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SEMI-SYMMETRIC GENERALIZED SASAKIAN-SPACE-FORMS

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Abstract. We show that semi symmetric and pseudo symmetric generalized Sasakian-space-forms are Einstein when $(0,6)$-tensors satisfy $R \cdot R = 0$, $R \cdot R = L_R Q(g,R)$, $R \cdot C = 0$, $R \cdot C = L_C Q(g,C)$, and $C \cdot C = 0$, where C is Quasi conformal curvature tensor. Further we discuss about Ricci solitons.

1. Introduction

Let (M,g) be a $(2n+1)$ dimensional Riemannian manifold and let ∇ be its Levi-Civita connection. The endomorphism $R(X,Y)Z$ of the Lie algebra of vector fields on M, named the curvature operator is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where X, Y, and Z are vector fields on M and $[X,Y]$ denotes the Lie bracket of X and Y. We also denote $R(X,Y)Z$ as the derivation induced by the curvature operator. For a symmetric $(0,2)$-tensor A and any vector fields X, Y, and Z on M, we define the endomorphism $(X \wedge_A Y)$ of M by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y.$$

The Riemannian Christoffel curvature tensor R and the $(0,4)$-tensor G are defined by

$$R(X,Y,W,Z) = g(R(X,Y)W,Z),$$
$$G(X,Y,W,Z) = g((X \wedge_g Y)Z,W).$$

respectively.

The $(0,6)$-tensor $R \cdot R$, obtained by the action of the curvature operators $R(X,Y)$ on the $(0,4)$-curvature tensor R, is given by [9]

(1)
$$(R \cdot R)(U,V,W,Z;X,Y) = -R(R(X,Y)U,V,W,Z) - R(U,R(X,Y)V,W,Z) - R(U,V,R(X,Y)W,Z) - R(U,V,W,R(X,Y)Z),$$

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where $U, V, W, Z, X, Y \in M$.

The tensor $R \cdot R$ has the following algebraic properties:

$$\begin{aligned}(R \cdot R)(U, V, W, Z; X, Y) &= -(R \cdot R)(V, U, W, Z; X, Y)\\ &= -(R \cdot R)(U, V, Z, W; X, Y),\end{aligned}$$

$$\begin{aligned}(R \cdot R)(U, V, W, Z; X, Y) &+ (R \cdot R)(U, W, Z, V; X, Y)\\ &+ (R \cdot R)(U, Z, V, W; X, Y) = 0,\end{aligned}$$

$$(R \cdot R)(U, V, W, Z; X, Y) = -(R \cdot R)(U, V, W, Z; Y, X),$$

$$\begin{aligned}(R \cdot R)(U, V, W, Z; X, Y) &+ (R \cdot R)(W, Z, X, Y; U, V)\\ &+ (R \cdot R)(X, Y, U, V; W, Z) = 0.\end{aligned}$$

The simplest $(0, 6)$-tensor having the same symmetry properties as $R \cdot R$ may well be the Tachibana tensor $Q(g, R)$ defined by [9]

(2)
$$\begin{aligned}Q(g, R)(U, V, W, Z; X, Y) = R((X \wedge Y)U, V, W, Z) &+ R(U, (X \wedge Y)V, W, Z)\\ + R(U, V, (X \wedge Y)W, Z) &+ R(U, V, W, (X \wedge Y)Z)\end{aligned}$$

In the context of generalized Sasakian-space-forms, U. K. Kim [7] studied locally symmetric properties of generalized Sasakian-space-forms. In [5] U. C. De and A. Sarkar have studied some symmetry properties of generalized Sasakian-space-forms regarding the projective curvature tensor. In [8] D. G. Prakasha studied some pseudosymmetric properties of generalized Sasakian-space-forms with Weyl conformal curvature tensor. In [3] C. S. Bagewadi and G. Ingalahalli studied some results of C-Bochner curvature tensor and τ-curvature tensor of a generalized Sasakian space forms. As a continuation of this, we plan to study generalized Sasakian space forms satisfying certain curvature conditions on Quasi conformal curvature tensor.

2. Preliminaries

A $(2n + 1)$ dimensional Riemannian manifold is called an almost contact metric manifold if the following results hold [4]

$$\begin{aligned}\phi^2 X &= -X + \eta(X)\xi, & \phi\xi &= 0,\\ \eta(\xi) &= 1, & g(X, \xi) &= \eta(X),\\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), & \eta(\phi X) &= 0,\\ g(\phi X, Y) &= -g(X, \phi Y), & g(\phi X, X) &= 0,\\ & & (\nabla_X \eta)(Y) &= g(\nabla_X \xi, Y).\end{aligned}$$

For a $(2n+1)$ dimensional generalized Sasakian-space-forms we have [1]

$$\begin{aligned} R(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} \\ + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{3}$$

For any vector fields X, Y, Z on M, where R denotes the curvature tensor of M,

$$\begin{aligned} S(X,Y) &= (2nf_1 + 3f_2 - f_3)g(X,Y) + (3f_2 - (2n-1)f_3)\eta(X)\eta(Y), \\ QX &= (2nf_1 + 3f_2 - f_3)X + (3f_2 - (2n-1)f_3)\eta(X)\xi, \\ r &= 2n(2n+1)f_1 + 6nf_2 - 4nf_3 \end{aligned}$$

are valid, where f_1, f_2, f_3 are differentiable functions on M and X, Y, Z are vector fields on M. In such case we will write the manifold as $M(f_1, f_2, f_3)$. This kind of manifolds appears as natural generalization of the Sasakian-space-forms by taking $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$, where c denotes constant ϕ-sectional curvature. The ϕ-sectional curvature of generalized Sasakian-space-forms $M(f_1, f_2, f_3)$ is $f_1 + 3f_2$. Moreover, cosymplectic space-forms and Kenmotsu space-forms are also particular case of generalized Sasakian-space-forms.

For generalized Sasakian-space-forms we also have

$$R(X,Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y],$$

$$R(\xi,X)Y = (f_1 - f_3)[g(X,Y)\xi - \eta(Y)X], \tag{4}$$

$$R(\xi,X)\xi = (f_1 - f_3)[\eta(X)\xi - X], \tag{5}$$

$$R(\xi,\xi)X = 0,$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X).$$

3. Semi symmetric generalized Sasakian-space-forms

Let $M(f_1, f_2, f_3)$ be a $(2n+1)$ dimensional semi symmetric generalized Sasakian-space-forms. Then from (1) we have

$$\begin{aligned} (R\cdot R)(U,V,W,Z;X,Y) &= 0, \\ (R(X,Y)\cdot R)(U,V,W,Z) &= 0, \\ -R(R(X,Y)U,V,W,Z) - R(U,R(X,Y)V,W,Z) & \\ -R(U,V,R(X,Y)W,Z) - R(U,V,W,R(X,Y)Z) &= 0, \end{aligned}$$

$$\begin{aligned} R(R(X,Y)U,V,W,Z) + R(U,R(X,Y)V,W,Z) & \\ +R(U,V,R(X,Y)W,Z) + R(U,V,W,R(X,Y)Z) &= 0. \end{aligned} \tag{6}$$

In view of (4) and (5), for $X = U = \xi$, (6) yields

$$(f_1 - f_3)[R(Y,V,W,Z) + (f_1 - f_3)g(Y,W)g(V,Z) - (f_1 - f_3)g(Y,Z)g(V,W)] = 0.$$

Since $(f_1 - f_3) \neq 0$, we have

$$R(Y,V,W,Z) = (f_1 - f_3)g(Y,Z)g(V,W) + (f_1 - f_3)g(Y,W)g(V,Z). \tag{7}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (7) and taking summation over $i, (1 \leq i \leq (2n+1))$, we get

$$S(V,W) = 2n(f_1 - f_3)g(V,W). \tag{8}$$

Therefore, $M(f_1, f_2, f_3)$ is an Einstein manifold. Hence we state the following.

Theorem 1. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$ dimensional generalized Sasakian-space-forms. If $M(f_1, f_2, f_3)$ is semi symmetric then $M(f_1, f_2, f_3)$ is an Einstein manifold.*

Corollary 1. *(g, V, λ) is Ricci soliton in semi symmetric generalized Sasakian space forms if and only if V is conformal killing vector field.*

Proof. From Theorem 1 and by the definition of Ricci soliton [6], we have

$$(L_V g)(V,W) + 2S(V,W) + 2\lambda g(V,W) = 0,$$

where λ is some constant. From (8) we get

$$(L_V g)(V,W) + 4n(f_1 - f_3)g(V,W) + 2\lambda g(V,W) = 0. \tag{9}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $V = W = e_i$ in (9) and taking summation over $i, (1 \leq i \leq (2n+1))$, we get

$$(L_V g)(e_i, e_i) + 4n(2n+1)(f_1 - f_3) + 2(2n+1)\lambda = 0.$$

Since $[e_i, e_j] = 0$, for all $1 \leq i, j \leq (2n+1)$, we obtain

$$\lambda = -2n(f_1 - f_3).$$

Thus, Ricci soliton in semi symmetric generalized Sasakian-space forms is shrinking if $f_1 > f_3$, i.e., $\lambda < 0$, steady if $f_1 = f_3$, i.e., $\lambda = 0$, and expands if $f_1 < f_3$, i.e., $\lambda > 0$. □

4. Pseudosymmetric generalized Sasakian-space-forms

Let $M(f_1, f_2, f_3)$ be a $(2n+1)$ dimensional generalized Sasakian-space-forms. Then from (1) and (2) we have

$$(R \cdot R)(U,V,W,Z;X,Y) = L_R Q(g,R)(U,V,W,Z;X,Y),$$

$$(R(X,Y) \cdot R)(U,V,W,Z) = -L_R((X \wedge Y) \cdot R)(U,V,W,Z),$$

(10)

$$-R(R(X,Y)U,V,W,Z) - R(U,R(X,Y)V,W,Z) - R(U,V,R(X,Y)W,Z)$$
$$-R(U,V,W,R(X,Y)Z) = L_R[R((X \wedge Y)U,V,W,Z) + R(U,(X \wedge Y)V,W,Z)$$
$$+R(U,V,(X \wedge Y)W,Z) + R(U,V,W,(X \wedge Y)Z)]$$

In view of (4) and (5), for $X = U = \xi$, (10) yields

$$(f_1 - f_3)[R(Y,V,W,Z) + (f_1 - f_3)g(Y,W)g(V,Z) - (f_1 - f_3)g(Y,Z)g(V,W)]$$
$$= -L_R[R(Y,V,W,Z) + (f_1 - f_3)g(Y,W)g(V,Z) - (f_1 - f_3)g(Y,Z)g(V,W)],$$

$$[L_R + (f_1 - f_3)][R(Y,V,W,Z) + (f_1 - f_3)g(Y,W)g(V,Z) - (f_1 - f_3)g(Y,Z)g(V,W)] = 0.$$

Therefore either $L_R = -(f_1 - f_3)$ or

$$R(Y,V,W,Z) = (f_1 - f_3)g(Y,Z)g(V,W) - (f_1 - f_3)g(Y,W)g(V,Z). \tag{11}$$

Let $\{e_1, e_2, ..., e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (11) and taking summation over $i, (1 \leq i \leq (2n+1))$, we get

$$S(V,W) = 2n(f_1 - f_3)g(V,W).$$

Therefore, $M(f_1, f_2, f_3)$ is an Einstein manifold. Hence we state the following.

Theorem 2. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$ dimensional generalized Sasakian-space-forms. If $M(f_1, f_2, f_3)$ is pseudosymmetric then $M(f_1, f_2, f_3)$ is an Einstein manifold provided that $L_R \neq -(f_1 - f_3)$.*

Corollary 2. *(g, V, λ) is Ricci soliton in pseudo symmetric generalized Sasakian space forms if and only if V is conformal killing vector field provided $L_R \neq -(f_1 - f_3)$.*

Proof. Proof follows from Theorem 2 and the definition of Ricci Soliton. □

5. Quasi conformal semi symmetric generalized Sasakian-space-forms

For a $(2n+1)$ dimensional almost contact metric manifold the Quasi conformal curvature tensor C is given by

$$\begin{aligned} &C(X,Y)Z \\ &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &\quad - \frac{r}{2n+1}\left[\frac{a}{2n} + 2b\right][g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{12}$$

Using equation (3) in (12) yields

$$C(X,Y)\xi = D[\eta(Y)X - \eta(X)Y],$$
$$C(\xi,X)Y = D[g(X,Y)\xi - \eta(X)Y], \tag{13}$$
$$C(\xi,X)\xi = D[\eta(X)\xi - X], \tag{14}$$

where

$$D = a(f_1 - f_3) + 2nb(f_1 - f_3) + b(2nf_1 + 3f_2 - f_3) - \frac{r}{2n+1}\left[\frac{a}{2n} + 2b\right].$$

Let $M(f_1, f_2, f_3)$ be a $(2n+1)$ dimensional Quasi conformal semi symmetric generalized Sasakian-space-forms. Then, from (1), we have

$$\begin{aligned}(R(X,Y)\cdot C)(U,V,W,Z) &= 0,\\ -C(R(X,Y)U,V,W,Z) - C(U,R(X,Y)V,W,Z)&\\ -C(U,V,R(X,Y)W,Z) - C(U,V,W,R(X,Y)Z) &= 0.\end{aligned}$$

In view of (4), (5), and (13), for $X = U = \xi$, (14) yields

$$(f_1 - f_3)\{C(Y,V,W,Z) + D[g(Y,W)g(V,Z) - g(Y,Z)g(V,W)]\} = 0.$$

Since $(f_1 - f_3) \neq 0$, we have

$$C(Y,V,W,Z) = D[g(Y,Z)g(V,W) - g(Y,W)g(V,Z)]. \tag{15}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (15) and taking summation over $i, (1 \leq i \leq (2n+1))$, using equation (12), we get

$$S(V,W) = D^{'}g(V,W),$$

where

$$D^{'} = \frac{2n[(a+2nb)(f_1 - f_3) + b(2nf_1 + 3f_2 - f_3)] - br}{a + b(2n+1)}.$$

Theorem 3. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$ dimensional generalized Sasakian-space-forms. If $M(f_1, f_2, f_3)$ is Quasi conformal semisymmetric then $M(f_1, f_2, f_3)$ is an Einstein manifold.*

6. Quasi conformal pseudo symmetric generalized Sasakian-space-forms

Let $M(f_1, f_2, f_3)$ be a $(2n+1)$ dimensional Quasi conformal pseudo symmetric generalized Sasakian-space-forms. Then, from (1) and (2), we have

$$\begin{aligned}&(R\cdot C)(U,V,W,Z;X,Y) = L_C Q(g,C)(U,V,W,Z;X,Y),\\ &(R(X,Y)\cdot C)(U,V,W,Z) = -L_C((X\wedge Y)\cdot R)(U,V,W,Z),\\ &-C(R(X,Y)U,V,W,Z) - C(U,R(X,Y)V,W,Z) - C(U,V,R(X,Y)W,Z)\\ &-C(U,V,W,R(X,Y)Z) = L_C[C((X\wedge Y)U,V,W,Z) + C(U,(X\wedge Y)V,W,Z)\\ &+C(U,V,(X\wedge Y)W,Z) + C(U,V,W,(X\wedge Y)Z)\end{aligned} \tag{16}$$

In view of (4), (5), and (13), for $X = U = \xi$, (16) yields

$$[L_C + (f_1 - f_3)]\{C(Y,V,W,Z) + D[g(Y,W)g(V,Z) - g(Y,Z)g(V,W)]\} = 0.$$

Therefore, either $L_C = -(f_1 - f_3)$ or

$$C(Y,V,W,Z) = D[g(Y,Z)g(V,W) - g(Y,W)g(V,Z)]. \tag{17}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (17) and taking summation over $i, (1 \leq i \leq (2n+1))$, using equation (12), we get

$$S(V, W) = D^{'} g(V, W). \tag{18}$$

Theorem 4. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$ dimensional generalized Sasakian-space-forms. If $M(f_1, f_2, f_3)$ is Quasi conformal pseudo symmetric then $M(f_1, f_2, f_3)$ is an Einstein manifold provided that $L_C \neq -(f_1 - f_3)$.*

7. Generalized Sasakian-space-forms satisfies the condition $C \cdot C = 0$

Let $M(f_1, f_2, f_3)$ be a $(2n+1)$ dimensional generalized Sasakian-space-forms. Let $C \cdot C$ be a $(0, 6)$-tensor and $C \cdot C = 0$. Then

$$(C(X, Y) \cdot C)(U, V, W, Z) = 0,$$
$$-C(C(X, Y)U, V, W, Z) - C(U, C(X, Y)V, W, Z)$$
$$-C(U, V, C(X, Y)W, Z) - C(U, V, W, C(X, Y)Z) = 0. \tag{19}$$

In view of (13) and (14), for $X = U = \xi$, (19) yields

$$-D[C(Y, V, W, Z) + D\{g(Y, W)g(V, Z) - g(Y, Z)g(V, W)\}] = 0.$$

Since $D \neq 0$, we have

$$C(Y, V, W, Z) = D[g(Y, Z)g(V, W) - g(Y, W)g(V, Z)]. \tag{20}$$

Let $\{e_1, e_2, \ldots, e_{2n+1}\}$ be an orthonormal basis of the tangent space at each point of the manifold. Putting $Y = Z = e_i$ in (20) and taking summation over $i, (1 \leq i \leq (2n+1))$, using equation (12), we get

$$S(V, W) = D^{'} g(V, W).$$

Theorem 5. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$ dimensional generalized Sasakian-space-forms. If $(0, 6)$-tensor $C \cdot C = 0$ holds on $M(f_1, f_2, f_3)$, then $M(f_1, f_2, f_3)$ is an Einstein manifold.*

Corollary 3. *Let $M(f_1, f_2, f_3)$ be a $(2n+1)$ dimensional Quasi conformal semi symmetric generalized Sasakian-space-forms. Then $R \cdot C = C \cdot C$ holds on $M(f_1, f_2, f_3)$.*

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