}(\widetilde{X},\widetilde{U},\widetilde{Z})) \\
&= 2g(\mathbf{B}(\widetilde{X},\widetilde{Y},\widetilde{Z}),\widetilde{U}),
\end{aligned}
$$

as desired. Finally, we prove (iii)⇒(i) by calculation:

$$
\begin{aligned}
-\nabla^{\mathsf{h}}g(\widetilde{Y},\widetilde{Z},\widetilde{U}) &\overset{(13)(c)}{=} \nabla^{\mathsf{h}}\mathcal{C}_\flat(\delta,\widetilde{Y},\widetilde{Z},\widetilde{U}) \\
&\overset{(15)}{=} g(\nabla^{\mathsf{h}}\mathcal{C}(\delta,\widetilde{Y},\widetilde{Z}),\widetilde{U}) + \nabla^{\mathsf{h}}g(\delta,\mathcal{C}(\widetilde{Y},\widetilde{Z}),\widetilde{U}) \\
&\overset{(13)(b),\ (iii)}{=} g(2\mathbf{B}(\delta,\widetilde{Y},\widetilde{Z}),\widetilde{U}) = 0.
\end{aligned}
$$

Since $\widetilde{\mathcal{L}}_{X^{\mathsf{h}}}g \overset{(9)}{=} \nabla^{\mathsf{h}}_{\widehat{X}}g$, Remark 3 implies (i)⇔(vii). Finally, the equivalence of (vii) and (ix) can be justified the same way as we proved (iv)⇔(v) in Proposition 8. □

It is easy to show that $D^{\mathsf{v}}g = 0$, and hence the covariant derivative D^p in (viii) is the Levi-Civita derivative of the Riemannian metric g^p in (ix). This explains why we needed Christoffel's trick to justify that $\nabla^{\mathsf{h}}_{\widehat{X}}g = 0$ implies $\widetilde{\mathcal{L}}_{X^{\mathsf{h}}}D^{\mathsf{v}} = 0$, because this step is in essence depends on the uniqueness of the Levi-Civita derivative.

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Institute of Mathematics,
University of Debrecen,
H–4010 Debrecen, P.O. Box 12, Hungary
E-mail address: kerteszd@science.unideb.hu